Extending Radiative Transfer Models by Use of Bayes' Rule[1]

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ABSTRACT

This paper presents a procedure that extends some existing radiative transfer modeling techniques to problems in atmospheric science where curvature and layering of the medium and dynamic range and angular resolution of the signal are important. Example problems include twilight and limb scan simulations. Techniques that are extended include successive orders of scattering, matrix operator, doubling, Gauss-Seidel iteration, discrete ordinates and spherical harmonics. The procedure for extending them is based on Bayes' rule from probability theory.

1. Introduction

Radiative transfer modeling is an important tool in atmospheric science, both for predicting radiative effects and for interpreting radiometric data. But the tool is often taxed because real atmospheres present so many difficult modeling problems. Typical problems with absorption and highly anisotropic scattering are impractical to solve analytically, and therefore require computer simulation. But even modeling by computer simulation can be excessively burdensome for those problems involving atmospheres with complicating features such as curvature and layering, and requiring results with features difficult to produce, such as large dynamic range and fine angular resolution. Examples include problems where the irradiating sun is near the horizon (twilight) or where observations are made near the horizon (limb scan). The geometries involved in these examples are illustrated in Figs. 1 and 2.

At present only two radiative transfer modeling techniques have been adapted to the type of problem cited above, while there are many others whose use in atmospheric science is currently restricted to a flat atmosphere. For the purposes of this paper, the various methods will simply be listed, with a single descriptive reference for each. But it should be recognized that the methods have a long and often interrelated history of use, and correspondingly voluminous literature. A recent full guide to this literature has been provided by Lenoble (1974).

The two methods which can handle twilight and limb scan situations are Monte Carlo (Collins and Wells, 1965) and DART[2] (Whitney, 1974). The methods which are currently restricted to a flat atmosphere include successive orders of scattering (Dave, 1964), matrix operator (Grant and Hunt, 1969), doubling (Hansen, 1969), Gauss-Seidel iteration (Herman and Browning, 1965), discrete ordinates (Liou, 1973) and spherical harmonics (Canosa and Penafiel, 1973).

Most of the currently restricted radiative transfer modeling techniques could, in principle, be extended to spherical problems by suitable modification of the equations that are computer programmed. But there are several factors deterring such an approach:

1) The algebraic techniques required would be unlikely to generalize to other problems that are nonspherical but otherwise similar, such as irregular finite clouds, rocket plumes, and comet tails.

2) There would likely be an impractical increase in the amount of computation required to carry out a complete simulation.

This paper presents an alternative approach to the problem that is computationally practical and potentially generalizable. Use is made of prior information, comprising some physically reasonable estimate of the general features of the solution being sought, apart from the radiative transfer simulation itself. The mathematical framework for incorporating such prior information is borrowed from the DART method, which was the first to use it, and which in turn borrowed it from probability theory. Probability theory offers Bayes' rule as a formalism for making use of prior information, and this paper adapts that formalism to generalize the above-mentioned radiative transfer modeling techniques.

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[2] An acronym for Dodecaton Approach to Radiative Transfer.

2. The radiative transfer problem

The general problem is to determine the radiance (or more generally, the Stokes' vector) S, expressed as a function of position vector $\mathbf{r}$ and propagation direction unit vector $\hat{\mathbf{k}}$. The function S is governed by the integro-differential equation of radiative transfer, which expresses the extinction of the radiance due to scattering and absorption, described by an extinction coefficient $\epsilon(\mathbf{r})$, the enhancement due to emission, described by a source term $J(\hat{\mathbf{k}},\mathbf{r})$, and coupling into $\hat{\mathbf{k}}$ from other propagation directions via scattering, described by a scattering coefficient $\sigma(\mathbf{r})$ and a phase function $P(\hat{\mathbf{k}}'\rightarrow\hat{\mathbf{k}},\,\mathbf{r})$ which integrates over 4π solid angle to unity. For notational economy, it is convenient to make the $\mathbf{r}$ dependence of all quantities implicit and write the transfer equation as

$$\hat{\mathbf{k}}\cdot\nabla S(\hat{\mathbf{k}})=-\epsilon S(\hat{\mathbf{k}})+J(\hat{\mathbf{k}})+\sigma\int_{4\pi}S(\hat{\mathbf{k}}')P(\hat{\mathbf{k}}'\rightarrow\hat{\mathbf{k}})d\Omega.$$

The transfer equation, along with source characteristics and boundary conditions, would provide $S(\hat{\mathbf{k}})$ exactly if it could be solved analytically. But often this is not possible, and computer modeling is used to provide an approximation to $S(\hat{\mathbf{k}})$. Approximation is unavoidable because the continuous $\mathbf{r}$ and $\hat{\mathbf{k}}$ variables mean that S constitutes an uncountable infinity of numbers, which has to be replaced by some finite set of numbers.

