The Noninteraction of Waves with the Zonally Averaged Flow on a Spherical Earth and the Interrelationships of Eddy Fluxes of Energy, Heat and Momentum

JOHN P. BOYD[1]

Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass. 02138

(Manuscript received 19 May 1976, in revised form 10 August 1976)

ABSTRACT

For linearized hydrostatic waves on a spherical earth with a zonal mean wind which is a function of latitude and pressure, I derive, without further approximations, expressions for the vertical and meridional energy fluxes in terms of the meridional heat flux and the vertical and meridional fluxes of zonal momentum. Using these expressions, I prove that in the absence of critical surfaces, dissipation, thermal heating and nonharmonic time dependence, that the waves and mean flow do not interact: the wave Reynold's stresses are exactly balanced by a mean meridional circulation whose streamfunction is simply the meridional heat flux divided by the static stability. In the presence of dissipation, thermal heating or transience, I am able to express the net forcing of the mean flow by the waves as expressions which are explicitly proportional to the coefficients of dissipation and heating and to the imaginary part of the phase speed. My work significantly extends earlier theorems on the noninteraction of waves with the zonally averaged flow and on the interrelationships of wave fluxes proved by Eliassen and Palm, Charney and Drazin, and Holton because my theorems eliminate some important restrictive assumptions and include all these previous results as special cases.

1. Introduction

The core of this work is a proof that a single, linearized, hydrostatic, time-harmonic wave will not interact with the zonally averaged flow in the absence of dissipation, critical surfaces and thermal forcing. In this context, the basic state is the zonally-averaged flow and "single wave" denotes a perturbation of this mean state by a disturbance of a particular zonal wavenumber. In Section 2 the conditions are described in more detail under which this theorem is true and a proof of the existence and uniqueness of a wave-mean steady state and several relationships between wave fluxes is given. In Section 3 a discussion is given as to how these flux theorems are modified by the presence of dissipation and transience and I show that it is possible to convert the *net* forcing of the zonally averaged flow by the wave into expressions which are explicitly proportional to the coefficients of dissipation or, for an exponentially growing or decaying wave, to the imaginary part of the phase speed. In the final section it is shown how this theorem should be interpreted when more than one wave or a wave packet is present.

Theorems that linearized waves and the zonally averaged flow will not interact and theorems relating wave energy fluxes to heat and momentum fluxes have a long history, but each of these previous works is limited by approximations I shall not make. All use a beta plane instead of the full spherical geometry except for Jones (1967) and Bretherton (1969b), who use an f-plane, and Bretherton (1969a) who ignores rotation entirely. Eliassen and Palm (1961) derive flux theorems with the fewest restrictions of all, but show that these theorems imply wave and mean will not interact only for nonrotating internal gravity waves. Charney and Drazin (1961), Dickinson (1969) and Uryu (1973, 1974a, b, 1975) assume quasi-geostrophy, while Uryu handles all effects of shear through multiscale perturbation theory. Holton (1974, 1975) extends this work to equatorial waves using scaling arguments to neglect the mean meridional velocity and vertical heat flux terms in the thermodynamic energy equation. As I show below these neglected terms cancel anyway. Eliassen and Palm (1961) and Holton (1974) were the twin springboards for my own work.

The importance of my theorems comes from their generality: for hydrostatic waves, my results include all previous work as special cases, and are applicable to a wide variety of specific problems such as the individual phenomena which motivated each of the large number of papers I have referenced above. When the necessary conditions are satisfied globally, the non-interaction theorem is a strong negative result in excluding linearized waves as a source of changes in the

[1] Present affiliation: Department of Atmospheric and Ocean Science, University of Michigan, Ann Arbor 48105.

mean. When, as is more often the case, one of these conditions is violated locally, my work (and previous specializations) show that wave-induced accelerations will be local also. Matsuno's (1971) theory of the stratospheric sudden warming, Holton and Lindzen's (1972) analysis of the quasi-biennial oscillation, Dickinson's analysis of the seminannual oscillation in the tropics, and Fels and Lindzen's (1974) theory of the 4-day rotation on Venus all depend on *local* breakdowns of the non-interaction theorem because of specific violations of one of the necessary conditions for its validity.

Bretherton (1976) and Andrews and McIntyre (1976a,b, 1977) have independently derived theories similar to those in this work (but somewhat more general) using a Lagrangian coordinate system. While their proofs require fewer steps and are easier to generalize further, their Lagrangian formalism is hard to follow for those not already familiar with wave action concepts and related ideas. Thus, their work and mine are complementary.

For clarity, major results will be indicated by using roman numerals in place of arabic to number important equations.

2. The wave-mean non-interaction theorem

In this section, I will assume the following *necessary* conditions are satisfied locally:

(i) No wave dissipation.

(ii) No thermal heating or other form of local wave forcing.

(iii) The Doppler-shifted frequency (the quantity $\bar{U}-c$ below) is not equal to zero.

(iv) The time dependence of the wave is pure harmonic, i.e., it can be described by a *real* phase speed (which may be 0).

