Aperiodic Trajectories and Stationary Points in a Three-Component Spectral Model of Atmospheric Flow

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ABSTRACT

Aperiodic solutions to spectrally truncated models based on the vorticity equation are considered for the case of a zonal flow interacting nonlinearly with two other components both having the same zonal wavenumber. It is shown that all such aperiodic trajectories proceed asymptotically to either a stationary point in the phase space of coefficients or to a periodic solution with steady amplitudes.

It is also shown that the set of such solutions is of measure zero on surfaces of constant energy in phase space. Thus if the initial coefficients for a nonlinear, three-component flow are selected at random, then the resulting flow will in all probability be periodic.

1. Introduction

The study of simple spectral models of atmospheric flow has provided considerable insight into some of the intricacies of nonlinear interactions on the global scale. A perhaps surprising result, apparently discovered by Lorenz (1960), is that two-dimensional flows modeled by spectral vorticity equations truncated to three-component systems are generally periodic, despite the presence of nonlinear advection.

The fact that solutions to such systems are elliptic functions was further noted by Platzman (1960, 1962), who extended the results to more complicated three-component systems involving a zonal component and two additional components with the same zonal wavenumber but different latitudinal structure. This five-component model has been investigated in detail by Baer (1970a,b; 1971) in a series of papers devoted to finding exact solutions and to various aspects of the truncated flows. The stability properties of the model have also been studied by Galin (1974a,b) who pointed out that aperiodic solutions may appear under certain conditions.

The purpose of this article is to examine the aperiodic solutions in greater detail, and it is shown that all aperiodic trajectories in the phase space of coefficients of three-component systems proceed asymptotically to a stationary point or to a periodic solution with steady amplitudes. It is also shown that the set of such trajectories is of measure zero on a surface of constant energy in phase space.

These results show that if a zonal current interacts nonlinearly with two-component disturbances, then in all probability the combined flow will oscillate periodically. The exception is the zero probability case that the initial coefficients of the zonal flow and the disturbance lie on a trajectory asymptotic to a stationary point or to a periodic solution.

The analysis is carried through for the barotropic model studied by Platzman and Galin. But Baer (1970a) showed that the same set of spectral equations can be obtained for baroclinic flows, such as those represented by the quasi-geostrophic equation. Hence the results all apply to such flows as well.

2. The spectral barotropic model

The vorticity equation for two-dimensional flow may be written upon scaling with the earth's radius a and rotation rate Ω as

$$\frac{\partial \nabla^2\psi}{\partial t}+\frac{1}{\cos\phi}\frac{\partial(\psi,\nabla^2\psi)}{\partial(\lambda,\phi)}+2\frac{\partial\psi}{\partial\lambda}=0, \tag{2.1}$$

where $\nabla^2\psi$ is the scaled relative vorticity ζ/Ω and the scaled absolute vorticity is $(\zeta+2\Omega\sin\phi)/\Omega$; λ is longitude and ϕ latitude.

The streamfunction ψ can be expanded in spherical harmonics with $\exp(i\lambda m)$ and associated Legendre functions $P_n^m(\sin\phi)$, and here we consider a model that has the following components: 1) a steady, solid rotation with angular velocity ω; 2) a zonal component $\psi_\alpha(t)P_{n_\alpha}(\sin\phi)$; and 3) two disturbance components $\psi_\beta(t)P_{n_\beta}^m(\sin\phi)e^{im\lambda}$ and $\psi_\gamma(t)P_n^m(\sin\phi)e^{im\lambda}$. Here ψ_α is real and ψ_β and ψ_γ are complex; conjugates will be denoted by an overbar.

