Planetary Waves in Horizontal and Vertical Shear: Asymptotic Theory for Equatorial Waves in Weak Shear

D. G. ANDREWS AND M. E. MCINTYRE

U. K. Universities' Atmospheric Modelling Group, Department of Geophysics, University of Reading and Department of Applied Mathematics and Theoretical Physics, University of Cambridge

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ABSTRACT

A simple multiple-scale expansion procedure is given for calculating corrections to the structure of equatorial planetary waves in the presence of weak shear and dissipation. For upward-propagating Rossby-gravity (Yanai-Maruyama) and Kelvin (Wallace-Kousky) waves, explicit results are obtained for the case of Newtonian cooling and Rayleigh friction, correct to the first two orders in the ratio μ of wave to mean-flow height scales. The results are used in a direct calculation of the horizontal Reynolds stress $\overline{u'v'}$ and demonstrate the strong dependence of $\overline{u'v'}$ on the ratio of friction to cooling coefficients.

In certain parameter regimes of interest in the tropical stratosphere, a slight north-south asymmetry in the y profile of $\bar{u}$ can cause changes in the wave structure such that the mean zonal acceleration $\partial\bar{u}/\partial t$ tends to have the *same* asymmetry. That is, there may be a tendency for asymmetries in $\bar{u}(y)$ to amplify in the presence of dissipating waves.

1. Introduction

In the previous paper (Andrews and McIntyre, 1976; hereafter referred to as I), we studied the mean zonal acceleration $\partial\bar{u}/\partial t$ due to dissipation, generation or transience of general wave-like disturbances. In the case of dissipating, upward-propagating, equatorial planetary waves the results showed that the y profile of $\partial\bar{u}/\partial t$ depends crucially upon whether the dissipation is thermal or mechanical in nature.

The possibility of more than one $\partial\bar{u}/\partial t$ profile arises from the sensitivity of the horizontal Reynolds stress $\overline{u'v'}$ to slight modifications of the wave structure induced by the dissipation terms. A direct calculation of $\overline{u'v'}$ requires the wave structure to be known correct to two orders in the corresponding small parameter μ. In this paper we give an appropriate asymptotic expansion procedure, and present explicit solutions for Kelvin and Rossby-gravity waves. These permit an independent calculation of $\partial\bar{u}/\partial t$, which has been checked with the results in I. We also give some results concerning the effect of slight horizontal shear $\partial\bar{u}/\partial y$ of the basic flow upon the wave structure and thus upon $\partial\bar{u}/\partial t$.

Our expansion is a straightforward application of known techniques [and Bretherton (1968) uses them in a similar but more general context] and is decisively simpler than the procedure adopted by Lindzen (1971, 1972). It is this that makes it practicable to obtain the wave-structure corrections. It is perhaps desirable to state our belief that some of the complication in Lindzen's approach arises from an unnecessary concern over secularity in the waveguide dimension y, and over the elimination of critical-latitude singularities. Our analysis is so arranged that these singularities never arise, showing immediately that they need not be explicitly considered for the present purpose. Further clarification results from expressing the basic procedure in terms of the original set of equations rather than a single, approximate, derived equation. There results a method which makes it clear in principle how to obtain solutions correct to any power in μ with reassurance that the expansion will not be ill-behaved at the critical latitudes. This is actually of some mathematical importance in the present state of the art of multiple-scale expansion procedures. Even though one may not want all the higher corrections explicitly, their existence and good behavior appears to be required by mathematical proofs of the uniform validity, at lower order, over a domain even *one* "slow" scale in size (Mahony, 1972).

2. The expansion procedure

Using the notation and scaling assumptions of I, Section 6, we write the inverse square root of the Richardson number as

$$\bar{u}_z/N \lesssim \mathbf{U}/\mathbf{NH} \lesssim \mu \ll 1,$$

where $\mathbf{N}$ is a scale for the buoyancy frequency (measuring the static stability), $\mathbf{U}$ a velocity scale for the basic zonal flow, and $\mathbf{H}$ a height scale for the vertical variation of all quantities, except wave phase. The latter varies on the scale

$$m^{-1}\sim \mathbf{h}=\mu\mathbf{H}\ll\mathbf{H}, \tag{2.1}$$

m being the local vertical wavenumber.