(v) The waves are linearizable about the zonally-averaged state.

In general, if any of the five conditions above is violated, all of the results proved in this section need modification. I will further assume that the zonally averaged meridional and vertical winds are negligible in the wave equations which generally follows from scaling. For consistency, we must allow a meridional circulation in the *mean* equations because such a flow will generally be driven by eddy flux terms in the mean thermodynamic energy equation. However, since this circulation will obviously be proportional to the square of the wave amplitude, it must be neglected in the wave equations to consistently linearize them.

I will also make one further assumption which is implicit in most earlier work on wave-mean interactions. As is usual in studies of stability, I will assume that the basic state is independent of time and I will not explicitly consider the sources and dissipation that created it. The derivation in the first part of this section follows that of Eliassen and Palm (1961, Section 10) very closely.

List of Symbols and Conventions

a	earth's radius	θ	latitude
λ	longitude	z	$-\ln(p/p_0)$

A subscript with respect to a coordinate variable denotes differentiation with respect to that variable. For any quantity q

$$L_\theta q \equiv \frac{1}{a\cos\theta}\frac{\partial}{\partial\theta}(\cos\theta q) \equiv \frac{1}{a}q_\theta - q\frac{\tan\theta}{a}$$

$$L_\theta^1 q \equiv \frac{1}{a\cos^2\theta}\frac{\partial}{\partial\theta}(\cos^2\theta q) \equiv \frac{1}{a}q_\theta - 2q\frac{\tan\theta}{a}$$

$$L_z q \equiv e^z\frac{\partial}{\partial z}(e^{-z}q)$$

$$\partial x = \frac{1}{a\cos\theta}\frac{\partial}{\partial\lambda}.$$

"Mean" will be equivalent to "zonally-averaged."

$\bar{U}$, $\bar{V}$, $\bar{W}$ and $\bar{\Phi}$ are the zonal mean velocities and geopotential, and u', v', w' and ϕ' are the corresponding perturbation quantities; in general an overbar will denote a zonally averaged quantity and a prime will denote the deviation of a quantity from its average. I will split the mean temperature into T_0, a function of z only, and T_1, a function of both pressure and latitude, and discard $\kappa T_1(\theta,z)$, where κ is a constant approximately equal to 2/7. As discussed in Holton (1975), to which the reader is referred for details, this approximation is necessary to obtain a set of energetically consistent equations. A detailed derivation of Eqs. (2.1)–(2.8) may be found in the same reference.

m	zonal wavenumber (integer)
Ω	earth's angular rotation frequency
$\bar{\omega}$	wave's angular frequency
c	phase speed $[=\bar{\omega}a\cos\theta/m]$
$\hat{f}$	coriolis parameter $[=2\Omega\sin\theta+(U/a)\tan\theta-a^{-1}\bar{U}_\theta]$
$\tilde{\tilde{f}}$	coriolis parameter $\{=[2\Omega+2U/(a\cos\theta)]\sin\theta\}$
N^2	static stability $\{=R[\kappa T_0(z)+T_{0z}(z)+T_{1z}(\theta,z)]\}$
R	gas constant for air

$$(\bar{U}-c)u'_x-\hat{f}v'+\bar{U}_zw'+\phi'_x=0 \tag{2.1}$$

$$\tilde{\tilde{f}}u'+(U-c)v'_x+\frac{1}{a}\phi_\theta'=0 \tag{2.2}$$

$$u'_x+L_\theta v'+L_zw'=0 \tag{2.3}$$

$$-\tilde{\tilde{f}}\bar{U}_zv'+N^2w'+(\bar{U}-c)\phi'_{zx}=0 \tag{2.4}$$

$$\bar{U}_t-\hat{f}\bar{V}+\bar{U}_z\bar{W}=-L_\theta^1\overline{u'v'}-L_z\overline{u'w'} \tag{2.5}$$

$$\bar{V}_{tz}+\bar{U}_z\tilde{\tilde{f}}+\frac{1}{a}\bar{\Phi}_{\theta z}=\text{eddy terms} \tag{2.6}$$

$$L_\theta\bar{V}+L_z\bar{W}=0 \tag{2.7}$$

$$\bar{\Phi}_{zt}-\tilde{\tilde{f}}\bar{U}_z\bar{V}+N^2\bar{W}=-L_\theta\overline{\phi'_zv'}-L_z\overline{\phi'_zw'}. \tag{2.8}$$