As shown by Platzman and restated by Baer and by

Galin, the spectral equations for this system are

$$\left.\begin{aligned}\frac{d\psi_\alpha}{dt}&=2c_\alpha^{-1}(c_\beta-c_\gamma)k_\alpha \operatorname{Im}(\psi_\beta\bar{\psi}_\gamma)\\ \frac{d\psi_\beta}{dt}&=i(I_{\beta\beta\alpha}\psi_\alpha-m\omega_\beta)\psi_\beta+ic_\beta^{-1}(c_\gamma-c_\alpha)k_\alpha\psi_\alpha\psi_\gamma\\ \frac{d\psi_\gamma}{dt}&=i(I_{\gamma\gamma\alpha}\psi_\alpha-m\omega_\gamma)\psi_\gamma+ic_\gamma^{-1}(c_\beta-c_\alpha)k_\alpha\psi_\alpha\psi_\beta\end{aligned}\right\}. \tag{2.2}$$

All coefficients of ψ_α, ψ_β and ψ_γ are constant; for the present purposes the values of these various constants (all real except for i) are immaterial, and for brevity they will not be redefined.

It follows immediately from the equations that the model has the kinetic energy and mean-square vorticity invariants

$$\left.\begin{aligned}E&=\tfrac{1}{2}c_\alpha\psi_\alpha^2+c_\beta|\psi_\beta|^2+c_\gamma|\psi_\gamma|^2\\ F&=\tfrac{1}{2}c_\alpha^2\psi_\alpha^2+c_\beta^2|\psi_\beta|^2+c_\gamma^2|\psi_\gamma|^2\end{aligned}\right\}, \tag{2.3}$$

which are independent if the coefficients c_ν are distinct.

The definitions

$$z=\psi_\alpha(t)-\psi_\alpha(0),\quad L+iM=2c_\alpha^{-1}(c_\beta-c_\gamma)k_\alpha\psi_\beta\bar{\psi}_\gamma \tag{2.4}$$

allow the system to be converted to real variables in the form

$$\left.\begin{aligned}\frac{dz}{dt}&=M\\ \frac{dL}{dt}&=-(g+hz)M\\ \frac{dM}{dt}&=(g+hz)L-(a_\beta+h_\beta z)k_\alpha P_\beta-(a_\gamma+b_\gamma z)k_\alpha P_\gamma\end{aligned}\right\}, \tag{2.5}$$

where all factors except for $P_\beta=2c_\alpha^{-1}(c_\beta-c_\gamma)|\psi_\beta|^2$ and $P_\gamma=2c_\alpha^{-1}(c_\gamma-c_\beta)|\psi_\gamma|^2$ are again constant.

3. Stationary points and multiple zeros

The two constraints (2.3) are crucial in developing a solution to the system (2.5). With them it is clear that $c_\beta\neq c_\gamma$ allows us to solve for $|\psi_\beta|^2$ and $|\psi_\gamma|^2$ as functions of E, F and ψ_α^2; hence P_β and P_γ become second-order polynomials in z with coefficients depending on constants and initial values. The inverse relations may then be expressed as

$$\left.\begin{aligned}\psi_\alpha(t)&=\psi_\alpha(0)+z\\ P_\beta&=P_{\beta 0}+(2\alpha_\gamma z+b_\gamma z^2)k_\alpha^{-1}\\ P_\gamma&=P_{\gamma 0}+(2\alpha_\beta z+b_\beta z^2)k_\alpha^{-1}\\ M/L&=\tan(\phi_\beta-\phi_\gamma)\end{aligned}\right\}, \tag{3.1}$$

where ϕ_β and ϕ_γ are phase angles, so that z, M and L determine ψ_α, $|\psi_\beta|$, $|\psi_\gamma|$ and $\phi_\beta-\phi_\gamma$ and thus determine ψ_α, ψ_β, ψ_γ to within one phase angle.