Our expansion is in powers of the small parameter μ and so, like Lindzen's, relies on the separation of scales, $\mathbf{h}\ll\mathbf{H}$. In the usual way we define "slow" independent variables

$$\mathcal{T}\equiv\mu t,\qquad \mathfrak{z}\equiv\mu z.$$

The horizontal and vertical shear of the basic flow are assumed "weak" in the sense that

$$\left.\begin{aligned}\bar{u}&=\bar{u}_0(\mathfrak{z})+\mu\bar{u}_1(y,\mathfrak{z})+O(a^2\mathrm{U})\\ \bar{\theta}_z&=N^2(\mathfrak{z})+O(\mu^2\mathrm{N}^2)+O(a^2\mathrm{N}^2)\end{aligned}\right\}. \tag{2.2}$$

It will be assumed that each disturbance variable in I, Eqs. (4.1), can be expanded in an asymptotic series in powers of μ:

$$u'=u_0+\mu u_1+\mu^2u_2+\cdots,\ \text{etc.},$$

where each term has the wave-like structure

$$u_r=\mathrm{Re}\{\hat{u}_r(y,\mathfrak{z},\mathcal{T})\exp i[k(x-ct)+\mu^{-1}\phi(\mathfrak{z})]\}. \tag{2.3}$$

We also assume that the dissipation is "weak," i.e.,

$$(X',Y',Q')=(\mu X',\mu Y',\mu Q'). \tag{2.4}$$

In Eqs. (4.1) of I we now make the substitutions

$$D_t\leftrightarrow -i(\omega_0-\mu k\bar{u}_1)+\mu\partial/\partial\mathcal{T},$$

$$\partial/\partial z\leftrightarrow im+\mu\partial/\partial\mathfrak{z},$$

where

$$\omega_0(\mathfrak{z})=k(c-\bar{u}_0)\quad\text{and}\quad m(\mathfrak{z})=d\phi/d\mathfrak{z}.$$

Equating the coefficients of successive powers of μ to zero, we obtain a hierarchy of sets of equations which, in matrix form, may be written

$$\mathbf{M}\hat{\mathbf{U}}_0=\mathbf{O}, \tag{2.5}$$

$$\mathbf{M}\hat{\mathbf{U}}_1=\mathbf{A}_0, \tag{2.6}$$

$$\cdots\cdots\cdots\cdots$$

$$\mathbf{M}\hat{\mathbf{U}}_r=\mathbf{A}_{r-1}, \tag{2.7}$$

where the transpose of the five-vector $\mathbf{U}_r$ is given by

$$\hat{\mathbf{U}}_r^{\text{transpose}}=\{\hat{u}_r,\hat{v}_r,-\hat{w}_r,\hat{\theta}_r/N,\hat{p}_r\}. \tag{2.8}$$

$\mathbf{A}_{r-1}$ is a vector whose components depend (in a non-singular manner) on the wave-structure functions $\hat{u}_s$, etc., for $s\leqslant r-1$, and $\mathbf{M}$ is a 5×5 matrix differential operator in y. $\mathbf{M}$ and $\mathbf{A}_0$ are given by

$$\mathbf{M}=\begin{bmatrix}\omega_0 & -i\beta y & 0 & 0 & -k\\ i\beta y & \omega_0 & 0 & 0 & i\partial/\partial y\\ 0 & 0 & 0 & iN & m\\ 0 & 0 & -iN & \omega_0 & 0\\ -k & i\partial/\partial y & m & 0 & 0\end{bmatrix}, \tag{2.9}$$

and

$$\mathbf{A}_0^{\text{transpose}}=-i\{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5\}, \tag{2.10}$$

