Open Boundary Conditions for Dispersive Waves

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ABSTRACT

Approximate outgoing radiation conditions have been widely used at open boundaries in dispersive wave computations. Exact outgoing radiation conditions are constructed here for infinitesimal surface inertia-gravity waves, barotropic Rossby waves, and non-hydrostatic internal gravity waves incident at infinite plane open boundaries. They are also constructed for infinitesimal two-dimensional surface gravity waves incident at a square open boundary, a finite straight open boundary in a channel, and a circular open boundary. The constructions use Laplace transforms, various Fourier transforms, and a wave field splitting technique. The exact outgoing radiation conditions all involve weighted space and time averages. Their numerical implementation would be most complex and would require computer storage approaching that which one is trying to avoid by the introduction of open boundaries. Alternatives are discussed.

1. Introduction

The finite memory capacity of computers limits the numerical simulation of waves propagating into an unbounded region after generation by an isolated disturbance. If a finite-difference approximation to the wave equations is used, then the computational domain must be truncated to reduce the required number of grid points.

Even if the dimensions of the truncated domain are substantially greater than those of the isolated disturbance contained within it, waves generated by the disturbance will eventually arrive at the boundary of the truncated domain. The boundary is "open": it has no effect on incident waves and this property must be properly represented in the numerical simulation. It is, in general, necessary to specify some condition on the wave field variables at the boundary in order to complete the finite-difference equations and thereby determine the wave field inside the domain at later times.

Any wave field variable may be specified at the boundary provided it is given values compatible with the incident wave field. If not, then the wave field will be improperly reflected back into the truncated domain. Compatible values of the wave field variables are usually not known in advance so only homogeneous relationships between boundary values of the wave field variables can be specified. This investigation finds the homogeneous relationships which must hold due to the fact that the incident wave field contains no incoming waves. The relationships are then assessed for the storage required for their numerical implementation.

It is well known that outgoing radiation conditions may be constructed for hydrostatic surface gravity waves propagating in one space dimension, using the theory of Riemann invariants (Courant and Hilbert, 1966). The construction is outlined here in Section 2. The conditions state that one of the Riemann invariants vanishes at the boundary because all the waves arriving at the boundary are outgoing and have the same phase velocity.

Boundary conditions involving Riemann invariants are readily implemented numerically. The invariants are linear combinations of boundary values of simple wave field variables such as wave height and water velocity at the point and time of interest. Although the theory of Riemann invariants has not been extended to several space dimensions, there have been attempts to use them to formulate outgoing radiation conditions for two-dimensional surface gravity waves (Wurtele *et al.*, 1971), surface inertia-gravity waves (Heaps, 1973), and hydrostatic internal inertia-gravity waves (Pearson, 1974). All these waves are dispersive, i.e., their phase velocities depend on frequency and wavenumber. Each outgoing harmonic wave annihilates a Riemann invariant, but only the one with the appropriate phase velocity. The resulting homogeneous condition is known as a Sommerfeld outgoing radiation condition (Courant and Hilbert, 1966).

When simulating waves generated by an isolated initial disturbance, there is the problem of selecting a single phase velocity as a representative of the entire wave spectrum. Wurtele *et al.* (1971) chose the one-dimensional phase velocity $C = (gH)^{\frac{1}{2}}$ (in the usual notation) which amounted to assuming that all the outgoing waves were plane and at normal incidence to the boundary. Heaps (1973) made the same choice, which amounted to assuming that all the waves were

Kelvin waves at normal incidence to the boundary. Pearson (1974) argued that the waves arriving at the boundary at time t, after having travelled a distance x, would be confined to a narrow frequency band and a narrow wavenumber band due to the effects of dispersion. By equating the group velocity to x/t, he was able to determine the common phase velocity of the waves. The argument is asymptotically valid if x and t are large. Pearson and Wurtele *et al.* presented numerical examples which appeared to justify their assumptions, but their simple, smooth initial disturbances may not be entirely representative of all interesting cases.

Exact outgoing radiation conditions may be obtained by summing the Sommerfeld conditions over all frequencies and wavenumbers. This construction is carried out in Section 3 for surface inertia-gravity waves, barotropic Rossby waves, and non-hydrostatic internal gravity waves at infinite plane boundaries. While such idealized boundaries are not met in practice, they do allow simple demonstrations of the nature of outgoing radiation conditions for dispersive waves. The conditions are linear combinations of boundary values of wave field variables averaged along the boundary and over the entire elapsed time. Accordingly the construction of outgoing radiation conditions for boundaries of various shapes and sizes requires a consideration of each boundary in turn.