Combination of the first two equations of (2.5) gives the integral $L=L_0-gz-hz^2/2$ and the combination of this result, the first and last equations of (2.5) and the new forms of P_β and P_γ presented in (3.1) gives

$$\frac{d^2z}{dt^2}=q_0+q_1z+q_2z^2+q_3z^3=R(z). \tag{3.2}$$

These constant coefficients q_i are given explicitly by Galin (1974b); here we need only note that q_3 is constant (but that its sign may change with different choices of components α, β and γ), that q_2 depends on α, β, γ and $\psi_\alpha(0)$, and that q_1 and q_0 depend on components α, β, γ and $\psi_\alpha(0)$, $\psi_\beta(0)$ and $\psi_\gamma(0)$.

If (3.2) is multiplied by dz/dt and integrated over the range $(0,t)$, then we obtain

$$(dz/dt)^2=M_0^2+2(q_0z+\tfrac{1}{2}q_1z^2+\tfrac{1}{3}q_2z^3+\tfrac{1}{4}q_3z^4)=P(z) \tag{3.3}$$

and this (upon taking the reciprocal and integrating again) yields

$$t(z)=\pm\int_0^z\frac{dz'}{[P(z')]^{\frac{1}{2}}}, \tag{3.4}$$

in which we have explicitly used $z=0$ at $t=0$ and affixed a sign choice depending on whether dz/dt is positive or negative so that t will increase monotonically.

Everything then depends on the roots of the quartic polynomial $P(z)$. If it has four distinct real roots $a_1<a_2<a_3<a_4$, then (3.4) gives solutions $z(t)$ as elliptic functions, as noted by Platzman (1962) and discussed in detail by Baer (1970a) and by Galin (1974b). In this case z, L and M and hence $|\psi_\beta|$, $|\psi_\gamma|$ and the phase angle difference $\phi_\beta-\phi_\gamma$ all become periodic functions of time.

But if the roots coalesce so that $P(z)$ has one or more multiple roots, then (3.4) is no longer an elliptic integral and either oscillatory or aperiodic solutions may be found. Galin (1974b) discussed various cases relative to results on stability of the zonal flow.

Now we come to the first result of this article:

THEOREM 3.1. *The necessary and sufficient condition that $z=a$ be a real multiple root of $P(z)$ is that $z=a$ be a point of the three-component flow that has steady amplitudes ψ_α, $|\psi_\beta|$, and $|\psi_\gamma|$.*

PROOF. We note that (3.3) can be rewritten as

$$\begin{aligned}\left(\frac{dz}{dt}\right)^2&=M_0^2+2\int_0^t\frac{dz}{dt}R(z)dt\\ &=M_0^2+2\int_0^z R(z)dz\\ &=M_0^2+Q(z)=P(z),\end{aligned} \tag{3.5}$$

where $\partial P/\partial z=2R(z)$.

Thus if a is a double zero of $P(z)$, then evaluating at $z=a$ gives $R(z)=dM/dt=0$ along with $dz/dt=M=dL/dt=0$; thus as is clear from (2.5), a is a stationary point for the coefficients z, L, M and hence for ψ_α, $|\psi_\beta|$, $|\psi_\gamma|$.

Conversely, if $|\psi_\beta|$ and $|\psi_\gamma|$ are constant, then (2.3) shows that ψ_α is constant and thus at such a point a we have $dz/dt=dL/dt=dM/dt=0$, which gives $P(a)=0$ and $R(a)=0$, so that both $\partial P(z)/\partial z$ and $P(z)$ are zero at a. Thus a is a multiple root of $P(z)$, which completes the proof.