where

$$\left.\begin{aligned}\sigma_1&=(\partial/\partial\mathcal{T}+ik\bar{u}_1)\hat{u}_0+\hat{X}_0+\bar{u}_{1_y}\hat{v}_0+\bar{u}_{0\mathfrak{z}}\hat{w}_0\\ \sigma_2&=(\partial/\partial\mathcal{T}+ik\bar{u}_1)\hat{v}_0+\hat{Y}_0\\ \sigma_3&=-\hat{p}_{0\mathfrak{z}}\\ N\sigma_4&=(\partial/\partial\mathcal{T}+ik\bar{u}_1)\hat{\theta}_0+Q_0-\beta y\bar{u}_{0\mathfrak{z}}\hat{v}_0\\ \sigma_5&=\hat{w}_{0\mathfrak{z}}\end{aligned}\right\}. \tag{2.11}$$

We require that $\hat{\mathbf{U}}_r$ lie in the space V of five-vectors whose components are square-integrable functions of y, and define an inner product $\{\ ,\ \}$ over V by

$$\{\mathbf{a},\mathbf{b}\}\equiv\langle\sum_{n=1}^{5}a_nb_n^*\rangle=\{\mathbf{b},\mathbf{a}\}^*,\quad \mathbf{a},\mathbf{b}\in V, \tag{2.12}$$

where

$$\langle(\ )\rangle\equiv\int_{-\infty}^{\infty}(\ )dy,\quad <\infty,$$

and an asterisk denotes the complex conjugate. It can be verified that $\mathbf{M}$ is self-adjoint with respect to $\{\ ,\ \}$, i.e.,

$$\{\mathbf{a},\mathbf{M}\mathbf{b}\}=\{\mathbf{M}\mathbf{a},\mathbf{b}\},\quad \mathbf{a},\mathbf{b}\in V. \tag{2.13}$$

The zero-order problem (2.5) is the standard problem for equatorial waves, and the solutions are quoted in dimensionless form in I, Section 9. However, (2.5) determines each solution only to within a multiplicative factor depending on $\mathfrak{z}$ and $\mathcal{T}$. For any given solution this factor is found by considering the next problem in the hierarchy, as follows.

Given that a solution $\hat{\mathbf{U}}_0$ of (2.5) exists in V, the self-adjointness property (2.13) implies that a necessary condition for a solution $\hat{\mathbf{U}}_1$ of (2.6) to exist in V is

$$\{\mathbf{A}_0,\hat{\mathbf{U}}_0\}=0. \tag{2.14}$$

Substitution from (2.8) and (2.10), (2.11) into the real part of (2.14) and use of (2.12), (2.2) and (2.4) then give

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial\mathcal{T}}\langle\overline{u_0^2}+\overline{v_0^2}+\overline{\theta_0^2}/N^2\rangle+\langle\overline{X_0u_0}+\overline{Y_0v_0}+\overline{Q_0\theta_0}/N^2\rangle\\ =-\left[\frac{\partial}{\partial\mathfrak{z}}\langle\overline{p_0w_0}\rangle+\langle\bar{u}_{1_y}\overline{u_0v_0}\rangle\right.\\ \left.+\langle\bar{u}_{0\mathfrak{z}}(\overline{u_0w_0}-f\overline{v_0\theta_0}/N^2)\rangle\right].\end{aligned} \tag{2.15}$$

This is just the y-integrated form of the leading-order version of the wave-energy equation [I, Eq. (4.2)]. Using the explicit zero-order wave solutions in Section 3, it can be shown to imply the y-integrated wave-action equation with forcing:

$$A_{\mathcal{T}}+(\gamma A)_{\mathfrak{z}}=-\mathfrak{F}. \tag{2.16}$$

Here $\gamma=\partial\omega_0/\partial m$, the vertical component of group velocity. The y integrated wave-action density

$$A\equiv\tfrac{1}{2}\langle\overline{u_0^2}+\overline{v_0^2}+\overline{\theta_0^2}/N^2\rangle/\omega_0(\mathfrak{z}),$$

and its flux γA take the simple forms implied by the fact that $\bar{u}_0$ and therefore ω_0 are here assumed independent of y, and slowly-varying in z (Bretherton and Garrett 1968, p. 545). The forcing term $\mathfrak{F}$ is given by ω_0^{-1} times the second group of terms on the left of (2.15), i.e.,

$$\mathfrak{F}\equiv\langle\overline{X_0u_0}+\overline{Y_0v_0}+\overline{Q_0\theta_0}/N^2\rangle/\omega_0(\mathfrak{z}).$$

Eq. (2.15) is also equivalent to the transport equation for the $(\mathfrak{z},\mathcal{T})$ dependence of the modulus of the zero-order wave amplitude. The transport equation for the phase is the imaginary part of (2.14).

