NOTES AND CORRESPONDENCE

On the Development of a Rossby Wave Critical Level

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ABSTRACT

The asymptotics of the analysis of Dickinson (1970) on the development of a Rossby wave critical level are reexamined. These linear results are used to discuss certain aspects of the development of the nonlinear critical level for Rossby waves. In particular it is shown that the critical layer becomes fully nonlinear in a (nondimensional) time scale of $O(\epsilon^{-\frac{1}{2}})$, where ϵ is a parameter measuring the amplitude of the forced wave, and that all harmonics are likely to be important in the critical layer after this time.

1. Introduction

The purpose of this note is to discuss some of the nonlinear aspects of the development of a Rossby wave critical level and at the same time give a simpler, more consistent analysis of the associated linear problem originally posed by Dickinson (1970). The new analysis indicates that very near the center of the critical layer, the perturbation zonal velocity component grows like $\ln t$. There is some confusing discussion in the text of Dickinson from which one might conclude that the large-time behavior should be as t although his Figs. 4 and 5 clearly support the conclusion of $\ln t$ growth. It is further demonstrated that the nonlinearities become of $O(1)$ importance within the critical layer for nondimensional $t\sim O(\epsilon^{-\frac{1}{2}})$, where ϵ is a parameter which measures the strength of the forced wave. This result is consistent with the results of both Benney and Bergeron (1969) and Davis (1969), who demonstrated that for steady parallel flows the critical layer has a thickness $O(\epsilon^{\frac{1}{2}})$ when nonlinearities dominate, and the linear result of Dickinson (1970) that the critical layer thickness diminishes as $O(t^{-1})$. It should be noted here that while the β term was omitted from the studies of Benney and Bergeron (1968) and Davis (1969), its inclusion results in quantitative but not qualitative changes in their solutions. The indicated behavior of the critical layer thickness as a function of time is given schematically in Fig. 1. For $t\sim O(1)$, the layer has yet to form, while for $1\ll t\ll\epsilon^{-\frac{1}{2}}$, the thickness is given by the linear $O(t^{-1})$ behavior quoted above. For $t\sim O(\epsilon^{-\frac{1}{2}})$, the layer thickness equilibrates at a thickness $O(\epsilon^{\frac{1}{2}})$. While this description is admittedly somewhat speculative, it is in substantial agreement with the recent numerical integrations of the nonlinear equations by Béland (1976).

Finally it is noted that scale arguments similar to those of Lin (1958) for the nonlinear-viscous parallel flow problem, suggest that *all* harmonics become as important as the mean flow distortion in the vicinity of the critical layer, casting doubt on the validity of "nonlinear" mean flow/single harmonic interaction models such as that used by Geisler and Dickinson (1974). Similar remarks are likely to apply to other wave problems involving critical layers.

2. The linear problem

The nonlinear Rossby wave equation for weakly forced long waves on a zonal flow with constant shear u' can be written in nondimensional form as

$$\Phi_{yyt}+y\Phi_{yyx}+\Phi_x+\epsilon(\Phi_x\Phi_{yyy}-\Phi_y\Phi_{yyx})=0,$$

where Φ is the perturbation streamfunction related to the nondimensional streamfunction by $\psi=-\frac{1}{2}y^2+\epsilon\Phi$. The nondimensional variables are related to the dimensional (primed) variables by

$$t'=\frac{\beta t}{ku'^2},\quad y'=\frac{u'y}{\beta},\quad x'=x/k,$$

and the amplitude parameter ϵ is defined by

$$\epsilon=\frac{\beta^2 A}{u'^3},$$

where A is the dimensional forcing amplitude. In the following it is assumed that $\epsilon\ll 1$; for $\epsilon\sim O(1)$ the concept of a critical layer fails to apply.