This relationship between amplitude-stationary points in phase space and multiple zeros of $P(z)$ has important implications for the aperiodic trajectories. However, for comparison, we first examine the case in which four distinct roots appear. By the fundamental theorem of algebra, we can write (3.3) as

$$(dz/dt)^2=\tfrac{1}{4}q_3(z-a_1)(z-a_2)(z-a_3)(z-a_4). \quad (3.6)$$

Since this result must hold for any initial conditions, we can conclude that the roots will vary with $\psi_\alpha(0)$, $\psi_\beta(0)$ and $\psi_\gamma(0)$ so that $z=0$ falls between roots in such a way that $(dz/dt)^2$ is positive. In particular, there must be at least one positive and one negative real root, as pointed out by Galin (1974b). To see this, note that $E=$ constant and $F=$ constant are ellipsoids in the variables $|\psi_\beta|$, $|\psi_\gamma|$ and $\psi_\alpha=z+\psi_\alpha(0)$. Their intersection is a closed curve passing through $z=0$ (in z, $|\psi_\beta|$, $|\psi_\gamma|$ space) since the point $\psi_\alpha(0)$, $|\psi_\beta(0)|$, $|\psi_\gamma(0)|$ is on the closed curve in the original space. Because $dz/dt=0$ at $\max(z)$ and $\min(z)$ on the curve, these real values of z are roots of $P(z)$. The exception would be the case $z\equiv 0$ or $\psi_\alpha=\psi_\alpha(0)$, which corresponds to a double root at $z=0$. Hence, if $q_3>0$ and all roots are real, then z must be confined between a_2 and a_3; for $q_3<0$, z must be between a_1 and a_2 or between a_3 and a_4. A similar analysis will locate $z=0$ and $z(t)$ when a pair of complex roots occurs.

Suppose, as an example, that $q_3>0$ and that z is between a_2 and a_3. Then the theory of elliptic functions shows that z oscillates between these two roots. The crucial point, for the comparison we intend to make, is that the integral (3.4) is finite at the limiting roots. To see this let a_3 be a limiting root and choose $\epsilon<\min[a_3-a_1, a_3-a_2]$. Then $(a_3-a_1-\xi)\geqslant(a_3-a_1-\epsilon)>0$ if $\xi\leqslant\epsilon$. The same is true for $(a_3-a_2-\xi)$. Now upon using the substitution $\xi=a_3-z$, we find

$$t(a_3)-t(a_3-\epsilon)$$

$$=\int_{a_3-\epsilon}^{a_3}\frac{dz}{[\tfrac{1}{4}q_3(z-a_3)(z-a_4)(z-a_1)(z-a_2)]^{\frac{1}{2}}}$$

$$=(\tfrac{1}{4}q_3)^{-\frac{1}{2}}\int_0^{\epsilon}\frac{d\xi}{\xi^{\frac{1}{2}}[(a_4-a_3+\xi)(a_3-a_1-\xi)(a_3-a_2-\xi)]^{\frac{1}{2}}}$$

$$\leqslant B\int_0^{\epsilon}\frac{d\xi}{\xi^{\frac{1}{2}}}=2B\epsilon^{\frac{1}{2}}, \quad (3.7)$$

where $B=[\tfrac{1}{4}q_3(a_4-a_3)(a_3-a_1-\epsilon)(a_3-a_2-\epsilon)]^{-\frac{1}{2}}$. Thus t remains finite as z approaches a_3 and departs again.

For the case of multiple roots, let $q_3>0$ and let the real roots be $a_1<a_2<a_4$, with a_4 a double root. (Note that should a_1 and a_2 be complex conjugates $a\pm ib$ rather than real, then $z\equiv a_3=a_4$ and the flow has steady amplitudes.) Now $z=0$ must fall between a_2 and a_4 and although $z(t)$ may approach and leave a_2 once, it must proceed toward a_4 eventually. This conclusion follows because once the trajectory has left a_2, we will have $dz/dt=+[P(z)]^{\frac{1}{2}}$ so that dz/dt cannot vanish until a_4 is reached. Hence we can let $t=0$ and $z=0$ with $dz/dt>0$.