Condition (2.14) is also sufficient[1] to ensure the existence of a solution $\hat{\mathbf{U}}_1$. Elimination from (2.6) yields

$$\mathfrak{L}\hat{v}_1=G_0(y,\mathfrak{z},\mathcal{T}),\tag{2.17}$$

where G_0 depends in a well-behaved manner on the zero-order solutions, and is given by

$$G_0=\frac{i}{m\omega_0^2}\left[\left(\frac{k\partial}{\partial y}+\frac{\beta m^2\omega_0 y}{N^2}\right)(kS+im\omega_0\sigma_1)+\left(\frac{m^2\omega_0^2}{N^2}-k^2\right)\left(\frac{\partial S}{\partial y}-m\omega_0\sigma_2\right)\right],\tag{2.18}$$

where $S=i\omega_0\sigma_3+N\sigma_4+iN^2\sigma_5/m$. Also

$$\mathfrak{L}\left(y,\frac{\partial}{\partial y},\mathfrak{z}\right)\equiv\frac{\partial^2}{\partial y^2}+\left(\frac{m^2\omega_0^2}{N^2}-k^2-\frac{\beta k}{\omega_0}\right)-\left(\frac{\beta^2m^2}{N^2}\right)y^2;\tag{2.19}$$

this operator, unlike that in the equation for $\hat{p}_0$ [Lindzen 1971, Eq. (10)], is not singular at the critical latitudes $y=\pm\omega_0/\beta$. In all cases except that in which the zero-order solution is a Kelvin wave, (2.17) provides the most convenient starting point for explicit solution. Some solutions for Kelvin and Rossby-gravity waves are quoted in Section 3.

The foregoing procedure may, in principle, be carried to any order in μ; at each stage the condition for the existence of a solution $\hat{\mathbf{U}}_r$ to (2.7) is

$$\{\mathbf{A}_{r-1},\hat{\mathbf{U}}_0\}=0,$$

and imposes constraints on the components of $\mathbf{A}_{r-1}$. No critical-latitude singularities occur anywhere in the analysis—e.g., no factors $(\omega_0^2-\beta^2y^2)^{-1}$ in (3.4), (3.6) and (3.8)–(3.10) below. In particular, the expressions giving $\hat{p}_r$ and the other fields in terms of $\hat{v}_r$ are bounded at the critical latitudes. [For modes other than Kelvin waves, this can be seen to follow from the fact that the determinant $N^2k^2-\omega_0^2m^2$ of the (2, 2)th minor of $\mathbf{M}$ does not vanish at any y.]

[1] Courant and Hilbert (1953, pp. 356, 394). No further statements are required, since for equatorial waves the eigenvalue problem (2.5) is nondegenerate. That is, there is never more than one linearly independent eigenvector $\hat{\mathbf{U}}_0$, for given ω_0 and k (cf. Courant and Hilbert, *loc. cit.*, p. 346).

3. Results for Kelvin and Rossby-gravity waves when N is constant

For convenience of presentation, we use here the dimensionless notation of I, Section 9, with the sign convention $\omega_0>0$. To ensure an upward group velocity this means that we must take $s=1$ in Eqs. (9.4) (9.6a) and (9.7a) of I. Formally, we put $m=-1$, $N=\beta=1$ after solving (2.5) and (2.6) or (2.17). The zero-order solutions [cf. I, Eqs. (9.6a), (9.7a)] are as follows:

Kelvin wave:

$$\{\hat{u}_0,\hat{v}_0,\hat{w}_0,\hat{\theta}_0\}=a(\mathfrak{z},\mathcal{T})\{i,0,i\omega_0,1\}e^{-\frac{1}{2}y^2}.\tag{3.1}$$

Rossby-gravity $(n=0)$ *wave:*

$$\{\hat{u}_0,\hat{v}_0,\hat{w}_0,\hat{\theta}_0\}=a(\mathfrak{z},\mathcal{T})\{i\omega_0y,1,i\omega_0^2y,\omega_0y\}e^{-\frac{1}{2}y^2}.\tag{3.2}$$

Here the behavior of $|a(\mathfrak{z},\mathcal{T})|$ is governed in each case by the wave-action equation (2.16), or equivalently by I, Eq. ⟨(6.26)⟩.

To determine the first-order solutions, we must specify the forcing terms in I, Eqs. (4.1); slightly generalizing I, Eq. (9.1), we take

$$(\hat{X}_0,\hat{Y}_0,\hat{Q}_0)=(\alpha_1\hat{u}_0,\ \alpha_2\hat{v}_0,\ \alpha_3\hat{\theta}_0).\tag{3.3}$$

Kelvin wave:

When solving for $\hat{\mathbf{U}}_1$, the Kelvin wave is an exceptional case in that it is simplest to calculate $\hat{\mathbf{U}}_1$ directly from (2.6), rather than by solving (2.17) first. We find

$$\hat{v}_1=-\tfrac{1}{2}ika\bar{u}_{0\mathfrak{z}}ye^{-\frac{1}{2}y^2}+(\text{terms involving }\bar{u}_1).\tag{3.4}$$

In the special case

$$(\alpha_1,\alpha_2,\alpha_3)=(0,0,\alpha),\quad \partial/\partial\mathcal{T}=0,\quad \bar{u}_0=0,\quad \bar{u}_1=\Lambda y,\tag{3.5}$$

we find

$$\hat{\theta}_1=a\omega_0^2\Lambda ye^{-\frac{1}{2}y^2}+\cdots,\tag{3.6}$$

where the dots represent terms symmetric in y and proportional to α.

Rossby-gravity $(n=0)$ *wave:*

In all cases except the Kelvin wave we use (2.17), which in dimensionless notation for $n=0$ becomes

$$\left(\frac{\partial}{\partial y}-y\right)\left(\frac{\partial}{\partial y}+y\right)\hat{v}_1=G_0.\tag{3.7}$$

This can readily be integrated in closed form once the appropriate zero-order solutions have been substituted in (2.18). The solubility condition (2.14) constrains G_0 to be such that (3.7) possesses a solution which is bounded as $|y|\to\infty$. In particular, we find

$$\hat{u}_1=\left[\left(1-\frac{k}{\omega_0}\right)a_T+\left(\alpha_1-\frac{k\alpha_2}{\omega_0}\right)a+\omega_0\bar{u}_{0z}a\right]ye^{-\frac{1}{2}y^2}$$
$$+\left[-\frac{(1+\omega_0^2)}{2\omega_0^2}a_T-\frac{(\alpha_1\omega_0^2+\alpha_2)}{2\omega_0^2}a+\frac{(\omega_0^3k^3+3\omega_0^2k^2+2\omega_0k-2)\bar{u}_{0z}a}{4\omega_0^3}\right]y^3e^{-\frac{1}{2}y^2}$$
$$+\left[-\frac{(1+\omega_0^2)}{8\omega_0^2}ak\bar{u}_{0z}\right]y^5e^{-\frac{1}{2}y^2}$$
$$+(\text{terms involving } \bar{u}_1) \quad (3.8)$$

$$\hat{v}_1=i\left[\frac{(1+\omega_0^2)}{2\omega_0^3}a_T+\frac{(\alpha_1\omega_0^2+\alpha_2)}{2\omega_0^3}a+\frac{(2-4\omega_0k-3\omega_0^2k^2)\bar{u}_{0z}a}{4\omega_0^2}\right]y^2e^{-\frac{1}{2}y^2}$$
$$+i\left[\frac{k(1+\omega_0^2)}{8\omega_0^3}\bar{u}_{0z}a\right]y^4e^{-\frac{1}{2}y^2}$$
$$+(\text{terms involving } \bar{u}_1). \quad (3.9)$$