Each of the radiative transfer modeling techniques mentioned in Section 1 uses some finite set of numbers to represent $S(\hat{\mathbf{k}})$, and there exist at least four such sets in common use. Members of each set can be regarded as integrals of the radiance over solid angles, but with different weighting functions in the integrals.

In the first case, the unit sphere of propagation direction is simply partitioned into small solid angle regions,

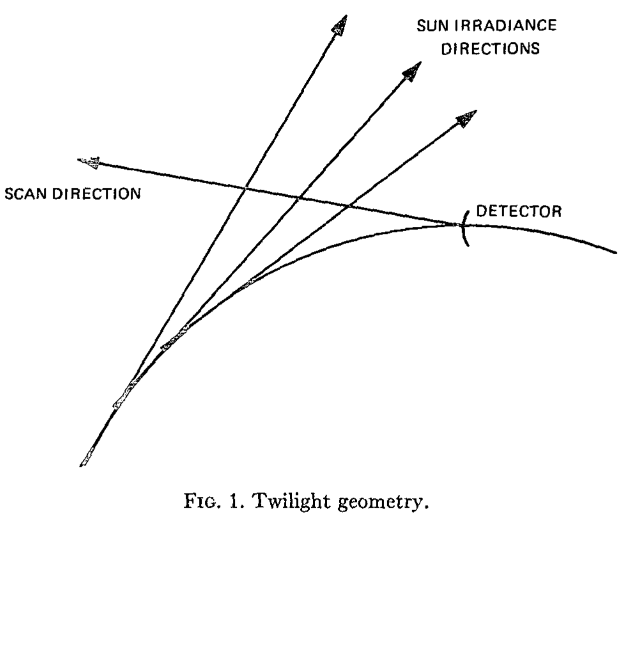

Fig. 1. Twilight geometry.

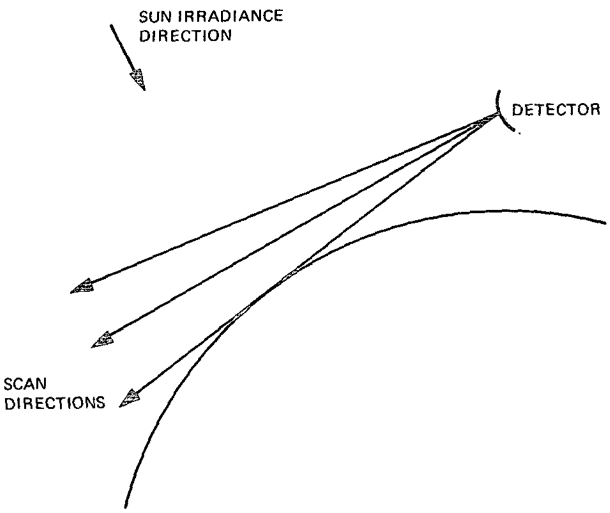

Fig. 2. Limb scan geometry.

and the weighting function is 1 within a particular region. This simple connecting rule is commonly used in the Monte Carlo, successive orders of scattering, matrix operator, doubling and Gauss-Seidel iteration methods.

For the second case, a spherical coordinate system with z axis normal to the plane of the flat atmosphere is defined. The radiance is approximated using values at discrete quadrature points in zenith, or more generally, by integrals over zenith bands that are usually vanishingly small but may in general be nonzero to allow some averaging of oscillation. The discrete numbers used are the Fourier coefficients for expansion in azimuth of the radiance integrated over a band of zeniths. This formulation is used in the discrete ordinates method, and often in conjunction with the matrix operator technique.

For the third case, the radiance is Fourier expanded in azimuth and Legendre polynomial expanded in zenith. This constitutes the spherical harmonics approach.

For the fourth case, the weighting function is highly peaked around a particular propagation direction, and decreases continuously to zero as a power of the cosine of the angle between $\hat{\mathbf{k}}$ and the particular propagation direction. This weighting function is used in the DART approach.