This time we shall use inequalities of the form $a_4-a_1-\xi<a_4-a_1$ and we have

$$t(a_4-\epsilon)=(\tfrac{1}{4}q_3)^{-\frac{1}{2}}\int_0^{a_4-\epsilon}\frac{dz}{(a_4-z)[(z-a_1)(z-a_2)]^{\frac{1}{2}}}$$

$$=(\tfrac{1}{4}q_3)^{-\frac{1}{2}}\int_{\epsilon}^{a_4}\frac{d\xi}{\xi[(a_4-a_1-\xi)(a_4-a_2-\xi)]^{\frac{1}{2}}}$$

$$\geqslant D_1\int_{\epsilon}^{a_4}\frac{d\xi}{\xi}=D_1\ln(a_4/\epsilon), \quad (3.8)$$

where $D_1=[\tfrac{1}{4}q_3(a_4-a_1)(a_4-a_2)]^{-\frac{1}{2}}$. Hence $t(a_4-\epsilon)\to\infty$ as $\epsilon\to 0$ or $t(z)\to\infty$ as $z\to a_4$. Because $(a_4-a_1-\xi)=(a_4+|a_1|-\xi)\geqslant|a_1|$ for $\epsilon\leqslant\xi\leqslant a_4$, the middle integral of (3.8) can be altered to give

$$t(a_4-\epsilon)\leqslant D_2\int_{\epsilon}^{a_4}\frac{d\xi}{\xi}=D_2\ln(a_4/\epsilon), \quad (3.9)$$

where $D_2=(\tfrac{1}{4}q_3|a_1a_2|)^{-\frac{1}{2}}$.

Thus as $t(z)\to\infty$, it is necessary that $\epsilon\to 0$ which implies that $z\to a_4$. This computation therefore proves:

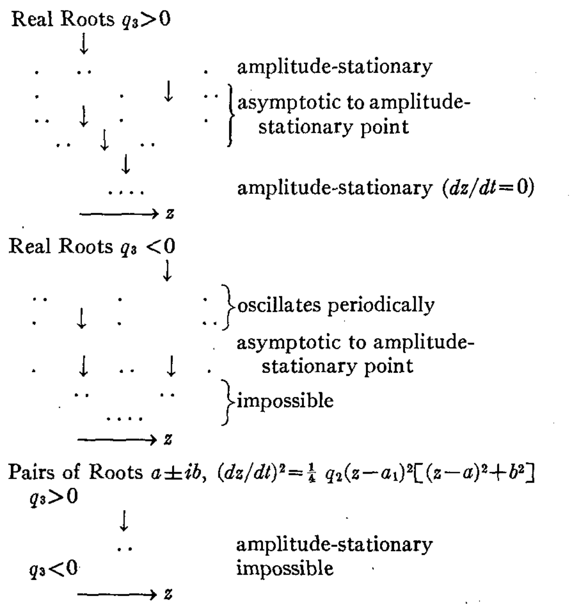
THEOREM 3.2. *If $P(z)$ has real multiple roots and $z(t)$ is aperiodic, then $z(t)$ is eventually asymptotic to the amplitude stationary point in phase space at the multiple root.*

The various cases are summarized in Table 1. It is worth noting that the asymptotic motion to an amplitude-stationary point represents an initial instability of the zonal flow to disturbances.

Next we consider the case in which z is periodic, with the aim of showing that then the flow is periodic. It is immediately clear that (3.1) will then give $|\psi_\beta|$, $|\psi_\gamma|$ and $\phi_\beta-\phi_\gamma$ as periodic functions. Thus the phase angle difference is periodic, as noted by Baer (1970a). Since $\phi_\beta=\tan^{-1}(\psi_\beta^I/\psi_\beta^R)$, where $\psi_\beta^R=\mathrm{Re}\psi_\beta$ and $\psi_\beta^I=\mathrm{Im}\psi_\beta$, we can use the real and imaginary parts of (2.2) to obtain ($\psi_\beta\neq 0$ and $C_{\alpha\gamma}=k_\alpha(c_\gamma-c_\alpha)/c_\beta$)

$$\frac{d\phi_\beta}{dt}=(I_{\beta\beta\alpha}\psi_\alpha-m\omega_\beta)+(C_{\alpha\gamma}\psi_\alpha/2|\psi_\beta|^2C_{\gamma\beta})L(z)=N(z). \quad (3.10)$$