When conditions (3.5) hold, we find

$$\hat{\theta}_1=-\frac{1}{2\omega_0}\Lambda a\left[(1-2\omega_0k-2\omega_0^2k^2)+\left(\frac{2\omega_0^3k^3+4\omega_0^2k^2-1}{1+\omega_0k}\right)y^2\right]e^{-\frac{1}{2}y^2}+\cdots, \quad (3.10)$$

where the dots represent terms antisymmetric in y and proportional to α.

The solutions (3.4), (3.6) and (3.8)–(3.10) will be required in Section 4. The calculation of the other components of $\hat{\mathbf{U}}_1$ for each wave is straightforward, but lengthy, and the results will not be presented here.

4. Discussion

The mean zonal acceleration is given by

$$\frac{\partial\bar{u}}{\partial t}=-\frac{\partial}{\partial y}\{\overline{u'v'}-\bar{u}_z\overline{v'\theta'}/\bar{\theta}_z\}-\frac{\partial}{\partial z}\{\overline{u'w'}-(f-\bar{u}_y)\overline{v'\theta'}/\bar{\theta}_z\}, \quad (4.1)$$

with relative error $O(\mu^2)$ [Holton (1975); I, Eq. (6.16)]. Our calculation of the first-order corrections (3.4), (3.8), (3.9) allows us to find $\partial\bar{u}/\partial t$ directly from (4.1) for Kelvin and Rossby-gravity waves, since the term

$$\overline{u'v'}=\mu(\overline{u_0v_1}+\overline{u_1v_0})+O(\mu^2)$$

can now be found to sufficient accuracy. [Note that (3.1) and (3.2) imply $\overline{u_0v_0}=0$ for each wave.]

For the Kelvin wave, (3.1) and (3.4) give

$$\overline{u'v'}=-\tfrac{1}{4}\mu k|a^2|\bar{u}_{0z}ye^{-y^2}+O(\mu^2), \quad (4.2)$$

while for the Rossby-gravity wave, (3.2), (3.8) and (3.9) yield

$$\overline{u'v'}=\mu\left[\frac{|a^2|_T}{4\omega_0^2}+\frac{1}{2}\left(\alpha_1-\frac{k}{\omega_0}\alpha_2\right)|a^2|+\tfrac{1}{2}\omega_0\bar{u}_{0z}|a^2|\right]ye^{-y^2}-\mu\left[\frac{k^2(1+\omega_0^2)}{4\omega_0}\bar{u}_{0z}|a^2|\right]y^3e^{-y^2}+O(\mu^2). \quad (4.3)$$

For each wave, it is found that the terms in $\hat{u}_1$ and $\hat{v}_1$ due to $\bar{u}_1$ are in quadrature with $\hat{v}_0$ and $\hat{u}_0$, respectively, and thus do not contribute to $\overline{u'v'}$ at leading order [as could have been deduced from Eq. (7.4) of I]. The remaining terms in (4.1) can be evaluated from (3.1) and (3.2). We have verified that the results for $\partial\bar{u}/\partial t$ so obtained agree with those calculated via the generalized Eliassen-Palm relation in I, Section 9.

If we now make use of the generalized Eliassen-Palm relation, together with the $O(\mu)$ corrections $\hat{\mathbf{U}}_1$, we can go one stage further in accuracy, and examine the effect of weak horizontal shear $\partial\bar{u}/\partial y$ upon $\partial\bar{u}/\partial t$ at the next order in μ. For example in the case, given by (3.5), of waves of steady amplitude dissipated only by thermal effects, with a basic flow

$$\bar{u}=\mu\Lambda y+O(\mu^2)+O(a^2), \quad \Lambda \text{ constant},$$

and with

$$X'=Y'=0, \qquad Q'=\alpha\theta',$$

we find, with the help of Eq. (7.1) of I,

$$\partial\bar{u}/\partial t=\frac{\mu\alpha}{c-\bar{u}}\frac{\overline{\theta'^2}}{N^2}+O(\mu^3)$$
$$=\left.\frac{\partial\bar{u}}{\partial t}\right|_{\Lambda=0}+\frac{\mu^2\alpha}{N^2c}\left(\frac{\Lambda y}{c}\overline{\theta_0^2}+2\overline{\theta_0\theta_1}\right)+O(\mu^3). \quad (4.4)$$