In Section 4 three types of boundary are considered: a square which is open on all four sides, a square with rigid barriers at a pair of opposite faces and open at the other pair, and a circle which is open all around. The nature of outgoing radiation conditions at these boundaries is shown by considering the simplest two-dimensional waves, namely, hydrostatic surface gravity waves. The storage required for implementation of the condition is estimated and compared with the requirement if the domains are not truncated.

The results are summarized in Section 5. Alternative approaches to formulating open boundary conditions are mentioned. The results are discussed for their significance to limited-area weather forecasting and mesoscale oceanic eddy simulation.

2. Hydrostatic surface gravity waves in one dimension

The conservation laws for mass and momentum governing the one-dimensional propagation of infinitesimal hydrostatic surface gravity waves are

$$u_t = -gh_x, \tag{1}$$

$$h_t = -Hu_x, \tag{2}$$

where $u(x,t)$ is the horizontal water velocity, $h(x,t)$ the water level disturbance, g the gravitational acceleration, and H the undisturbed water depth. The subscripts x and t denote partial differentiation with respect to space and time, respectively. Eqs. (1) and (2) may be decoupled to yield

$$h_{tt} - c^2 h_{xx} = 0, \tag{3}$$

where $c = (gH)^{\frac{1}{2}}$.

The wave equation (3) may be factored to yield

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)h(x,t)=0, \tag{4}$$

from which it may be deduced that the linearized Riemann invariants (Courant and Hilbert, 1966)

$$h_t \pm ch_x \qquad (5), (6)$$

are conserved on characteristic lines in the x,t plane given by

$$\frac{dx}{dt} = \mp c. \qquad (7), (8)$$

In particular, if the initial disturbance is confined to the region $x > 0$, then

$$h_t(0,t) - ch_x(0,t) = h_t(-ct,0) - ch_x(-ct,0) = 0. \tag{9}$$

The homogeneous condition (9) is satisfied by only the solutions of (3) of the form $h(x+ct)$, that is, waves leaving the region $x>0$. Hence (9) is an outgoing radiation condition at the boundary of that region.

Eqs. (1) and (9) are equivalent to

$$h(0,t) + (H/g)^{\frac{1}{2}} u(0,t) = 0. \tag{10}$$

The outgoing radiation condition (10) is particularly convenient since it relates values of the wave field variables themselves rather than their gradients. Its effectiveness has been demonstrated numerically by Wurtele *et al.* (1971).

3. Dispersive waves

In this section outgoing radiation conditions are constructed for three types of dispersive waves incident upon infinite plane boundaries.

a. *Infinitesimal hydrostatic surface inertia-gravity waves*

The conservation laws for mass and momentum are

$$h_t = -H(u_x + v_y), \tag{11}$$

$$u_t - fv = -gh_x, \tag{12}$$

$$v_t + fu = -gh_y, \tag{13}$$

where (u,v) is the horizontal water velocity, and the Coriolis parameter f is twice the uniform rate at which the non-inertial reference frame is rotating about the vertical. The other symbols have the same meaning as in Section 2. The additional subscript y denotes partial differentiation with respect to the second Cartesian coordinate which is tangential to the infinite straight open boundary at $x=0$.

Decoupling (11)–(13) yields

$$[h_{tt}+f^2h-c^2(h_{xx}+h_{yy})]_t=0, \qquad (14)$$

but (14) is not factorable in the same manner as (3). However a generalization of (9) may be constructed using spectral analysis and synthesis.

Introducing the Laplace transform and inverse

$$\bar{h}(x,y,s)=\int_0^\infty dte^{-st}h(x,y,t), \qquad (15)$$

$$h(x,y,t)=(2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty} dse^{st}\bar{h}(x,y,s), \qquad (16)$$

and the Fourier transform and inverse

$$\tilde{h}(x,\lambda,t)=\int_{-\infty}^{\infty} dye^{-i\lambda y}h(x,y,t), \qquad (17)$$

$$h(x,y,t)=(2\pi)^{-1}\int_{-\infty}^{\infty} d\lambda e^{i\lambda y}\tilde{h}(x,\lambda,t), \qquad (18)$$

reduces (14) to

$$(s^2+f^2+c^2\lambda^2)\tilde{\bar{h}}-c^2\tilde{\bar{h}}_{xx}=\tilde{I}, \qquad (19)$$

where

$$\tilde{I}(x,\lambda,s)=s^{-1}[\tilde{h}_{tt}+s\tilde{h}_t+(s^2+f^2+c^2\lambda^2)\tilde{h}-c^2\tilde{h}_{xx}]_{t=0}. \qquad (20)$$