TABLE 1. Multiple roots, initial conditions and type of solution. [The dots represent the relative location of the roots of the polynomial $(dz/dt)^2=P(z)=\frac{1}{4}\,q_3\,(z-a_1)(z-a_2)(z-a_3)(z-a_4)$ on the z axis, the arrows indicate the intervals in which the initial value $z=0$ may occur for $(dz/dt)^2\geqslant 0$. Single roots are denoted by a single dot, double roots by two dots. Only cases with multiple roots are shown.]



Both ψ_β and ψ_γ vanish simultaneously only at a stationary point, as can be seen from (2.2). Thus if ψ_β were to vanish at $t=t_1$, we would switch to an analysis of ϕ_γ for an interval including t_1 and use the ϕ_γ analogue of (3.10). With this option available, we continue to solve (3.10) on the assumption that $N(z)$ is bounded, and we have

$$\phi_\beta(t)-\phi_\beta(0)=\int_0^t N(z)dt. \qquad (3.11)$$

Baer (1970a) evaluated this integral in terms of various elliptic functions, and concluded that the flow might not be periodic due to the "probable disparity" of periods of ϕ_β and $|\psi_\beta|$.[1] However, for $z=z(t)$ we can use (3.3) to change variables in (3.11) so that

$$\phi_\beta(z)-\phi_\beta(0)=\int_0^z \frac{N(z')}{\pm[P(z')]^{\frac{1}{2}}}dz', \quad dz/dt\not\equiv 0, \qquad (3.12)$$

and as shown by (3.7) this integral is bounded for periodic z as long as $N(z)$ is bounded. Hence, from the fundamental theorem of calculus, (3.12) defines a differentiable function Φ such that

$$\phi_\beta(z)=\Phi(z)+\phi_\beta(0)-\Phi(0). \qquad (3.13)$$

But now if z is periodic, then ϕ_β is periodic and so ψ_α, ψ_β and ψ_γ are all periodic. The amplitude-stationary points are also periodic solutions since (3.10) for $z=$ constant gives

$$\phi_\beta=\sigma t+\sigma_0, \quad \phi_\gamma=\sigma t+\sigma_0+\pi n. \qquad (3.14)$$

Combining these conclusions with the two previous theorems, we have

THEOREM 3.3. *If z is periodic, then the three-component flow is periodic. If z is not periodic, then z is asymptotic to an amplitude stationary point in phase space, and the flow is aperiodic but asymptotic to a periodic flow.*

4. Characterization of stationary points

The previous results suggest investigation of the location and character of the amplitude-stationary points in phase space. With (3.14), (2.2) becomes

$$\left.\begin{aligned} \mathrm{Im}\{\psi_\beta\bar{\psi}_\gamma\}=0\\ (I_{\beta\beta\alpha}\psi_\alpha-m\omega_\beta-\sigma)|\psi_\beta|+C_{\alpha\gamma}\psi_\alpha|\psi_\gamma|=0\\ C_{\alpha\beta}\psi_\alpha|\psi_\beta|+(I_{\gamma\gamma\alpha}\psi_\alpha-m\omega_\gamma-\sigma)|\psi_\gamma|=0 \end{aligned}\right\}. \qquad (4.1)$$

We propose to solve the last two equations for $|\psi_\beta|$ and $|\psi_\gamma|$. Toward this end, we have either the case $|\psi_\beta|=|\psi_\gamma|=0$ so that the flow has only a stationary zonal component or it must be true that

$$(I_{\beta\beta\alpha}\psi_\alpha-m\omega_\beta-\sigma)(I_{\gamma\gamma\alpha}\psi_\alpha-m\omega_\gamma-\sigma)-C_{\alpha\beta}C_{\alpha\gamma}\psi_\alpha^2=0. \qquad (4.2)$$