Eq. (2.6), neglecting $\bar{V}_{tz}$ and the eddy terms because they are of second order in the wave amplitude, has been used to replace $a^{-1}\bar{\Phi}_{\theta z}$ in the coefficient of $\bar{V}$ in (2.8) and (2.4). [Note that the coefficient of $\bar{U}$ in the z-integrated form of (2.6) is $2\Omega\sin\theta+U(\tan\theta)/a$. We recover the factor of 2 that appears in the definition of $\tilde{\tilde{f}}$ when we differentiate $U^2\tan\theta/a$.] The quadratic eddy terms in (2.5) and (2.8) should be interpreted as

$$\overline{e'f'}=2[\mathrm{Re}(e')\mathrm{Re}(f')+\mathrm{Im}(e')\mathrm{Im}(f')] \qquad (2.9)$$

for any perturbation quantities e', f', which implies

$$\overline{e'e_x'}=0 \qquad (2.10)$$

and also the integration by parts rule (note the sign change)

$$\overline{e_x'f'}=-\overline{e'f_x'}. \qquad (2.11)$$

I will prove the existence of a wave-mean steady state in three stages. First, I will derive expressions for the wave energy fluxes in terms of the meridional heat flux and the vertical and meridional momentum fluxes. Second, I will show that a two-dimensional vector whose components are the meridional and vertical energy fluxes divided by the Doppler-shifted frequency ("wave action fluxes") is nondivergent. This implies conservation of wave action flux and can be used to replace the perturbation momentum fluxes that appear as forcing terms in the mean zonal momentum equation by expressions involving the meridional heat flux. Finally, I will show that a mean circulation whose streamfunction is proportional to the meridional heat flux will balance the wave forcing terms in the zonally-averaged equations and thus prove the existence of a coupled wave-mean steady state. It is further shown that this is unique.

To begin the first part, we eliminate w' between (2.1) and (2.4) by multiplying (2.4) by $-U_z/N^2$ and adding it to (2.1) to get

$$[(\bar{U}-c)u'+\phi'-(\bar{U}-c)\bar{U}z\phi_z'/N^2]_x = [\hat{f}-\tilde{\tilde{f}}\bar{U}_z^2/N^2]v'. \qquad (2.12)$$

Next, we multiply both sides of (2.12) by

$$(\bar{U}-c)u'+\phi'-(\bar{U}-c)\bar{U}_z\phi_z'/N^2$$

and average to find

$$\overline{\phi'v'}=-(\bar{U}-c)\overline{u'v'}+(\bar{U}-c)\bar{U}_z\overline{\phi_z'v'}/N^2. \qquad \text{(I)}$$

Next, we multiply (2.4) by ϕ_z' and by $u'(\bar{U}-c)+\phi'$ in turn and average, integrating the last term of the second equation by parts and using (2.1) to simplify it to obtain

$$\overline{w'\phi_z'}=\frac{\tilde{\tilde{f}}\bar{U}_z}{N^2}\overline{v'\phi_z'} \qquad (2.14)$$

$$-\tilde{\tilde{f}}\bar{U}_z[\overline{\phi'v'}+(\bar{U}-c)\overline{u'v'}]+N^2[\overline{\phi'w'}+(\bar{U}-c)\overline{u'w'}] -(\bar{U}-c)\hat{f}\overline{v'\phi_z'}+(\bar{U}-c)\bar{U}_z\overline{w'\phi_z'}=0. \qquad (2.15)$$

Substituting (I) and (2.14) into (2.15) yields

$$\overline{\phi'w'}=-(\bar{U}-c)\overline{u'w'}+\hat{f}(\bar{U}-c)\overline{\phi_z'v'}/N^2. \qquad \text{(II)}$$

The physical significance of (I) and (II) is discussed in the Appendix.

To begin the second stage, we multiply (2.1) by u', (2.2) by v', (2.3) by ϕ' and (2.4) by ϕ_z'/N^2, add the results together and average, and then use (I) and (II) to obtain

$$L_\theta[-(\bar{U}-c)\overline{u'v'}+(\bar{U}-c)(\bar{U}_z/N^2)\overline{\phi_z'v'}] +L_z[-(\bar{U}-c)\overline{u'w'}+\hat{f}(\bar{U}-c)\overline{\phi_z'v'}/N^2]+(\tilde{\tilde{f}}-\hat{f})\overline{u'v'} +\bar{U}_z\overline{u'w'}-\tilde{\tilde{f}}\bar{U}_z\overline{v'\phi_z'}/N^2=0. \qquad (2.17)$$

The final step is to bring the $(\bar{U}-c)$ factors out of the operands of L_θ and L_z. We find that the terms left over from this process will identically cancel the last three terms of (2.17) since for any quantity q [note the change from L_θ to L_θ^1 in (2.18) and (III)]

$$L_\theta[(\bar{U}-c)q]=(\bar{U}-c)L_\theta^1q+(f-\hat{f})q, \qquad (2.18)$$

$$L_z[(\bar{U}-c)q]=(\bar{U}-c)L_zq+\bar{U}_zq. \qquad (2.19)$$

Using these identities in (2.17) and dividing out $(U-c)$, we find

$$L_\theta^1[-\overline{u'v'}+(\bar{U}_z/N^2)\overline{\phi_z'v'}] +L_z[-\overline{u'w'}+\hat{f}(\overline{\phi_z'v'}/N^2)]=0. \qquad \text{(III)}$$

This is the form I shall actually use, but (I) and (II) make it possible to rewrite (III) in the equivalent form

$$L_\theta\left[\frac{\overline{\phi'v'}}{(\bar{U}-c)/(ma\cos\theta)}\right] +L_z\left[\frac{\overline{\phi'w'}}{(\bar{U}-c)/(ma\cos\theta)}\right]=0. \qquad \text{(III')}$$

The terms $\overline{\phi'v'}$ and $\overline{\phi'w'}$ can be interpreted as energy fluxes so that (III′) can be interpreted as an equation of conservation of wave action flux, but the reader is referred to Bretherton and Garrett (1969) for a full discussion of its significance.