The solution of (19) which behaves like $\tilde{I}$ as $|x|\to\infty$ is

$$\tilde{\bar{h}}(x,\lambda,s)=(2kc^2)^{-1}\int_{-\infty}^{\infty} dx'e^{-k|x-x'|}\tilde{I}(x',\lambda,s), \qquad (21)$$

where

$$k^2=(s^2+f^2+c^2\lambda^2)c^{-2}. \qquad (22)$$

Using the result

$$\frac{d}{d\xi}|\xi|=\begin{cases}1, & \xi>0\\ -1, & \xi<0,\end{cases} \qquad (23)$$

it follows from (21) that if the initial disturbance is confined to the region $x>0$ then

$$\tilde{\bar{h}}(0,\lambda,s)-k^{-1}\tilde{\bar{h}}_x(0,\lambda,s)=(kc^2)^{-1}\times\int_{-\infty}^{0} dx'e^{kx'}\tilde{I}(x',\lambda,s)=0. \qquad (24)$$

Eq. (24) is satisfied by each wave leaving the region but $k(s,\lambda)$ takes a different value for each wave. These Sommerfeld outgoing radiation conditions (Courant and Hilbert, 1966) may be summed (inverted) to yield a Riemann outgoing radiation condition satisfied by the entire outgoing wave field:

$$h(0,y,t)-c\int_0^t dt'\int_{y-c(t-t')}^{y+c(t-t')} dy'F_1(y-y',t-t')\times h_x(0,y',t')=0, \qquad (25)$$

where

$$F_1(y,t)=\pi^{-1}(c^2t^2-y^2)^{-\frac{1}{2}}\cos[(f/c)(c^2t^2-y^2)^{\frac{1}{2}}]. \qquad (26)$$

Instead of relating values of the wave field variables only at the point $(0,y)$ and time of interest t, the condition (25) relates values averaged along the boundary and over the entire elapsed time. Since the weight $F_1(y-y',\ t-t')$ depends on the point and time of interest, the averages cannot be calculated by updating the averages at earlier times but must be re-totalled at each point and time with the appropriate weight. For this purpose all the elapsed values at every boundary point must be stored. The storage requirement grows like t^2 which is the same as the requirement which was to be avoided by truncation of the domain.

The Sommerfeld conditions (24) and hence the Riemann condition (25) are not unique. If (24) is true then

$$\tilde{\bar{G}}(\lambda,s)(\tilde{\bar{h}}-k^{-1}\tilde{\bar{h}}_x)=0, \qquad (27)$$

where $\tilde{\bar{G}}$ is any function of λ and s. However, since the ratio of the terms in (27) is always k^{-1} or $k=k^2k^{-1}$, the inverse of (27) must contain a convolution integral with a weight related to F_1. For example, if $\tilde{\bar{G}}(\lambda,s)=s$ then the inverse of (27) becomes

$$h_t(0,y,t)-ch_x(0,y,t)+c\int_0^t dt'\int_{y-c(t-t')}^{y+c(t-t')} dy'\times[F_2(y-y',t-t')h_{xy}(0,y',t')+F_3(y-y',t-t')\times(f/c)h_x(0,y',t')]=0, \qquad (28)$$

where

$$F_2(y,t)=\pi^{-1}(y/t)(c^2t^2-y^2)^{-\frac{1}{2}}\cos[(f/c)(c^2t^2-y^2)^{\frac{1}{2}}], \qquad (29)$$

$$F_3(y,t)=\pi^{-1}t^{-1}\sin[(f/c)(c^2t^2-y^2)^{\frac{1}{2}}]. \qquad (30)$$

Eq. (28) is the generalization of (9) to inertia-gravity waves.