Thus the values of σ and $\psi_{\alpha s}$ giving energetically stationary solutions are linked by a quadratic equation and only those choices of components α, β and γ that give real roots for both will have energetically stationary points. For those cases, we have the solutions

$$|\psi_{\beta s}|=K(I_{\gamma\gamma\alpha}\psi_{\alpha s}-m\omega_\gamma-\sigma), \quad |\psi_{\gamma s}|=-KC_{\alpha\beta}\psi_{\alpha s}, \qquad (4.3)$$

where K is a constant.

Upon inserting this solution in the equation for E in (2.3) we have

$$K^2=[E-\tfrac{1}{2}c_\alpha\psi_{\alpha s}^2][c_\beta(I_{\gamma\gamma\alpha}\psi_{\alpha s}-m\omega_\gamma-\sigma)^2+c_\gamma(C_{\alpha\beta}\psi_{\alpha s})^2]^{-1}, \qquad (4.4)$$

so that K^2 is determined by the choice of components through $\psi_{\alpha s}$, σ, and by the initial energy. If for the fixed values of α, β and γ determining components, there is a range of σ giving real values of $\psi_{\alpha s}$, then we have the solutions $\psi_\alpha=\psi_{\alpha s}(\sigma)$, $|\psi_\beta|=|\psi_{\beta s}(\sigma)|$, $|\psi_\gamma|=|\psi_{\gamma s}(\sigma)|$ (where there may be a finite number of multiple values), which gives a finite set of curves on $E=$ constant as functions of the parameter σ. The sur-

[1] This conclusion would appear to be confirmed by numerical results of Galin and Kurbatkin (1975) in which phase angles are shown that do not return to their original values in the period of oscillation of $|\psi_\beta|$ and $|\psi_\gamma|$. Careful analysis of photographic enlargements of their two figures yield for the differences between initial and final values of $\phi_\beta-\phi_\gamma$ the two values 95° and −13°. Hence the computed results violate the condition that $\phi_\beta-\phi_\gamma$ be periodic, and thus are apparently in error. Since $|\psi_\beta|$ and $|\psi_\gamma|$ become quite small in both cases, we speculate that this is the source of the error, especially since $|\psi_\beta|$ nearly vanishes in the 95° case.

face $E=$ constant encloses a three-dimensional volume in ψ_α, $|\psi_\beta|$, $|\psi_\gamma|$ space, so it is a two-dimensional surface. The curves on which the amplitude stationary points lie are thus a set of measure zero on $E=$ constant in ψ_α, $|\psi_\beta|$, $|\psi_\gamma|$ space. Upon expanding into ψ_α, ψ_β^R, ψ_β^I, ψ_γ^R, ψ_γ^I space, the surface $E=$ constant is an ellipsoid and, enclosing a five-dimensional volume, is a four-dimensional surface.

The set of amplitude-stationary points is described in this space on $E=$ constant by the equations

$$\left.\begin{aligned}\psi_\alpha&=\psi_{\alpha s}(\sigma)\\(\psi_\beta^R)^2+(\psi_\beta^I)^2&=|\psi_{\beta s}(\sigma)|\\(\psi_\gamma^R)^2+(\psi_\gamma^I)^2&=|\psi_{\gamma s}(\sigma)|\end{aligned}\right\}, \tag{4.5}$$

which is at most a three-dimensional set.