Using the equation of nondivergence of wave energy flux [(III) and (2.14)] we can rewrite the eddy divergences of the mean zonal momentum and thermodynamic energy equations in terms of the meridional heat flux alone. To further simplify the mean equations, we can define a meridional streamfunction χ so that (2.7), the mean equation of continuity, is satisfied by

$$\bar{V}=L_z\chi, \qquad (2.25)$$

$$\bar{W}=-L_\theta\chi. \qquad (2.26)$$

Eqs. (2.5) and (2.8) become

$$\bar{U}_t-\hat{f}L_z\chi-\bar{U}_zL_\theta\chi=-L_z(\hat{f}\overline{\phi_z'v'}/N^2)-L_\theta'(\bar{U}_z\overline{\phi_z^1v'}/N^2) \quad (2.27)$$

$$\bar{\Phi}_{zt}-\tilde{\tilde{f}}\bar{U}_zL_z\chi-N^2L_\theta\chi=-L_z(\tilde{\tilde{f}}\bar{U}_z\overline{\phi_z'v'}/N^2)-L_\theta(N^2\overline{\phi_z'v'}/N^2), \quad (2.28)$$

while (2.6), the definition of N^2 given earlier and the hydrostatic relation

$$RT^1(\theta,z)=\Phi_z^1(\theta,z)$$

imply

$$\frac{\partial}{\partial z}(\tilde{\tilde{f}}\bar{U}_z)=\frac{\partial}{\partial z}\left(-\frac{\bar{\Phi}_{\theta z}}{a}\right), \quad (2.29)$$

$$=-\frac{1}{a}\frac{\partial}{\partial\theta}N^2. \quad (2.30)$$

Using this identity, we can factor out the coefficients of the ratio of the meridional heat flux to the static stability in each of the parentheses in (2.27) and (2.28) by employing the Leibnitz derivative of a product rule to find

$$\bar{U}_t-\hat{f}L_z\chi-\bar{U}_zL_\theta\chi$$

$$=\left\{-\hat{f}L_z\left(\frac{\overline{\phi_z'v'}}{N^2}\right)-\left[\frac{\overline{\phi_z'v'}}{N^2}\frac{\partial}{\partial z}\left(\frac{\bar{U}\tan\theta}{a}-\frac{\bar{U}_\theta}{a}+2\Omega\sin\theta\right)\right]\right\}$$

$$+\left\{-\bar{U}_zL_\theta\left(\frac{\overline{\phi_z'v'}}{N^2}\right)-\left[\frac{\overline{\phi_z'v'}}{N^2}\left(\frac{1}{a}\frac{\partial}{\partial\theta}\bar{U}_z-\frac{\bar{U}_z\tan\theta}{a}\right)\right]\right\}, \quad (2.31)$$

$$=-\hat{f}L_z(\overline{\phi_z'v'}/N^2)-\bar{U}_zL_\theta(\overline{\phi_z'v'}/N^2), \quad (2.32)$$

$$\Phi_{zt}-\tilde{\tilde{f}}\bar{U}_zL_z\chi-N^2L_\theta\chi$$

$$=\left\{-\tilde{\tilde{f}}\bar{U}_zL_z\left(\frac{\overline{\phi_z'v'}}{N^2}\right)-\left(\frac{\overline{\phi_z'v'}}{N^2}\frac{\partial}{\partial z}(\tilde{\tilde{f}}\bar{U}_z)\right)\right]$$

$$+\left\{-N^2L_\theta\left(\frac{\overline{\phi_z'v'}}{N^2}\right)-\left(\frac{\overline{\phi_z'v'}}{N^2}\frac{1}{a}\frac{\partial}{\partial\theta}(N^2)\right)\right], \quad (2.33)$$

$$=-\tilde{\tilde{f}}\bar{U}_zL_z\left(\frac{\overline{\phi_z'v'}}{N^2}\right)-N^2L_\theta\left(\frac{\overline{\phi_z'v'}}{N^2}\right), \quad (2.34)$$

as the terms in square brackets in (2.31) and (2.33) sum to zero.