b. Barotropic Rossby waves

If it is assumed that the Coriolis parameter f in (12) and (13) has a uniform gradient β in the y direction (northward), then an additional mode of propagation is introduced into the system. The surface waves discussed in Subsection 3a may be filtered out, leaving only the Rossby waves, by assuming that the velocity field is solenoidal, i.e.,

$$u=-\psi_y,\ v=\psi_x, \qquad (31)$$

where $\psi(x,y,t)$ is a streamfunction. It follows from (12) and (13) that

$$(\psi_{xx}+\psi_{yy})_t+\beta\psi_x=0. \qquad (32)$$

It may be shown, as in Subsection 3a, that if the initial disturbance is confined to the region $x>0$ then an outgoing radiation condition at the meridional boundary $x=0$ is

$$\psi(0,y,t)-\int_0^t dt'\int_{-\infty}^{\infty} dy' F_4(y-y',\, t-t')\psi_x(0,y',t')$$
$$-(\beta/2)\int_0^t dt'\int_0^{t'} dt''\int_{-\infty}^{\infty} dy' F_4(y-y',\, t'-t'')\times\psi(0,y',t'')=0, \quad (33)$$

where

$$F_4(y,t)=\pi^{-1}\frac{\partial}{\partial t}\int_1^{\infty} du(u^2-1)^{-\frac{1}{2}}J_0[2u^{\frac{1}{2}}(\beta y t/2)^{\frac{1}{2}}] \quad (34)$$

and J_0 is the regular Bessel function of order zero.

If the initial disturbance is confined to the region $y>0$ then an outgoing radiation condition at the zonal boundary $y=0$ is

$$\psi(x,0,t)-\int_0^t dt'\int_{-\infty}^{\infty} dx' F_5(x-x',\, t-t')\times\psi_{yt}(x',0,t')=0, \quad (35)$$

where

$$F_5(x,t)=-\pi^{-1}\int_0^t dt'(t-t')^{-\frac{1}{2}}(t')^{\frac{1}{2}}Y_0[(4\beta x t')^{\frac{1}{2}}] \quad (36)$$

and Y_0 is the singular Bessel function of order zero.

Conditions (33) and (35) are relationships between average boundary values of the wave field variables. Due to the essentially elliptic character of (32) the ranges of integration in (33) and (35) extend along the entire boundary for all $t>0$.

c. Non-hydrostatic internal gravity waves

The conservation equations for mass, momentum and internal energy in a fluid weakly stratified in the vertical direction may be reduced to

$$(w_{xx}+w_{yy}+w_{zz})_{tt}+N^2(w_{xx}+w_{yy})=0, \quad (37)$$

where w is the vertical velocity, the subscript z denotes partial differentiation in the vertical direction, and N is the buoyant frequency (Phillips, 1966).

If the initial disturbance is confined to the region $x>0$, then an outgoing radiation condition at $x=0$ is

$$\int_0^t dt' J_0[N(t-t')]w(0,y,z,t')-\int_0^t dt'\int_{-\infty}^{\infty} dz'\int_{-\infty}^{\infty} dy'\times F_6(y-y',\, z-z',\, t-t')w_x(0,y',z',t')=0, \quad (38)$$

where

$$F_6(y,z,t)=(2\pi)^{-1}(y^2+z^2)^{-\frac{1}{2}}J_0[zNt(y^2+z^2)^{-\frac{1}{2}}]. \quad (39)$$

If the initial disturbance is confined to the region $z>0$, then an outgoing radiation condition at $z=0$ is

$$w(0,y,z,t)-\int_0^t dt'\int_{-\infty}^{\infty} dx'\int_{-\infty}^{\infty} dy' F_7(x-x',\, y-y',\, t-t')\times w_z(x',y',0,t')=0, \quad (40)$$

where

$$F_7(x,y,t)=(2\pi)^{-1}(x^2+y^2)^{-\frac{1}{2}}\frac{\partial}{\partial t}J_0(Nt). \quad (41)$$

The remarks in Subsection 3b about (33) and (35) apply also the (38) and (40).

For the three types of waves discussed in Subsections 3a–c, the outgoing radiation conditions relate boundary values of the wave field variables averaged along the boundary and over the elapsed time. If the boundaries are of finite length or not straight then outgoing radiation conditions must be constructed for each boundary in turn.