To discuss the four cases listed above, we need a common notation for describing them. Although they differ in detail, the discrete numbers in all cases have units of irradiance, and can all be referred to as "streams." For identifying the streams, the first case suggests a particularly simple notation. We can let a unit vector $\hat{\mathbf{p}}$ identify a particular propagation direction associated with a solid angle region $\Delta\mu$, and use $\hat{\mathbf{p}}$ as a subscript to identify the stream associated with that solid angle. The discrete stream is represented by $S_{\hat{\mathbf{p}}}$, notation analogous to $S(\hat{\mathbf{k}})$ for the continuous radiance.

To use this notation in general, we have to modify the interpretation of $\hat{\mathbf{p}}$ in other cases. In the discrete ordinates case, we let the θ coordinate of $\hat{\mathbf{p}}$ identify the band in zenith that is integrated over, called $\Delta\mu$ because μ=cosine (zenith) is the variable of integration. Then we let the ϕ coordinate of $\hat{\mathbf{p}}$ identify the Fourier index m. In the spherical harmonics case we let the ϕ coordinate of $\hat{\mathbf{p}}$ identify the Fourier index m, and the θ coordinate identify the Legendre index M. In the DART case we let $\hat{\mathbf{p}}$ be the unit vector pointing to the weighting function peak. For each of the cases we let the weighting function be $w(\hat{\mathbf{p}}|\hat{\mathbf{k}})$. In terms of the above notation, the four types of radiative transfer models deal with weighting functions and streams that are defined as follows:

In the first case, common to most of the radiative transfer models, $w(\hat{\mathbf{p}}|\hat{\mathbf{k}})$ is simply 1 for $\hat{\mathbf{k}}$ inside the solid angle region $\Delta\mu$ specified by $\hat{\mathbf{p}}$. The stream is

$$S_{\hat{p}}=\int_{\Delta\Omega} S(\hat{\mathbf{k}})d\Omega.$$

In the discrete ordinates case, for $\hat{\mathbf{k}}$ inside the band $\Delta\mu$ specified by $\hat{\mathbf{p}}$,

$$w(\hat{\mathbf{p}}|\hat{\mathbf{k}})=\frac{1}{2\pi}\cos[m(\phi-\phi_0)],$$

where ϕ is azimuth and the reference ϕ_0 is the input sun irradiance azimuth. The stream is

$$S_{\hat{p}}=\frac{1}{2\pi}\int_{2\pi}\cos[m(\phi-\phi_0)]\int_{\Delta\mu} S(\hat{\mathbf{k}})d\mu d\phi.$$

For the spherical harmonics case

$$w(\hat{\mathbf{p}}|\hat{\mathbf{k}})=\frac{1}{4\pi}\cos[m(\phi-\phi_0)]P_M^m(\mu),$$

where $P_M^m(\mu)$ is an associated Legendre polynomial with $M>m$, defined with normalization

$$\int_{-1}^{1}[P_M^m(\mu)]^2d\mu=\frac{2}{(2M+1)}.$$

The stream is

$$S_{\hat{p}}=\frac{1}{4\pi}\int_{4\pi}\cos[m(\phi-\phi_0)]P_M^m(\mu)S(\hat{\mathbf{k}})d\mu d\phi.$$

In the DART case, for $\hat{\mathbf{k}}$ in the hemisphere centered on $\hat{\mathbf{p}}$,

$$w(\hat{\mathbf{p}}|\hat{\mathbf{k}})=(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})^n,$$

where the power n is an arbitrary positive integer (although usually set to 2). The stream is

$$S_{\hat{p}}=\int_{2\pi}(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})^nS(\hat{\mathbf{k}})d\Omega,$$

where the integration is over the hemisphere centered on $\hat{\mathbf{p}}$.

It is extremely important to realize that although the various radiative transfer modeling techniques have been associated with particular weighting functions in the past, the same choices are not necessary in the future. There is nothing to prevent, say, a doubling code that uses a DART weighting function. Section 4 will show that there would be an advantage in developing hybrid techniques of this sort.

The reason why a variety of such combinations of techniques is possible is that, with the exception of the spherical harmonics method, *all* the discrete transfer models are really dealing with the same discretized equations. The discrete approximations for the transfer equation generally have the form

$$\hat{\mathbf{p}}\cdot\nabla S_{\hat{p}}=-\epsilon S_{\cdot}+J_{\hat{p}}+\sigma\sum_{\hat{p}'}S_{\hat{p}'}T_{\hat{p}'\to\hat{p}},$$

where $J_{\hat{p}}$ is a discrete representation for emission, the sum on $\hat{\mathbf{p}}'$ replaces integration over $\hat{\mathbf{k}}'$, and $T_{\hat{p}'\to\hat{p}}$ is a discrete representation for the phase function. Changing the definition of $S_{\hat{p}}$ has the effect of changing the definition of the $T_{\hat{p}'\to\hat{p}}$ but it need not change the basic form of the transfer equation.