We now assume that mean flow is a balanced vortex, which for a rotating fluid is equivalent to assuming that the thermal wind equation holds. The derivation that follows is similar to that of Eliassen (1951). Differentiating this equation with respect to time, we obtain the identity

$$\frac{\partial}{\partial z}(\tilde{\tilde{f}}\bar{U}_t)=\frac{-\bar{\Phi}_{\theta zt}}{a}. \quad (2.35)$$

Multiplying (2.32) by $\tilde{\tilde{f}}$ and differentiating the results with respect to z, differentiating (2.34) with respect to $a\theta$, adding these two equations, and then invoking (2.35) to eliminate the time derivatives, we are left with

$$\frac{\partial}{\partial z}[\tilde{\tilde{f}}\hat{f}L_z\chi+\tilde{\tilde{f}}\bar{U}_zL_\theta\chi]+\frac{1}{a}\frac{\partial}{\partial\theta}[\tilde{\tilde{f}}\bar{U}_zL_z\chi+N^2L_\theta\chi]$$

$$=\frac{\partial}{\partial z}\left[\tilde{\tilde{f}}\hat{f}L_z\left(\frac{\overline{v'\phi_z'}}{N^2}\right)+\tilde{\tilde{f}}\bar{U}_zL_\theta\left(\frac{\overline{v'\phi_z'}}{N_2}\right)\right]$$

$$+\frac{1}{a}\frac{\partial}{\partial\theta}\left[\tilde{\tilde{f}}\bar{U}_zL_z\left(\frac{\overline{v'\phi_z'}}{N^2}\right)+N^2L_\theta\left(\frac{\overline{v'\phi_z'}}{N^2}\right)\right]. \quad (2.36)$$

Comparing the left and right sides of this elliptic equation for the streamfunction, it is obvious that the *unique* solution is

$$\chi=\overline{\phi_z'v'}/N^2. \quad (2.37)$$

Substituting this back in (2.32) and (2.33), we find

$$\bar{U}_t=0,\qquad \bar{\Phi}_{zt}=0, \quad (2.38)$$

i.e., the mean meridional circulation exactly balances the eddy fluxes and the wave does not change the zonally-averaged flow.

3. Flux theorems with dissipation, transience and forcing

When the conditions of the previous section are violated by a combination of dissipation, transience and eddy forcing, the equation of continuity is unchanged, but (2.1), (2.2) and (2.4) become, respectively,

$$(\bar{U}-c)u_x'-\hat{f}v'+\bar{U}_zw'+\phi_x'+M=0, \quad (3.1)$$

$$\tilde{\tilde{f}}u'+(\bar{U}-c)v_x'+\frac{1}{a}\phi_\theta'+P=0, \quad (3.2)$$

$$-\tilde{\tilde{f}}\bar{U}_zv'+N^2w'+(\bar{U}-c)\phi_{zx}'+J=0, \quad (3.3)$$

where

M = eddy zonal momentum forcing + eddy zonal dissipation $+[c_im/(a\cos\theta)]u'$,
P = eddy meridional momentum forcing + eddy meridional dissipation $+[c_im/(a\cos\theta)]v'$,
J = eddy thermal forcing + thermal dissipation $+[c_im/(a\cos\theta)]\phi_z'$,

where c_i is the imaginary part of the phase speed.

To derive the new version of (I), we eliminate w' between (3.1) and (3.3) as before to find

$$[(\bar U-c)u'+\phi'-(\bar U-c)\bar U_z\phi'_z/N^2]_x + M-(\bar U_z/N^2)J=[\hat f-\tilde{\tilde{f}}\bar U_z/N^2]v'. \quad (3.4)$$

Defining

$$\left.\begin{aligned}\hat M&=\left(\frac{a\cos\theta}{im}\right)M\\ \hat P&=\left(\frac{a\cos\theta}{im}\right)P\end{aligned}\right\},$$

we can rewrite (3.4) as

$$[(\bar U-c)u'+\phi'-(\bar U-c)\bar U_z\phi'_z/N^2+\hat M-(\bar U_z/N^2)\hat J]_x =[\hat f-\tilde{\tilde{f}}\bar U_z/N^2]v'. \quad (3.5)$$

If we multiply through by the expression inside the square brackets on the left-hand side of (3.5) and average, we find

$$\overline{\phi'v'}=-(\bar U-c)\overline{u'v'}+(\bar U-c)(\bar U_z/N^2)\overline{v'\phi'_z}-\overline{\hat M v'} +(\bar U_z/N^2)\overline{\hat J v'}. \quad \text{(IV)}$$

Similarly, eliminating v' between (3.1) and (3.3) and averaging and then using the identity

$$(\bar U-c)\overline{w'\phi'_z}+\overline{\hat J w'}=(\tilde{\tilde{f}}\bar U_z/N^2)(\bar U-c)\overline{v'\phi'_z} +(\tilde{\tilde{f}}\bar U_z/N^2)\overline{\hat J v'} \quad (3.6)$$

which follows from (3.3) by multiplying it by

$$(\bar U-c)\phi'_z+J$$

and averaging, we find

$$\overline{\phi'w'}=-(\bar U-c)\overline{u'w'}+(\bar U-c)\hat f(\overline{v'\phi'_z}/N^2) -\overline{\hat M w'}+\hat f\overline{\hat J v'}/N^2. \quad \text{(V)}$$