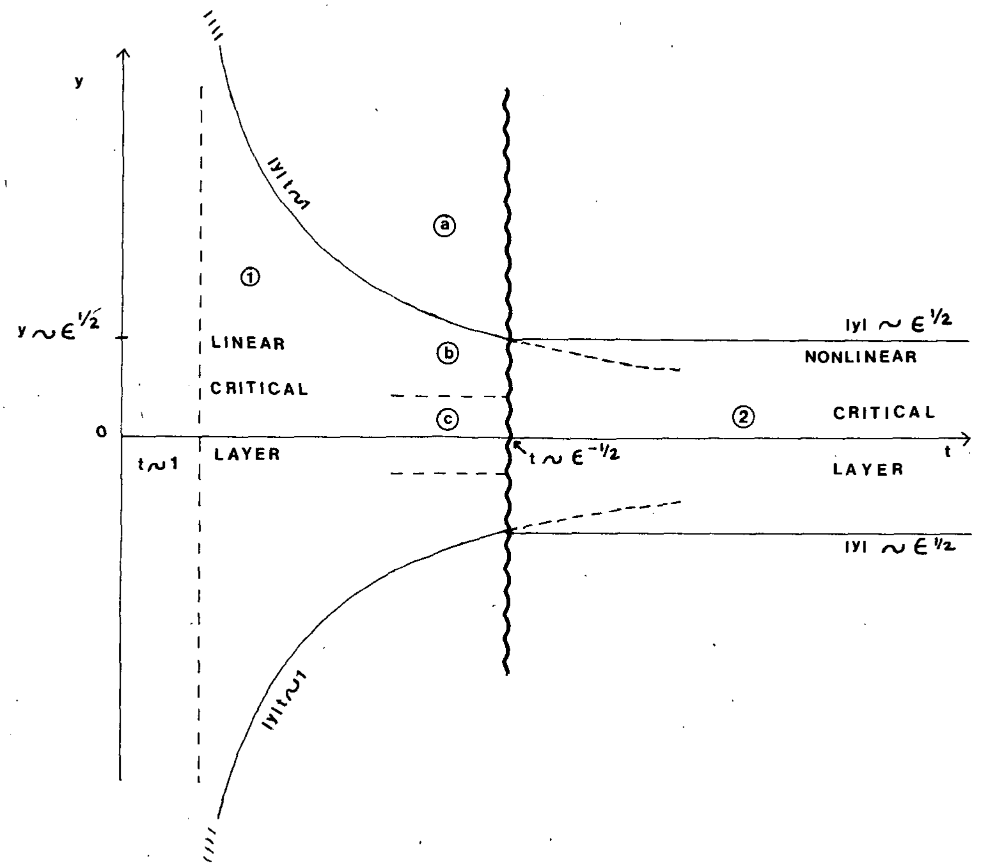

FIG. 1. Schematic representation of the development of a nonlinear Rossby wave critical layer. In region ①, $1 \ll t \ll \epsilon^{-\frac{1}{2}}$, $|y|t<1$, the critical layer is linear, and its width decreases as $O(t^{-1})$. In region ②, $t>\epsilon^{-\frac{1}{2}}$, and the critical layer is nonlinear with thickness of $O(\epsilon^{\frac{1}{2}})$. The linear asymptotic regions are: (a) the outer region: $|y|t \gg 1$, $t \gg 1$, solution given by Eq. (3); (b) critical layer: $|y|t \sim O(1)$, $t \gg 1$, solution given by Eq. (4); (c) the center of the layer: $|y|t \ll 1$, $t \gg 1$, solution given by Eq. (5).

The linear theory studied by Dickinson (1970) results from neglecting the $O(\epsilon)$ terms in the above equation. Clearly this approach is valid if the nonlinear terms remain uniformly small in both space and time. It is necessary to check this *a posteriori* by using the linear solutions to evaluate the neglected terms; for the moment we suppose that these are small. The resulting problem was solved by Dickinson (1970) for the semi-infinite domain $y_0 \geq y > -\infty$ for a switched-on forcing

$$\Phi(x,t) = \mathrm{Re}[H(t)e^{ix}] \quad \text{at} \quad y = y_0.$$

For sufficiently large times he demonstrated that the solution was adequately represented by

$$\phi(y,t) = \frac{ia}{2}\int_0^t \frac{d\tau}{\tau^2}\exp[-i(y\tau+\tau^{-1})] + O(t^{-2}). \quad (1)$$

Dickinson then proceeded to examine the asymptotic limits of this expression. One of the ranges he considered

$$|y|^{\frac{1}{2}} \sim O(1), \quad |y|^{\frac{1}{2}}t \ll 1,$$

[see his Eq. (27)] is, however, inconsistent with the requirement that t be large in (1). We now demonstrate how the various relevant asymptotic limits