But regardless of what weighting function is used, computer simulation of radiative transfer produces only the finite array of streams ($S_{\hat{p}}$'s) where the complete function $S(\hat{\mathbf{k}})$ is desired. Thus it presents the following problem: Given the array of streams, formulate the best possible estimate of the radiance function.

The solution to the general problem stated will be based on some analogies. The important point to note about $S(\hat{\mathbf{k}})$ is that, apart from an overall scale factor carrying physical dimensions, it is like a probability density function. That is, it expresses the density of probability that a differential solid angle centered on $\hat{\mathbf{k}}$ includes the propagation direction of a random photon traveling through point $\mathbf{r}$. The domain of this probability density function is 4π steradians. Similarly, the important point to note about $S_{\hat{p}}$ is that it is proportional to a probability; namely, that a photon traveling through location $\mathbf{r}$ is counted in the stream identified by $\hat{\mathbf{p}}$. This probability function is discrete and its domain is the countable number N of streams.

With the analogies stated above, the radiative transfer modeling problem becomes analogous to a standard problem in probability theory: Given the probability density function for one random variable, estimate the probability density function for another random variable, using a theoretical model correlating the two.

3. The probabilistic solution

Bayes' rule is the tool used to solve the hypothetical problem from probability theory: given two random variables which are theoretically correlated, and given a probability density function for one of them, estimate the probability density function for the other one. This section reviews the statement of Bayes' rule and its application to the posed problem.

The basis for Bayes' rule is a straightforward relationship between joint probability, conditional probabilities and unconditioned probabilities. Let a and b represent values of two random variables α and β. The various probabilities concerning α and β are all designated by a "p" for "probability," so it is helpful to distinguish them by subscripts involving α and β. The joint probability that α lies in the range $(a, a+da)$ and β lies in the range $(b, b+db)$ is expressed by $da\,db$ times a density function as

$$p_{\alpha,\beta}(a,b)da\,db.$$

The unconditioned probabilities that α lies in the range $(a, a+da)$ or β lies in the range $(b, b+db)$ individually are expressed by differentials times density functions as

$$p_{0\alpha}(a)da \quad \text{and} \quad p_{0\beta}(b)db.$$

The conditional probability that α lies in the range $(a, a+da)$ if it is already known that $\beta=b$ is expressed by da times a density function as

$$p_{\alpha|\beta}(a|b)da.$$

Similarly, the conditional probability that β lies in the range $(b, b+db)$ if it is already known that $\alpha=a$ is expressed by db times a density function as

$$p_{\beta|\alpha}(b|a)db.$$

The relationship between the various probability densities is

$$p_{\alpha,\beta}(a,b)=p_{0\alpha}(a)p_{\beta|\alpha}(b|a)=p_{\alpha|\beta}(a|b)p_{0\beta}(b).$$

This relationship rearranges to

$$p_{\alpha|\beta}(a|b)=p_{0\alpha}(a)p_{\beta|\alpha}(b|a)/p_{0\beta}(b)$$

and the above is the usual statement of Bayes' rule [see section 2.2 of Gelb (1974)].

If all the probability density functions are known, then Bayes' rule is no more than a logical identity. Its practical utility arises only in the case where something is unknown and in fact not directly knowable. Suppose knowledge about the distribution function for random variable α is incomplete, and α cannot even be sampled to provide such knowledge. But β can be sampled, and there is at least a theoretical model correlating β with α so that $p_{\beta|\alpha}(b|a)$ can be formulated. Then one can make some progress by letting $p_{0\alpha}(a)$ stand for a prior estimate of the desired function, to be updated by sampling β. This idea of prior estimate and update is one conceptual element in a major trend in modern estimation theory (see e.g., Kalman and Bucy, 1961; Deutsch, 1965; Jazwinski, 1970; Gelb, 1974). It is implemented by Bayes' rule as follows: Given $p_{\beta|\alpha}(b|a)$ and $p_{0\alpha}(a)$, the prior estimate function for β has to be

$$p_{0\beta}(b)=\int p_{\beta/\alpha}(b|a)p_{0\alpha}(a)da.$$

Bayes' rule for updating $p_{0\alpha}(a)$ from a single sample $\beta=b$ is thus

$$p_{\alpha|\beta}(a|b)=\frac{p_{0\alpha}(a)p_{\beta|\alpha}(b|a)}{\int p_{\beta|\alpha}(b|a)p_{0\alpha}(a)da}.$$