4. Boundaries of finite length

This section examines the influence of boundary geometry upon outgoing radiation conditions. The conclusions do not depend on the particular wave type, so only simple two-dimensional hydrostatic surface gravity waves are considered. These are described by (11)–(13) with $f=0$. It follows that

$$h_{tt}-c^2(h_{xx}+h_{yy})=0. \quad (42)$$

a. Square boundary, open on all sides

The boundary is the unit square with its four corners at

$$(x,y)=(0,0),\ (1,0),\ (1,1),\ (0,1). \quad (43)$$

All four sides are open: incident waves can cross them unhindered. An initial disturbance is isolated inside the square so outgoing radiation conditions hold at all four sides. Due to the symmetry of (42), conditions at the three sides $x=1$, $y=0$, $y=1$ may be obtained from the condition at the side $x=0$ by simple transformations of the coordinates.

The side $x=0$ is of unit length so the infinite range Fourier transform (17) and inverse (18) must be replaced by the finite Fourier transform and inverse:

$$h_n(x,t)=\int_0^1 dy e^{-2\pi i n y}h(x,y,t), \quad (44)$$

$$h(x,y,t)=\sum_{-\infty}^{\infty} e^{2\pi i n y}h_n(x,t). \quad (45)$$

The Fourier-Laplace transform of (42) then becomes

$$[s^2+(2\pi n c)^2]\bar{h}_n-c^2\bar{h}_{n_{xx}}=\bar{I}_n+\bar{T}_n, \quad (46)$$

where

$$\bar{I}_n(x,s)=\left[\left(\frac{\partial h}{\partial t}\right)_n+sh_n\right]_{t=0} \quad (47)$$

is the Fourier transform of the initial data, and

$$\bar{T}_n(x,s)=c^2[\bar{h}_y+(2\pi n i)\bar{h}]_{y=0}^{y=1} \quad (48)$$

is the Laplace transform of terminal values of the wave field variables. Tiedtke (1973) ignored the terminal values in his construction of outgoing radiation conditions for these waves. An equivalent assumption is that the wave field is periodic in y with period unity, but there is no justification for such an assumption. No alternative choice of phase range in (44) eliminates the terminal values entirely from (46), unless there are physical boundaries at $y=0$, $y=1$. Such a situation is examined in Subsection 4b.

It may be shown from (46) that

$$\bar{h}_n(x,s)-k^{-1}\bar{h}_{n_x}(x,s)=(kc^2)^{-1}\times\int_{-\infty}^{x}dx'e^{-k|x-x'|}[\bar{I}_n(x',s)+\bar{T}_n(x',s)], \quad (49)$$

where

$$k^2=[s^2+(2\pi nc)^2]c^{-2}. \quad (50)$$

The initial disturbance is confined to the region $x>0$ so $\bar{I}_n(x')=0$ for $x'<0$. However the waves will eventually arrive at those parts of the lines $y=0$, $y=1$ at which $x'\leqslant 0$ so $\bar{T}_n(x',s)\neq 0$ for $x'\leqslant 0$. The only conclusion is that at $x=0$,

$$\bar{h}_n(0,s)-k^{-1}\bar{h}_{n_x}(0,s)=(kc^2)^{-1}\times\int_{-\infty}^{0}dx'e^{kx'}\bar{T}_n(x',s)\neq 0. \quad (51)$$

The outgoing radiation condition obtained by inverting (51) is not suitable as a boundary condition because it involves terminal values of h outside the square. These values can be eliminated only by an application of the operator $(1+k^{-1}\partial/\partial x)$ on (49), yielding at $x=0$:

$$\bar{h}_n(0,s)-k^{-2}\bar{h}_{n_{xx}}(0,s)=c^{-2}k^{-2}\bar{T}_n(0,s). \quad (52)$$

Eq. (52) could have been deduced directly from (46). The crucial advantage of (52) over (51) is that it involves only boundary values of the wave field variables. Following a convenient multiplication by s, (52) may be inverted to yield

$$h_t(0,y,t)-(c^2/2)\int_0^t dt'\sum_{\pm} h_{xx}[0,\, y\pm c(t-t')-N_\pm,\, t']$$
$$=\tfrac{1}{2}\sum_{\pm}\sum_{M_\pm}\{ch_y[0,\, y',\, t\pm(y-M_\pm)/c]$$
$$\mp h_t[0,\, y',\, t\pm(y-M_\pm)/c]\}_{y'=0}^{y'=1}, \quad (53)$$

where $N_\pm$ are integers such that

$$0<y\pm c(t-t')-N_\pm<1, \quad (54)$$

and $M_\pm$ are integers such that

$$0<t\pm(y-M_\pm)/c<t. \quad (55)$$

Numerical implementation of (53) requires storage for all values of h_{xx} on the side $x=0$ and values of h_y, h_t at the corners (0,0), (0,1) for all elapsed times up to t. The requirement increases by $O(t)$ as t increases.

b. Wave propagation between rigid walls

The lines $y=0$, $y=1$ are rigid barriers at which the normal velocity v and hence the surface slope h_y vanish. The line segment $\{x=0,\ 0<y<1\}$ is an open boundary. An initial disturbance is isolated to the region $x>0$ so an outgoing radiation condition holds at the open boundary.