If we define

$$\overline{\overline{\phi'v'}}=-(\bar U-c)\overline{u'v'}+\frac{(\bar U-c)}{N^2}\bar U_z\overline{\phi'_z v'}, \quad (3.7)$$

$$\overline{\overline{\phi'w'}}=-(\bar U-c)\overline{u'w'}+\frac{(\bar U-c)}{N^2}\hat f\overline{\phi'_z v'}, \quad (3.8)$$

we can, by executing the same steps as for the non-interacting case, show that

$$L_\theta^1\left[\frac{\overline{\overline{\phi'v'}}}{(\bar U-c)}\right]+L_z\left[\frac{\overline{\overline{\phi'w'}}}{(\bar U-c)}\right]=\frac{1}{(\bar U-c)} \times\left[-L_\theta\left(-\overline{\hat M v'}+\frac{\bar U_z}{N^2}\overline{\hat J v'}\right)-L_z\left(-\overline{\hat M w'}+\hat f\frac{\overline{\hat J v'}}{N^2}\right) -\overline{Mu'}-\overline{Pv'}-\frac{\overline{J\phi'_z}}{N^2}\right]. \quad (3.9)$$

Using this in (2.5) and (2.7) as before and making the streamfunction decomposition

$$\chi=\frac{\overline{v'\phi'_z}}{N^2}+\psi,$$

we obtain

$$\bar U_t-\hat f L_z\psi-\bar U_z L_\theta\psi=\text{right-hand side of (3.9)}, \quad \text{(VI)}$$

$$\Phi_{zt}-\tilde{\tilde{f}}\bar U_z L_z\psi-N^2L_\theta\psi=L_z(\overline{J\phi'_z}). \quad \text{(VII)}$$

The forcing term in (VI) is a consequence of our use of the wave action flux equation [(3.9) or (III)] to convert momentum fluxes into heat fluxes [compare with (2.28)]. The forcing term in (VII) is a consequence of using (3.3), multiplied by ϕ'_z and averaged, to convert the $\overline{w'\phi'_z}$ term in the thermodynamic equation to one in $\overline{v'\phi'_z}$: the $\overline{J\phi'_z}$ term is left over.

Obviously, if we solve the wave equations with and without forcing, transience and dissipation, we will calculate different values of $\overline{v'\phi'_z}/N^2$ for these two cases, so the meridional circulation with the streamfunction $\overline{v'\phi'_z}/N^2$ is no longer in any physical balance with the wave flux divergences. We can still, however, use the *mathematical identities* relating such a component of the total streamfunction to the wave flux divergences so as to leave us with forcing terms which depend explicitly on M, P and J.

The motivation for this is simple: In weakly damped wave phenomena, the net acceleration of the mean by the wave will be the small difference of large quantities since the wave-induced meridional circulation will almost balance the momentum flux divergence. This near-cancellation may be disastrous if the wave equations are solved by numerical or approximate analytical means.

Fels and Lindzen (1974) have discussed in detail some examples of heating-induced wave-mean flow interactions and Dickinson (1969) has derived quasi-geostrophic expressions for the case of linear cooling only. The results I discuss here are more general than those of either of these previous papers.

Transience and dissipation play similar roles in the sense that I will now explain. If we can represent the nonharmonic time dependence by an imaginary phase speed c_i and model dissipation as linear friction and linear cooling, then we can combine these effects into a single term in each of (3.1)–(3.3) by merely adding $[c_i m/(a\cos\theta)]$ to the damping coefficients. For easterly Rossby waves, this implies that a damped, harmonic-in-time wave, an undamped, locally growing-in-time wave, and the leading edge of a wave packet will produce easterly (negative) accelerations of the mean wind; and a locally decaying-in-time wave and the trailing edge of a wave packet will produce westerly accelerations. Since a Rossby wave carries easterly momentum away from where it is forced, it is obvious that when a Rossby wave dissipates, it must surrender its easterly momentum to the zonal mean. Similarly, a locally growing wave or the leading edge of a wave packet is clearly making the total zonal momentum of the region more and more easterly with time.

4. Conclusions

In the preceding sections, I have always used "wave" to describe the perturbation, but these results hold true in most situations where the plural would be more appropriate.

If waves of different phase speeds are present, we can apply my theorems as before to show that $-L_\theta^1\overline{u_1'v_1'}$ and $-L_z\overline{u_1'w_1'}$ can be balanced by a mean circulation with streamfunction $\overline{\phi_{z1}'v_1'}/N^2$ and similarly with component 2, 3, etc., but we are left with cross terms of the form $\overline{u_1'v_2'}$ and $\overline{u_2'v_1'}$. However, ignoring phase variation in θ and z for simplicity only, these cross terms will be proportional to

$$\begin{aligned}\mathrm{Re}[e^{ik(x-c_1t)}]\,\mathrm{Re}[e^{ik(x-c_2t)}] &+ \mathrm{Im}[e^{ik(x-c_1t)}]\,\mathrm{Im}[e^{ik(x-c_2t)}]\\ =\cos[k(x-c_1t)]\cos[k(x-c_2t)] &+\sin[k(x-c_1t)]\sin[k(x-c_2t)] \quad (4.1)\\ =\cos[k(c_1-c_2)t] & \quad (4.2)\end{aligned}$$

by use of a trigonometric identity. Unless $c_1=c_2$, i.e.' unless we are considering quadratic terms involving quantities of the same wave, the cross terms are periodic in time and vanish when averaged over an appropriate time interval. In this *time-averaged* sense, the wave-mean non-interaction theorem still applies to groups of waves of different phase speeds.