If β is sampled infinitely many times, the frequency of occurrence of different values is determined, providing a probability density function $p_\beta(b)$. Clearly, the best posterior estimate for the distribution for α is the average over values of b of the individual $p_{\alpha|\beta}(a|b)$ functions:

$$p_\alpha(a)=p_{0\alpha}(a)\int\left[\frac{p_{\beta|\alpha}(b|a)p_\beta(b)}{\int p_{\beta|\alpha}(b|a)p_{0\alpha}(a)da}\right]db.$$

This is the equation which is adapted to solve the radiative transfer problem.

4. The radiative transfer solution

Adapting the probabilistic solution to the radiative transfer problem requires that the proper analogs be identified. They are as follows:

(i) The variable a becomes the continuous propagation direction $\hat{\mathbf{k}}$.
(ii) The variable b becomes the discrete stream label $\hat{\mathbf{p}}$.
(iii) The estimated function $p_\alpha(a)$ becomes the desired radiance function $S(\hat{\mathbf{k}})$.
(iv) The prior estimate function $p_{0\alpha}(a)$ becomes a prior estimate of the radiance $S_0(\hat{\mathbf{k}})$.
(v) Integration over the continuous variable b becomes summation over the discrete $\hat{\mathbf{p}}$.
(vi) The measured function $p_\beta(b)$ becomes the array of streams $S_{\hat{\mathbf{p}}}$ generated by a discrete radiative transfer simulation.
(vii) The theoretical connecting function $p_{\beta|\alpha}(b|a)$ becomes the weighting function $w(\hat{\mathbf{p}}|\hat{\mathbf{k}})$.

The radiative transfer solution is obtained by substituting the above analogs in the probabilistic solution. Taking account of the fact that $w(\hat{\mathbf{p}}|\hat{\mathbf{k}})$ is not necessarily normalized over $\hat{\mathbf{p}}$ as $p_{\beta|\alpha}(b|a)$ is over b, we put

$$S(\hat{\mathbf{k}})=\frac{S_0(\hat{\mathbf{k}})}{\sum_{\hat{\mathbf{p}}} w(\hat{\mathbf{p}}|\hat{\mathbf{k}})}\sum_{\hat{\mathbf{p}}}\left[\frac{w(\hat{\mathbf{p}}|\hat{\mathbf{k}})S_{\hat{\mathbf{p}}}}{\int_{4\pi} w(\hat{\mathbf{p}}|\hat{\mathbf{k}})S_0(\hat{\mathbf{k}})d\Omega}\right]$$

The above formula applies for each of the particular definitions of $\hat{\mathbf{p}}$ mentioned in Section 2, given the proper function $w(\hat{\mathbf{p}}|\hat{\mathbf{k}})$ for each. The fact that at least four cases occur means, essentially, that there are at least four different random variables that $\hat{\mathbf{k}}$ is correlated with, and information on any one of them can be used to update $S_0(\hat{\mathbf{k}})$ via Bayes' rule. However, the character of the updated $S(\hat{\mathbf{k}})$ will naturally depend, to some extent, on which of the four is chosen.

Substituting the formulas given in Section 2 for $w(\hat{\mathbf{p}}|\hat{\mathbf{k}})$ into the expression for $S(\hat{\mathbf{k}})$ reveals that the continuous spherical harmonics and DART weighting functions always allow a continuous function for $S(\hat{\mathbf{k}})$, whereas the orders can impose artificial discontinuities where the boundaries of zenith bands or solid angle regions occur. The discontinuities are removed in the discrete ordinates method by using the above $S(\hat{\mathbf{k}})$ only for $\hat{\mathbf{k}}$ at the discrete quadrature zeniths and using polynomial interpolation between. But that can artificially impose continuity instead of discontinuity. Since neither discontinuity nor continuity ought to be imposed by the choice of weighting function, it appears that the inherently continuous weighting functions are preferable. Fortunately, this observation does not mean that methods other than spherical harmonics or DART need be avoided, since the idea of continuous weighting function really can be used in general, even where it has not been adopted yet.