No terminal values at $y=0,1$ will appear in the Laplace-Fourier transform of (42) if (44) and (45) are replaced by the finite Fourier cosine transform and inverse, i.e.,

$$h_n(x,t)=\int_0^1 dy\,\cos n\pi y\,h(x,y,t), \quad (56)$$

$$h(x,y,t)=\sum_0^\infty \epsilon_n h_n(x,t)\cos n\pi y, \quad (57)$$

where

$$\epsilon_n=\begin{cases}1, & n=0\\ 2, & n=1,2,3,\ldots\end{cases} \quad (58)$$

It may be shown in the usual way that an outgoing radiation condition at $x=0$ is

$$h(0,y,t)-c\int_0^t dt'\int_0^1 dy'F_8(y,\,y',\,t-t')h_x(0,y',t')=0, \quad (59)$$

where

$$F_8(y,y',t)=\tfrac{1}{2}\sum_{\pm}\{\sum_{M_\pm}[\pi^2c^2t^2-(2\pi M_\pm+\pi(y\pm y'))^2]^{-\frac{1}{2}}$$
$$+(\pi ct)^{-1}[1-(y\pm y')^2(ct)^{-2}]^{-\frac{1}{2}}$$
$$+\sum_{N_\pm}[\pi^2c^2t^2-(2\pi N_\pm-\pi(y\pm y'))^2]^{-\frac{1}{2}}\}. \quad (60)$$

The summations over $M_\pm$ and $N_\pm$ in (60) include only those integers for which the radicands are positive. Storage requirements for the numerical implementation of (59) increase by $O(t)$ as t increases.

c. Circular open boundaries

The boundary is a circle of radius R and is open all around. An initial disturbance is isolated inside the circle so an outgoing radiation condition holds at the circle. Polar coordinates (r,θ) with origin at the center of the circle are convenient.

The wave field is periodic in θ with period 2π so no terminal values will appear in the Fourier-Laplace transform of (42) if (56) and (57) are replaced by the finite Fourier transform and inverse, i.e.,

$$h_n(r,t)=(2\pi)^{-1}\int_0^{2\pi}d\theta e^{-in\theta}h(r,\theta,t), \quad (61)$$

$$h(r,\theta,t)=\sum_{-\infty}^{\infty} e^{in\theta}h_n(r,t). \quad (62)$$

The Sommerfeld outgoing radiation conditions are

$$\frac{\partial}{\partial R}[K_n(sR/c)]\bar{h}_n(R,s)-K_n(sR/c)\bar{h}_{nr}(R,s)=0, \quad (63)$$

where K_n is the modified Bessel function of the third kind of order n. Eq. (63) cannot be inverted to yield a meaningful Riemann outgoing radiation condition, for the following reason. The inverse Laplace transform of $K_n(sR/c)$ is

$$a_n(t)=\begin{cases} 0, & 0<t<R/c \\ [t^2-(R/c)^2]^{-\frac{1}{2}} & \\ \quad \times\cosh[n\cosh^{-1}(ct/R)], & R/c<t \end{cases} \quad (64)$$

Since $a_n(t)$ grows exponentially as $|n|\rightarrow\infty$, the Fourier series

$$a(\theta,t)=\sum_{-\infty}^{\infty} a_n(t)e^{in\theta} \quad (65)$$

cannot be interpreted even as a generalized function (Lighthill, 1964). Formal attempts to sum (65) lead to the result $a(\theta,t)\equiv 0$.

The difficulty can be overcome in principle by dividing (63) by $K_n(sR/c)$; the inverse becomes

$$(2\pi)^{-1}\int_0^{2\pi} d\theta' \int_0^t dt' F_9(\theta-\theta', t-t')h_t(R,\theta',t') - ch_r(R,\theta,t)=0, \quad (66)$$

where

$$F_9(\theta,t)=\sum_{-\infty}^{\infty} e^{in\theta}(2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty} ds e^{st}K_n^{(1)} \times(sR/c)/K_n(sR/c). \quad (67)$$

It may be shown that the Fourier coefficients in (67) do not have exponential growth but I have been unable to express $F_9(\theta,t)$ in closed form.