This periodic cycling of energy and momentum from mean to wave and back again may be observationally significant. Van Loon *et al.* (1975) and Madden (1975) interpret the quasi- two-week oscillation in the amplitude of the zonally averaged wind and its perturbations in the lower stratosphere as the passage of a spectrum of travelling waves of 1–3 weeks period through quasi-stationary waves of the same zonal wavenumber. The changes in wave amplitude (factor of 2) and mean wind (10–15 m s^{-1}) associated with these cross-term interactions are quite large, but in agreement with what I have said above, Madden and his co-workers find that the time-averaged *net* transfer of energy and momentum from wave to mean flow is quite small.

Matsuno (1971) and Uryu (1973, etc.) for the quasi-geostrophic wave and Bretherton (1969a) for the internal gravity wave have discussed wave-induced accelerations for transients and wave packets. The latter pair have shown through explicit expressions for their respective cases that the leading edge and trailing edge wave packet accelerations are nonzero but of opposite signs and the same magnitude, so that after the packet has passed, the zonal mean is in the same state as before; Dickinson (1969), Fourier transforming the time dependence of the wave into frequency space, came to a similar conclusion. These earlier results are implicit in what I have said above: if we apply the argument behind Eq. (4.2) to a wave packet, then time-averaging over a period long enough for the packet to have come and gone at a fixed location will essentially eliminate the cross terms associated with the packet and again give us the result (if the necessary conditions of Section 2 are satisfied) that the mean is only temporarily affected by the perturbation.

APPENDIX

The Physical Interpretation of Eddy Energy Fluxes

Here, I will discuss the interpretation of the heat flux terms in the energy flux expressions (I) and (II). Since the heat flux term in (I) depends explicitly only on U_z, we take the limit $\hat{f}\rightarrow 0$. Then (2.4) reduces to

$$N^2w'+(\bar{U}-c)\phi_{zx'}=0, \qquad (2.4')$$

or equivalently, letting ξ' represent the vertical displacement

$$\xi'=-\phi_z'/N^2, \qquad (2.4'')$$

giving

$$\overline{\phi'v'}=-(\bar{U}-c)(\overline{u'v'}+\bar{U}_z\overline{\xi'v'}). \qquad (\mathrm{I}')$$

At a given level, the total change in zonal velocity due to the presence of the wave is not simply u'. The vertical displacement associated with the wave will cause an additional change u_2' which for small ξ' is given by

$$u_2'=U_z\xi'. \qquad (\mathrm{A1})$$

The meridional displacement of η' will have a similar effect. As pointed out by Uryu (1973), if we use a Lagrangian coordinate system in which the equilibrium position of a fluid particle is used as its label, the Lagrangian perturbation zonal velocity u_L' is given by

$$u_L'=u'+\bar{U}_z\xi'+\cos\theta\frac{\partial}{\partial\theta}\left(\frac{\bar{U}}{a\cos\theta}\right)\eta' \qquad (\mathrm{A2})$$

and the total energy flux is simply

$$\overline{\phi'v'}=-(\bar{U}-c)\overline{u_L'v'}. \qquad (\mathrm{I}'')$$

(Note that η', because of its definition, must be out of phase with the corresponding velocity v'.) In a similar way, if we allow $\bar{U}_z\rightarrow 0$ while restoring $\hat{f}$ to a nonzero value, (2.4″) is still valid. If we transfer a factor $[im/(a\cos\theta)](\bar{U}-c)$ from ξ' to η', where η' is the meridional displacement, and use (2.11), the integration by parts rule, we can write (II) as

$$\overline{\phi'w'}=-(\bar{U}-c)(\overline{u'w'}-\hat{f}\overline{\eta'w'}). \qquad (\mathrm{II}')$$

We can split $\hat{f}$ into

$$\hat{f}=-\cos\theta\frac{d}{d\theta}\left(\frac{\bar{U}}{a\cos\theta}\right)+\left(2\Omega+\frac{2\bar{U}}{a\cos\theta}\right)\sin\theta. \quad \text{(A3)}$$

The first term of $\hat{f}$ will contribute a Lagrangian correction which is identical in nature to that for (I) except for the direction. The second term is necessary because of conservation of angular momentum. If we displace a fluid particle a distance η' in the north-south direction, it will acquire a zonal velocity increment u_3' where

$$u_3'=\left(2\Omega+\frac{2\bar{U}}{a\cos\theta}\right)\sin\theta\eta'. \quad \text{(A4)}$$

Again the total change in zonal velocity at a point from its equilibrium value, not simply u', is what enters the energy flux expressions.