This is clear since as E expands from two dimensions in ψ_α, $|\psi_\beta|$, $|\psi_\gamma|$ space to four in ψ_α, ψ_β^R, $\ldots$, ψ_γ^I space, the line $\psi_{\alpha s}(\sigma)$, $|\psi_{\beta s}(\sigma)|$, $|\psi_{\gamma s}(\sigma)|$ on $E=$ constant cannot expand into more than two more dimensions. As a more formal argument, we use the fact that $\dim(\mathrm{A}\times\mathrm{B})\leqslant \dim\mathrm{A}+\dim\mathrm{B}$ where $\mathrm{A}\times\mathrm{B}$ is the Cartesian product of A and B. Eqs. (4.5) each describe lines on $E=$ constant, hence the product set does not exceed dimension three.

The next step is to observe [for $(\dot{\ })=d(\ )/dt$] that

$$\frac{\partial\dot\psi_\alpha}{\partial\psi_\alpha}+\frac{\partial\dot\psi_\beta^R}{\partial\psi_\beta^R}+\ldots+\frac{\partial\dot\psi_\gamma^I}{\partial\psi_\gamma^I}=0, \tag{4.6}$$

and hence the transformation in phase space generated by (2.2) is measure-preserving. It then follows that for dA being the element of area on $E=$ constant and dn being an increment of distance normal to $E=$ constant that a volume integral in phase space is

$$\int(\ )dV=\int(\ )\,dAdn=\int\frac{(\ )}{\partial E/\partial n}dAdE, \tag{4.7}$$

since $dE=(\partial E/\partial n)dn$. Thus

$$\nu(S)=\int_S\frac{dA}{\partial E/\partial n} \tag{4.8}$$

is an invariant measure on $E=$ constant. This implies that if the set S is an image of a set S_0 of initial points, then $\nu(S)=\nu(S_0)$.

We now let I_0 be the set of points $\boldsymbol{\psi}_0$, where

$$\boldsymbol{\psi}=(\psi_\alpha,\psi_\beta^R,\psi_\beta^I,\psi_\gamma^R,\psi_\gamma^I),$$

that have trajectories $\boldsymbol{\psi}(\boldsymbol{\psi}_0,t)$ proceeding asymptotically to the set P of amplitude-stationary points. Since the measure ν of (4.8) is invariant, $\nu(I_0)=\nu(P)=0$, where the equality to zero follows from the conclusion stated above that P is a set of measure zero. Because I_0 includes all points on all trajectories that proceed asymptotically to amplitude-stationary points, we have proved:

THEOREM 4.1. *The aperiodic trajectories on any energy surface occupy a set of measure zero.*

It is in this precise sense that we can say that aperiodic trajectories are rare.

The explanation for the scarcity of aperiodic trajectories lies in the necessity of initial conditions for such trajectories producing double roots of $P(z)$. In order that $z=a$ be a root of both $P(z)$ and $R(z)$, it is necessary that the eliminant of the equations vanish. This is the determinant of the coefficients of powers of a in the equations $P(a)=0$, $aP(a)=0$, $a^2P(a)=0$, $R(a)=0$, $\ldots$, $a^3R(a)=0$. This condition gives a relation of the form $F(\psi_\alpha(0),\psi_\beta^R(0),\psi_\beta^I(0),\psi_\gamma^R(0),\psi_\gamma^I(0))=0$ which is a four-dimensional figure. The intersection of this hypersurface with the four-dimensional set $E=$ constant will be a three-dimensional set except where the two surfaces coincide. Although proving from the eliminant that they never coincide would be difficult, we see from the theorem that the set of initial points for aperiodic trajectories is of measure zero, and the argument above indicates the geometry of the situation.

5. Conclusion

The motions possible in a three-component spectral model based on the vorticity equation (in either barotropic or potential vorticity form) divide into the class of periodic flows and the class of flows that are asymptotic to amplitude-stationary points. The latter class is of measure zero on surfaces of constant energy in phase space. Thus if a zonal flow is disturbed by two additional components chosen at random (but satisfying the selection rules), then the probability is zero that the resulting flow will reach a steady equilibrium.

The conclusion reached here for three-component flow can be stated in the form that all solutions, except for those occurring on a set of measure zero, are periodic.

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