The leading-order contribution is as indicated by the case $\lambda=0$ in I, Section 9. However, the small shear $\mu\Lambda y$ makes itself felt at the next order in μ both through the factor $(c-\bar{u})^{-1}$ [the first $O(\mu^2)$ term in (4.4)] and by producing a non-zero $\overline{\theta_0\theta_1}$ (the second such term). The factor $(c-\bar{u})^{-1}$ contributes a tendency for the y profile of $\bar{u}$ to be (algebraically) unstable to incipient irregularities and asymmetries, i.e., a tendency for $|c-\bar{u}|$ to diminish more rapidly where it is already smallest. The change in wave structure due to the small shear $\mu\Lambda y$, represented by the second $O(\mu^2)$ term in (4.4), may or may not contribute in a similar sense. Upon substituting (3.1), (3.6) and (3.2), (3.10) into (4.4) we find that the two effects reinforce, at least near the equator, under conditions (3.5) when

$$\left.\begin{array}{ll} |\omega_0|<2^{-\frac{1}{2}} & \text{[Kelvin wave]} \\ -1<k\omega_0<0 \text{ or } k\omega_0>\frac{1}{2}(\sqrt{3}-1) & \\ & \text{[Rossby-gravity wave]} \end{array}\right\}. \quad (4.5)$$

The intervention of barotropic instability would mean that if such self-enhancing profile irregularities occurred on too small a latitudinal scale, they would be too short-lived to be easily observable. Those on a larger scale, however, might play a role in the side-to-side shifting seen on time sequences of meridional sections of $\bar{u}$ such as Fig. 2 of Wallace (1967) and Fig. 10.24 of Newell *et al.* (1974). For example, in the latter's sections for September 1957–July 1958 (reproduced in Fig. 5 of I), observe the anomalously high westward accelerations, first to the south and then to the north of the equator. Of course the annual cycle is clearly important, whether as a trigger or as a dominant effect in itself, and further investigation is needed.

REFERENCES

Andrews, D. G., and M. E. McIntyre, 1976: Planetary waves in horizontal and vertical shear: The generalized Eliassen-Palm relation and the mean zonal acceleration. *J. Atmos. Sci.*, **33**, 2031–2048.

Bretherton, F. P., 1968: Propagation in slowly-varying waveguides. *Proc. Roy. Soc. London*, **A302**, 555–576.

——, and C. J. R. Garrett, 1968 Wavetrains in inhomogeneous moving media. *Proc. Roy. Soc. London*, **A302**, 529–554.

Courant, R., and D. Hilbert, 1953: *Methods of Mathematical Physics*, Vol. 1. Interscience, 561 pp.

Holton, J. R., 1975: *The Dynamic Meteorology of the Stratosphere and Mesosphere*. Amer. Meteor. Soc., 218 pp.

Lindzen, R. S., 1971: Equatorial planetary waves in shear: Part I. *J. Atmos. Sci.*, **28**, 609–622.

——, 1972: Equatorial planetary waves in shear: Part II. *J. Atmos. Sci.*, **29**, 1452–1463.

Mahony, J. J., 1972: Validity of averaging methods for certain systems with periodic solutions. *Proc. Roy. Soc. London*, **A330**, 349–371.

Newell, R. E., J. W. Kidson, D. G. Vincent and G. J. Boer, 1974: *The General Circulation of the Tropical Atmosphere*, Vol. 2. The MIT Press, 371 pp.

Wallace, J. M., 1967: On the role of mean meridional motions in the biennial wind oscillation. *Quart. J. Roy. Meteor. Soc.*, **93**, 176–185.