When both f and $\bar{U}_z$ are nonzero, (2.4″) is not valid. From the proof of the noninteraction theorem, however, we know that the wave will drive a mean meridional circulation which through Coriolis torques and advection will enter the mean zonal momentum equation. In the general case when the simplifying limits I have used are not valid, it is still true that the presence of the heat flux terms in (I) and (II) are due to this mean meridional circulation.

Acknowledgments. This work is part of a thesis to be submitted to Harvard University for the degree of Doctor of Philosophy. Financial support was provided by a University Corporation for Atmospheric Research Graduate Fellowship and a research assistantship under grant ATM-75-20156 from the National Science Foundation.

I wish to thank Professor R. S. Lindzen, my thesis advisor, for his comments and criticism, and also Francis Bretherton and Takashi Sasamori, James Holton, Michael McIntyre and an anonymous referee for helpful discussions.

REFERENCES

Andrews, D. G., and M. E. McIntyre, 1976a: Planetary waves in horizontal and vertical shear: The generalized Eliassen-Palm relation and the mean zonal acceleration. *J. Atmos. Sci.*, **33**, 2031–2048.

——, and ——, 1976b: Planetary waves in horizontal and vertical shear: Asymptotic theory for equatorial waves in weak shear. *J. Atmos. Sci.*, **33**, 2049–2053.

——, and ——, 1977: On Lagrangian-mean flow and radiation stress, with application to stratified, rotating fluids. Submitted to *J. Fluid Mech.*

Bretherton, Francis P., 1969a: On the mean motion induced by internal gravity waves. *J. Fluid. Mech.*, **36**, 785–803.

——, 1969b: Momentum transport by gravity waves. *Quart. J. Roy. Meteor. Soc.*, **95**, 213–243.

——, 1976: Conservation of wave action and angular momentum in a spherical atmosphere. *J. Fluid Mech.* (in press).

——, and C. J. R. Garrett, 1968: Wave trains in inhomogenous moving media. *Proc. Roy. Soc. London*, **A302**, 529–554.

Charney, J. G., 1973: Lectures on planetary fluid dynamics. *Dynamical Meteorology*, P. Morel, Ed., D. Reidel, Chaps. 4 and 5.

——, and P. G. Drazin, 1961: Propagation of planetary scale disturbances from the lower into the upper atmosphere. *J. Geophys. Res.*, **66**, 83–109.

Dickinson, R. E., 1969: Theory of planetary wave-zonal flow interaction. *J. Atmos. Sci.*, **26**, 73–81.

Eliassen, Arnt, 1951: Slow thermally or frictionally controlled meridional circulation in a circular vortex. *Astrophys. Norv.*, **5**, 19–61.

——, and E. Palm, 1961: On the transfer of energy in stationary mountain waves. *Geofys. Publ.*, **22**, No. 3, 22 pp.

Fels, S., and R. Lindzen, 1974: The interaction of thermally excited gravity waves with mean flows. *Geophys. Fluid Dyn.*, **6**, 149–192.

Holton, J. R., 1974: Forcing of mean flows by stationary waves. *J. Atmos. Sci.*, **31**, 942–945.

——, 1975: *The Dynamic Meteorology of the Stratosphere and Mesosphere.* Amer. Meteor. Soc., 216 pp.

——, and Richard S. Lindzen, 1972: An updated theory for the quasi-biennial cycle of the tropical stratosphere. *J. Atmos. Sci.*, **29**, 1076–1080.

Jones, Walter L., 1967: Propagation of internal gravity waves in fluids with shear flow and rotation. *J. Fluid Mech.*, **30**, 439–448.

Madden, Roland A., 1975: Oscillations in the winter stratosphere: Part 2. The role of horizontal eddy heat transport and the interaction of transient and stationary planetary-scale waves. *Mon. Wea. Rev.*, **103**, 717–729.

Matsuno, T., 1971: A dynamical model of the stratospheric sudden warming. *J. Atmos. Sci.*, **28**, 1479–1494.

Uryu, M., 1973: On the transport of energy and momentum in stationary waves in a rotating stratified fluid. *J. Meteor. Soc. Japan*, **51**, 86–92.

——, 1974a: Induction and transmission of mean zonal flow by quasi-geostrophic disturbances. *J. Meteor. Soc. Japan*, **52**, 341–364.

——, 1974b: Mean zonal flows induced by a vertically propagating Rossby wave packet. *J. Meteor. Soc. Japan*, **52**, 481–490.

——, 1975: On the mean motions induced around a planetary wave packet on a rotating sphere. *J. Meteor. Soc. Japan*, **53**, 45–54.

van Loon, H., R. A. Madden and R. L. Jenne, 1975: Oscillations in the winter stratosphere, Part 1: Description. *Mon. Wea. Rev.*, **103**, 154–162.