Considering the weighting function which is both continuous and peaked in solid angle (DART) suggests that the Bayesian formula for $S(\hat{\mathbf{k}})$ can be thought of as a kind of interpolation for the uncountable infinity of $\hat{\mathbf{k}}$'s between the countable $\hat{\mathbf{p}}$'s actually handled in a discrete simulation. The word "interpolation" seems less applicable in the other cases though. For the weighting function that is constant over $\Delta\Omega$, only one $\hat{\mathbf{p}}$ affects $S(\hat{\mathbf{k}})$ for any particular $\hat{\mathbf{k}}$. For the discrete ordinates and spherical harmonics cases, the $\hat{\mathbf{p}}$'s do not correspond to specific propagation directions that are interpolated between.

Examining the limit as the number N of streams becomes very large reveals that the two weighting functions highly peaked in solid angle (first and fourth cases) provide an obvious convergence property. The prior estimate $S_0(\hat{\mathbf{k}})$ in the numerator and the denominator tend to cancel, leaving the posterior estimate totally dominated by the streams at $\hat{\mathbf{p}}$ near $\hat{\mathbf{k}}$. This behavior reproduces the fundamental behavior of Bayes' rule: extensive measurement of random variable β that is totally correlated with α will totally determine $p_\alpha(a)$ regardless of $p_{0\alpha}(a)$.

However, except in limiting cases, the prior estimate $S_0(\hat{\mathbf{k}})$ generally affects the nature of the posterior $S(\hat{\mathbf{k}})$. One interesting observation is that, for each of the four cases considered, it is possible to find a particular choice of prior estimate function that reduces the Bayesian formula for $S(\hat{\mathbf{k}})$ to its commonly used non-Bayesian counterpart. In other words, the Bayesian formula is general; in special cases it reduces to more familiar formulas.

For the simple integration rule used by most of the radiative transfer models, we introduce a prior estimate which is constant over each solid angle region. Then $S_0(\hat{\mathbf{k}})$ cancels out of the numerator and denominator of the $S(\hat{\mathbf{k}})$ formula, leaving

$$S(\hat{\mathbf{k}})=S_{\hat{\mathbf{p}}}/\Delta\Omega$$

for $\hat{\mathbf{k}}$ inside the solid angle $\Delta\Omega$ defined by $\hat{\mathbf{p}}$. This is the standard non-Bayesian formula for this case.

For the discrete ordinates case, we introduce a prior estimate which is constant in θ within each zenith band, and which has equal amounts of each frequency component in ϕ. That is, we set

$$S_0(\hat{\mathbf{k}})=S_0F(\phi),$$

where

$$F(\phi)=\sum_m \cos[m(\phi-\phi_0)].$$

Substituting this in the Bayesian formula gives for $\hat{\mathbf{k}}$ inside the zenith band $\Delta\mu$ specified by $\hat{\mathbf{p}}$

$$S(\hat{\mathbf{k}})=\sum_m [2-\delta_{0,m}]\cos[m(\phi-\phi_0)]S_{\hat{\mathbf{p}}}/\Delta\mu,$$

where $\delta_{0,m}$ is the Kronecker delta function. This is the standard non-Bayesian formula for the discrete ordinates method.

For the spherical harmonics case, we let the prior estimate have equal amounts of each frequency and each Legendre polynomial:

$$S_0(\hat{\mathbf{k}})=\sum_m \{\cos[m(\phi-\phi_0)]\sum_M P_M^m(\mu)\}.$$

Substituting this in the Bayesian formula for $S(\hat{\mathbf{k}})$ gives

$$S(\hat{\mathbf{k}})=\sum_m (2-\delta_{0,m})\cos[m(\phi-\phi_0)]\sum_M (2M+1)P_M^m(\mu)S_{\hat{\mathbf{p}}}.$$

This is the standard non-Bayesian formula used in the spherical harmonics method.

To summarize the situation, the standard formulas used in the currently available radiative transfer models can be obtained from the Bayesian formulas by substituting prior estimates that are uniform in the streams, whatever the streams are. By setting all the streams equal, such a prior estimate represents a maximum of mathematical uncertainty. But the user should be aware that it can also correspond to a minimum of