Storage requirements for the numerical implementation of (66) increase by $O(t)$ as t increases.

Outgoing radiation conditions at the open boundaries considered in Subsections 4a–c may also be constructed for the dispersive waves considered in Subsections 3a–c. Second-order differential-integral conditions are needed for the open square if exterior terminal values are to be avoided. First-order differential-integral conditions suffice at the open ends of the channel, but the configuration is of little real interest. The Fourier series summation problem is always present at the open circle. The storage requirements are independent of wave type.

5. Discussion

Uniformly spaced finite differences are often used in simple techniques for the simulation of wave fields generated by isolated initial disturbances. If the domain is partially or totally unbounded then it must be truncated in order to reduce computer storage requirements. The boundaries introduced by truncation are open in the sense that they have no effect on incident waves. Any condition needed at the open boundaries in order to close the finite difference equations must not reflect incident waves. Homogeneous conditions are preferable since they may be specified *a priori*.

Conditions specifying the absence of incoming radiation have been constructed here for several types of dispersive waves and several types of open boundaries. The conditions relate values of the wave field variables averaged over the elapsed time t and around the boundary. The weights in the averages depend on the point and time of interest and so must be computed in entirety at each point and time. The corresponding storage requirements (and computation time) increase by $O(t)$ as t increases.

For (two-dimensional) surface inertia-gravity waves propagating from an isolated disturbance into a domain unbounded in one direction only, the region of influence and hence the storage requirement increases by $O(t)$. If the domain is unbounded in both directions then the requirement increases by $O(t^2)$. There is clearly no advantage in truncating the domain and specifying outgoing radiation conditions in the first situation, and a marginal advantage in the second case which is overweighed by the complexity of the outgoing radiation conditions.

For barotropic Rossby waves and non-hydrostatic internal gravity waves, the regions of influence include the entire domain at any instant after the initial one, even if the initial disturbance is isolated. This is due to the essentially elliptic character of their respective "wave" equations. Infinite domains must be truncated, but outgoing radiation conditions are too complex for numerical implementation.

The weights in the time averages decay at most slowly in time, typically $O(t^{-1})$ or $O(t^{-\frac{1}{2}})$. For hydrostatic surface gravity waves in an open square [Eq. (53)] the weighting is *uniform* in time. Hence any attempt to reduce the storage requirements, by restricting the time averaging to a short interval prior to the time of interest, would lead to significant error.

There are two alternatives. First, truncation could be avoided by the use of a singular coordinate transformation which maps the infinite domain into a finite image domain. This in turn may be covered by a finite uniformly spaced grid. The computational resolution would become increasingly poorer as the image of the point at infinity is approached.

The second alternative is to truncate the domain and use homogeneous open boundary conditions based on the fact that the wave field is to some degree smooth in space and time. Boundary values of the wave field variables may be obtained by extrapolation of past boundary values or present or past interior values. The order of extrapolation is dictated by the smoothness of the wave field. Chen (1973) has considered a variety of extrapolation techniques.

The boundary-value problems considered here are related to limited-area weather forecasting and mesoscale oceanic eddy simulation. In the latter two types of computation, data are available at the open boundaries, so boundary conditions are readily prescribed. The problem is the avoidance of error trapping inside the truncated domains. These errors are generated by finite-difference approximations to the wave equations inside the domains and by inconsistencies between boundary and initial data. They must be allowed to escape via the open boundaries. Specification of outgoing radiation conditions at the open boundaries would permit them to do so. The correct wave field is not purely outgoing so the conditions would be nonhomogeneous. For example, (10) would become

$$h(0,t)+(H/g)^{\frac{1}{2}}u(0,t)=h_B(t)+(H/g)^{\frac{1}{2}}u_B(t), \quad (68)$$

where h_B and u_B are boundary data. Note that both wave heights and fluid velocities must be specified. Incident waves due to errors generated inside the domain and at other parts of the boundary would be subjected to (68) in homogeneous form [i.e., Eq. (10)] and so would pass out of the truncated domain.

If the wave fields are dispersive then the outgoing radiation conditions are too complex to implement numerically. The only alternative in this context is an extrapolation technique.

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