(a) $|y|t \gg 1$, the outer region
(b) $|y|t \sim O(1)$, the critical layer
(c) $|y|t \ll 1$, the center of the critical layer

can be obtained simply from (1), with a minimum of analysis. Since (1) can be rewritten as

$$\phi(y,t) = \phi(y,\infty) - \frac{ia}{2}\int_t^\infty \frac{d\tau}{\tau^2}\exp[-i(y\tau+\tau^{-1})] + O(t^{-2}),$$

where $\phi(y,\infty)$ is the steady solution (see Dickinson, 1970) and since t is large, we can expand the $e^{-i/\tau}$ term in the integrand and integrate term by term. We find that

$$\phi(y,t) = \phi(y,\infty) - \frac{ia|y|}{2}\sum_{n=0}^{\infty}\frac{(-i|y|)^n}{n!} \times I_{n+2}^{\mp}(|y|t) + O(t^{-2}), \quad (2)$$

where

$$I^{\mp}(x)=\int_x^{\infty}\frac{e^{\mp iu}}{u^n}du,$$

the upper sign pertaining for $y>0$.

(a) The outer region ($|y|t\gg1$, $t\gg1$)

From the asymptotic expression

$$I_n^{\mp}(x)=\mp\frac{ie^{\mp ix}}{x^n}\left\{1\pm\frac{in}{x}-\frac{n(n+1)}{x^2}+\cdots\right\},$$

which can be easily obtained by integration by parts, we find on using (2) that

$$\phi(y,t)=\phi(y,\infty)-\frac{ae^{-iyt}}{2yt^2}+O(t^{-2})+O(y^{-2}t^{-3})+O(y^{-1}t^{-3}). \quad (3)$$

This is essentially Dickinson's result (25), except that his results contains some erroneous factors of 2.

(b) The critical layer ($|y|t\sim O(1)$, $t\gg1$)

In this case since $|y|\ll1$ the first term in the series (2) is sufficient and

$$\phi(y,t)=\phi(y,\infty)-\frac{ia|y|}{2}I_2^{\mp}(|y|t)+O(|y|^2)+O(t^{-2}). \quad (4)$$

(c) The center of the layer ($|y|t\ll1$, $t\gg1$)

In this case the asymptotics could be obtained from (2) by considering the limit for $|y|t\ll1$. However, it is easier to return to (1) and simplify by expanding $e^{-iy\tau}$ in the integrand. Thus for $|y|t\ll1$, $t\gg1$

$$\phi(y,t)=\frac{ia}{2}\sum_{n=0}^{\infty}\frac{(-iy)^n}{n!}I_n^{-}(1/t)+O(t^{-2}).$$

With the use of

$$I_0^{-}(1/t)=-ie^{-i/t},\quad I_1^{-}(1/t)=\ln t-\left(\gamma+\frac{i\pi}{2}\right)+O(t^{-1}),$$

where γ is Euler's constant, the above expression can be written

$$\phi(y,t)\sim\frac{ia}{2}\left\{-ie^{-i/t}-iy\left[\ln t-\left(\gamma+\frac{i\pi}{2}\right)+O(t^{-1})\right]-\frac{y^2}{2}[te^{-i/t}+O(\ln t)]+\frac{iy^3}{6}\left[t^2\frac{e^{-i/t}}{2}+O(t)\right]\right\}+O(t^{-2}). \quad (5)$$

The expressions (3)–(5) describe fairly completely the flow throughout the entire domain. Further, in an analogous fashion, expressions can also be obtained for the various derivatives of ϕ. These turn out to be identical to those obtained by direct differentiation of (3)–(5). Thus, for example, within the critical layer the zonal velocity component is found from (4) to be

$$\phi_y(y,t)\sim\phi_y(y,\infty)+\frac{ia}{2yt}e^{-iyt}\mp\frac{ia}{2}I_2^{\mp}(|y|t)$$

for $|y|t\sim O(1)$ and $t\gg1$. Near the center of the layer (5) gives

$$\phi_y(y,t)\sim\frac{a}{2}\ln t+O(1),$$

i.e. the zonal velocity component grows like $\ln t$ near the center of the critical layer.