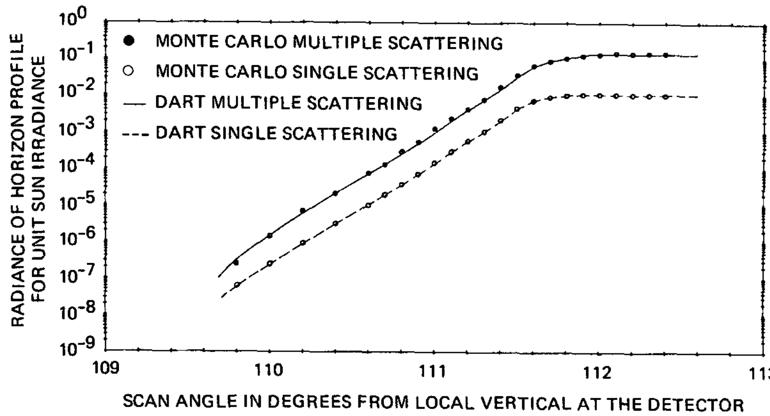

FIG. 3. Single and multiple scattering by Monte Carlo and DART methods in limb scan geometry. The lines arise from a 12-stream DART simulation with Bayesian estimation of the radiance described in the text. The points represent the Monte Carlo Flash code of Radiation Research Associates (Collins and Wells, 1965), with 40 000 photon histories followed so that there would be accuracy to 10% even at the low end of the dynamic range of the signal.

For both programs, the detector is 500 km above the earth, scanning to the left. Sunlight enters from the right, traveling in a direction lying 201.4° from the detector local vertical. The atmosphere of the earth as seen at 7000 Å is used. Both programs model it with 51 concentric spherical layers of 2 km thickness. To make the scattering problem more difficult, surface albedo is set to unity and the optical depths of the various atmospheric constituents are all artificially ascribed to aerosols having a real refractive index of 1.5 and ranging in radius r from 0.01 to 1.0 μm, according to r^{-3}. This aerosol distribution has a rather complicated phase function, with forward and backward peaks separated by two orders of magnitude. Single scattering is treated exactly in both codes.

physical reality. A prior estimate function with, say, all equal Fourier and Legendre coefficients has a very strange, highly structured and physically unlikely shape!

It is normally not true that the user has no knowledge at all of the radiance function he is trying to simulate. He may know, for instance, that the radiance is zero in the upper hemisphere if the detector is above the atmosphere, or that the radiance is uniform in the lower hemisphere if the detector rests on a Lambertian surface. He can and should use the prior estimate to express whatever characteristics of the solution are intuitively obvious physical requirements. Naturally, this has to be done with recognition of a risk. There is nothing in the formalism to actually prevent the chosen prior estimate from being very wrong, enough to make the posterior estimate in some sense "worse" than the non-Bayesian estimate. Therefore, whenever possible, representative problems should be checked against some type of benchmark to provide confidence in the Bayesian approach.

5. Application to the non-flat atmosphere

The simulation of radiative transfer in a curved atmosphere is a natural application for Bayesian estimation. Use of the method allows incorporation of the curvature through the prior estimate $S_0(\hat{\mathbf{k}})$, so that the discrete computer simulation can be done solely for a flat atmosphere. This means that existing techniques for handling the discrete equation can be used directly without the necessity for any generalization. Otherwise the $\mathbf{r}$ variable in the transfer equation would have to be represented by a full three variables (r,θ_r,ϕ_r) instead of one (z) as is done now. The algebraic and computational problems inherent in such a generalization have been a major factor inhibiting the adaptation of most radiative transfer modeling techniques to the curved problem.

The incorporation of curvature in the overall transfer simulation is mainly a matter of producing results with realistic features. The transfer problems of interest in a curved atmosphere include twilight and limb scan simulation, where radiance varies over several orders of magnitude within a few degrees of variation in illumination or scan direction, and where significant details due to atmospheric layers appear over even smaller angular ranges.

On the face of it, the twilight and limb scan situations seem to require radiative transfer simulation with extreme angular resolution. This would mean a computer model with a large number of streams, and hence a large requirement for computer time.

The requirement for large amounts of computer time is overcome by using Bayesian estimation. The dynamic range and angular resolution required in the final solution is conveyed in the prior estimate function $S_0(\hat{\mathbf{k}})$. The discrete streams are used only to carry information about the absolute amount of flux and the angular shaping due to phase functions.

To apply the method, one must formulate a prior estimate function $S_0(\hat{\mathbf{k}})$. A simple procedure for doing this is to make use of information obtained by simulating just single scattering. The single-scattering solution has all the features of dynamic range and angular resolution required in the final total solution. Furthermore, it is generally easy to calculate with both accuracy and speed, regardless of realistic features (such as curvature, fine atmospheric layering and highly oscillatory phase functions) that make multiple scattering so difficult to compute.

As a first approximation, it would be very reasonable to set $S_0(\hat{\mathbf{k}})$ simply equal to the single scattering result, except with the phase function set to $1/(4\pi)$. Experience with the DART model has shown that this is fairly good as judged by comparison to Monte Carlo simulations. Some simple modifications make it even better. In calculating the prior estimate, it is advantageous to delete a fraction of optical depths that can be attributed to extreme forward scattering into the diffraction peak of the phase function. It is also advantageous to neglect extinction prior to scattering into the scan direction.

Results obtained in the manner described above have been reported by Whitney (1974), and are reproduced in Fig. 3 to illustrate several points. The figure shows limb scan simulations done by the 12-stream DART method and by the Monte Carlo method (the only two radiative transfer models presently adapted to a curved geometry). Details concerning the problem and the two simulations are provided in the figure caption.

Examination of Fig. 3 confirms that both single and multiple scattering as calculated by the two methods agree very well. The Bayesian estimation thus appears to function properly. It offers an advantage, in that the DART run time is approximately 1 min on an IBM/360-75 machine, whereas the Monte Carlo run is a benchmark standard that was allowed nearly an hour of computing time on a CDC/6600 machine.

Even for scan angles which do not penetrate deeply into the atmosphere, both Monte Carlo and DART results indicate strongly that *the total radiance does not converge to the single-scattering radiance!* This is a result which has surprised many researchers, since it is a fairly common misconception that the limb scan approaches single scattering high in the atmosphere. But the result is actually physically reasonable, since every parcel of air, even high in the atmosphere, has the lower atmosphere as a source of multiple scattered photons which, like primary photons from the sun, are scattered into the detector.

6. Conclusions

This paper has demonstrated a procedure for extending the scope of a variety of radiative transfer computer modeling techniques to include more general problems. The procedure can handle features such as atmospheric layering and curvature, and can efficiently produce results with features such as large dynamic range and fine angular resolution. It constitutes an application of Bayes' rule from probability theory, which allows incorporation of prior information concerning physically necessary features of the solution being sought.

Example problems where incorporation of such prior information is helpful include twilight and limb scan simulations. In these problems, use of Bayes' rule allows estimation of the detailed continuous radiance function of propagation direction for the curved atmosphere from a coarse array of discrete streams modeled for a flat atmosphere. The detailed properties of the radiance obtained depend on the details of the mathematical connection between radiance and stream, and freedom from artificial effects requires a connection that integrates radiance with a continuous weighting function. Most of the available radiative transfer modeling techniques are not currently using such a connecting rule, but there is no reason why they could not. Thus eventually excellent results in a curved geometry can be expected from any of the currently available flat radiative transfer models when fully adapted to the problem.

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REFERENCES

Canosa, J., and H. R. Penafiel, 1973: A direct solution of the radiative transfer equation: Application to Rayleigh and Mie atmospheres. *J. Quant. Spectrosc. Radiat. Transfer*, **13**, 21–39.

Collins, D. G., and M. B. Wells, 1965: Monte Carlo codes for study of light transport in the atmosphere. Rep. RRA-T54, Vols. 1 and 2, Fort Worth, Tex., Radiation Research Associates.

Dave, J. V., 1964: Meaning of successive iteration of the auxiliary equation in the theory of radiative transfer. *Astrophys. J.*, **140**, 1292–1303.

Deutsch, R., 1965: *Estimation Theory*. Prentice Hall.

Gelb, A., Ed., 1974: *Applied Optimal Estimation*. The MIT Press.

Grant, I. P., and G. E. Hunt, 1969: Discrete space theory of radiative transfer and its application to problems in planetary atmospheres. *J. Atmos Sci.*, **26**, 963–972.

Hansen, J. E., 1969: Radiative transfer by doubling very thin layers. *Astrophys. J.*, **155**, 565–573.

Herman, B. M., and S. R. Browning, 1965: A numerical solution to the equation of radiative transfer. *J. Atmos. Sci.*, **22**, 559–566.

Jazwinski, A. H., 1970: *Stochastic Processes and Filtering Theory*. Academic Press.

Kalman, R. E., and R. Bucy, 1961, New results in linear filtering and prediction. *J. Basic Eng.*, **83D**, 95–108.

Lenoble, J., Ed., 1974: *Standard Procedures to Compute Atmospheric Radiative Transfer in a Scattering Atmosphere*, Vol. II. Lille, France, International Association of Meteorology and Atmospheric Physics, IUGG Radiation Commission.

Liou, K. N., 1973: A numerical experiment on Chandrasekhar's discrete ordinate method for radiative transfer. *J. Atmos. Sci.*, **30**, 1303–1326.

Whitney, C., 1974: Efficient stream distributions in radiative transfer theory. *J. Quant. Spectrosc. Radiat. Transfer*, **14**, 591–611.