In the scale analysis given in the following section the various derivatives of ϕ appearing in the vorticity equation are required. These are easily obtained from (3)–(5); in particular it is found that near the center of the critical layer

$$\phi\sim O(1),\quad \phi_y O\sim(\ln t),\quad \phi_{yy}\sim O(t)$$
$$\phi_{yyy}\sim O(t^2),\quad \phi_{yyt}\sim O(1).$$

3. Remarks on the nonlinear problem

As mentioned in the Introduction the linear solution can only be uniformly valid providing the nonlinear terms remain sufficiently small for all time and all space. From the results of the previous section this is clearly not the case since the vorticity equation scales as

$$\phi_{yyt}+y\phi_{yyx}+\phi_x+\epsilon(\phi_x\phi_{yyy}-\phi_y\phi_{yyx})=0$$
$$O(1)\quad O(yt)\quad O(1)\quad O(\epsilon t^2)\quad O(\epsilon t\ln t)$$

for $yt\ll1$, $t\gg1$. The nonlinear terms become $O(1)$ when $\epsilon t^2\sim O(1)$, that is, the critical layer becomes nonlinear when $t\sim O(\epsilon^{-\frac{1}{2}})$. Further since the nonlinear terms become of $O(1)$ importance at this time all harmonics will be generated and more importantly they will not be ordered. This result is identical to the conclusions of Lin (1958) for the viscous critical layer and is in agreement with the result of Benney and Bergeron (1969) for the steady nonlinear-viscous case when nonlinearities dominate. One must conclude therefore that single harmonic "nonlinear" models such as that used by Geisler and Dickinson (1974) are probably not rational approximations to the complete equations.

The three principal results quoted here, namely 1) the $\ln t$ behavior of the perturbation zonal wind, 2) the $\epsilon^{-\frac{1}{2}}$ time to nonlinearity and 3) the necessity of including many harmonics, have been verified by numerical integrations of the complete equations (Béland, 1976). The authors are presently engaged in performing a more complete investigation of the fully nonlinear Rossby wave critical layer.

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Comments on "The Role of Electric Forces in Charge Separation by Falling Precipitation in Thunderclouds"

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Kamra (1970, 1975) has used simple mathematical models to set the upper limits of cloud electrification by several gravitational mechanisms of charge separation. In his latest model, he requires that the size of the cloud particles, the number of cloud particles, the size of the precipitation particles, the number of precipitation particles, the charge per cloud particle and the charge per precipitation particle do not change with time. He has assumed that the cloud has already developed precipitation-sized particles and that the particles have acquired charges before he begins the computations, presuming that these assumptions will not affect the maximum value of the electric field that can be attained.

In his earlier paper he allowed the charges on the particles to vary with the electric field to approximate the effect of polarization charging. In either case an estimate of the maximum attainable electric field was atttained through an integration of Maxwell's differential equation for the growth of the electric field due to the contributing currents. As a result of these calculations, Kamra (1975) has concluded that electrification of clouds by any mechanism involving precipitation can only be effective as a secondary mechanism. Within the limitations of the models used, this conclusion appears valid. However, we have recently refined this treatment to allow for expected variations in particle sizes and numbers as well as charge sizes and numbers in a reasonably complete, interactive, stochastic model (Scott and Levin, 1975). Our model considers the electrical forces on all the particles and their altered fall velocities, the recombination of charge due to capture processes, the redistribution of charge during bouncing collisions, and the efficiency of contact and the contact angle.

Results from our model indicate field growth to over 4 kV cm^{-1} in realistic growth times even with precipitation rates well below 1 cm h^{-1} (Levin and Scott, 1975). Such fields are in agreement with most measurements of electric field strengths in thunderclouds (Winn and Moore, 1971, 1972; Gunn, 1954). One must point out that all the models to date only address themselves to average fields in the cloud and it is possible that lightning is triggered by higher fields that exist for short times due to inhomogenities in the cloud electrical structure. Such inhomogenities cannot be predicted by infinite cloud models since the field is averaged; full two-dimensional or even three-dimensional models are needed to resolve them. However, the electric fields reported are also electronically