A Two-Layer Model Study of the Combined Barotropic and Baroclinic Instability in the Tropics

H. L. Kuo

Department of Geophysical Sciences, The University of Chicago, Chicago, IL 60637

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ABSTRACT

The stabilities of shear zone and jet type mean currents with both horizontal and vertical shears in the tropics are investigated by the use of two-level linearized primitive equations. The results show that the main source of instability is the horizontal shear, and the influences of the mean vertical shear and stable stratification are to restrict the disturbance either to the upper or to the lower level. The $\beta$-effect stabilizes the shear flow and the westerly jet but it destabilizes the easterly jet, and the CISK mechanism is significant only when the surface layer is well mixed.

The jet profile is most unstable when its center is located from $6^\circ$ to $10^\circ$ from the equator, and the maximum intensity of the disturbance occurs on the side of the jet where the absolute vorticity of the mean flow is small. The first effect mentioned above is significant for the mean position of the ITCZ through the feedback mechanism.

1. Introduction

Observations show that synoptic-scale disturbances in the tropical troposphere are usually characterized during their early history by a relatively cold ascending region and a warm descending region, and those in the upper troposphere tend to behave independently from those in the lower troposphere, except when sufficient latent heat is released by the deep cumulus imbedded in them, which tends to unify the upper and lower systems into a single system and produces a warm core. These observations indicate that the origin of most incipient disturbances in the tropics is mainly the barotropic instability associated with the horizontal shear of the basic flow, as has been discussed by the author (Kuo, 1973a) and by many other investigators. However, almost all the past investigations on the barotropic stability problem are based on the quasi-geostrophic model and therefore reached the result that the most unstable disturbance is purely horizontal and non-divergent and hence independent of height (Kuo, 1949, 1951, 1973a; Jacobs and Win-Nielsen, 1966; Williams et al. (1971), contrary to the observed facts mentioned above. Judging from the smallness of the Coriolis parameter and its prominent variation with latitude in tropics, one expects that large-scale disturbances in low latitude will be three-dimensional and essentially ageostrophic and hence they can only be represented by the primitive equations. The purpose of this paper is to investigate the influences of the horizontal and the vertical shears of the basic current, the stable stratification and the CISK mechanism on the development and structure of unstable disturbances in the tropics, especially those along the Intertropical Convergence Zone (ITCZ). For simplicity of presentation, we shall first consider the system which is composed of two layers of homogeneous incompressible fluid, with the lighter layer 1 on top of the heavier layer 2, and then demonstrate that the equations for this system are identical to the two-level approximation to the continuous and compressible atmospheric system provided the vertical shear term in the zonal momentum equation is neglected, which is permissible when the Richardson number is much greater than 1. It will be shown by using this simple model that the almost independency of the motions in the lower and the upper layers of the atmosphere in low latitudes is attributable to the influences of the mean vertical shear and the stable stratification, especially the latter, and that the CISK effect on the growth rate of the disturbance is important only when the lapse rate in the surface layer is nearly adiabatic, such as when this layer is thoroughly mixed by convection.

2. The governing equations

a. Two homogeneous layers system

As has been mentioned above, we shall at first consider that the system is composed of two homogeneous layers of incompressible fluid, with the lighter layer of density $\rho_1$ and geostrophic mean flow $U_1$ lying above the heavier layer of density $\rho_2$ and geostrophic mean flow $U_2$, where $U_1$ and $U_2$ are functions of the latitude $\gamma$. Let the undisturbed depths of the two layers be $H_1$ and $H_2$, and the external pressure on the top surface $z = H = H_1 + H_2$ be $\rho^*$. Then the pressures $\tilde{p}_1$...
and $\bar{p}_2$ of the undisturbed state in the two layers are given by

$$\bar{p}_1 = p^* + g_1(H - z), \quad H_2 \leq z \leq H, \quad (1a)$$

$$\bar{p}_2 = p^* + g_1H_1 + g_2(H_2 - z), \quad 0 \leq z \leq H_2, \quad (1b)$$

while the slopes of the total depth $H = H_1 + H_2$ and $H_2$ are given by

$$\frac{d\bar{p}^*}{dy} + g_1 \frac{dH}{dy} = -f_1u_1, \quad (1c)$$

$$\frac{dH_2}{dy} = f_1(p_1u_1 - p_2u_2) \approx f_1(p_1u_1 - u_2), \quad (1d)$$

where $\rho$ is the mean density of the two layers and the approximation $\rho_1 \approx \rho \approx \rho_2$ is used. Since $0 < (\rho_2 - \rho_1) \rho$, the relations (1c) and (1d) show that, when $d\bar{p}^*/dy = 0$ and $U_1$ is either zero or is in opposite direction as $U_1$, then the slopes of $H$ and $H_2$ are of opposite signs and the magnitude of $dH/\rho dy$ is much smaller than that of $dH_2/\rho dy$. Thus the approximation $H = H_2$ constant can be used and $U_1$ can be attributed to $d\bar{p}^*/dy$ completely.

Next we consider that the disturbed state is represented by the total pressures $\bar{p}_1$ and the depth of the lower layer $H_2 = H_2 + z'$, and the perturbation velocities $u'$, $v'$ and $w'$ in the zonal, meridional and vertical directions, respectively, where $i = 1$ or 2. Since the pressure distribution in the large-scale systems is hydrostatic, we have

$$\rho_1 = \bar{p}_1 + \bar{p}_1', \quad H_2 \leq z \leq H, \quad (2a)$$

$$\rho_2 = \bar{p}_2 + \bar{p}_2', \quad 0 \leq z \leq H_2, \quad (2b)$$

The linearized equations of motion and the continuity equation are then given by

$$\frac{\partial u'}{\partial t} + U_1 \frac{\partial u'}{\partial x} - f'v' = -\frac{1}{\rho} \frac{\partial \bar{p}'}{\partial x}, \quad (3)$$

$$\frac{\partial v'}{\partial t} + U_1 \frac{\partial v'}{\partial x} + f'w' = \frac{1}{\rho} \frac{\partial \bar{p}'}{\partial y}, \quad (4)$$

$$\frac{\partial w'}{\partial t} + \frac{\partial \bar{p}'}{\partial y} \frac{\partial \bar{p}'}{\partial x} + \frac{\partial \bar{p}'}{\partial z} = 0, \quad (5)$$

where $f = \beta y$ is the Coriolis parameter, $f' = -f_1u_1/\rho_1$, and the superscripts and subscripts $i$, with $i = 1$ or 2, refer to the upper and the lower layer.

From the continuity of the total pressure across the interface $z = H_2 + z'$ and the relations (1a,b) and (2a,b), we find

$$\rho_1 - \rho_1' = g(\rho_2 - \rho_1)z'. \quad (6a)$$

Further, the relations (2a,b) and (1a,b) also give

$$\frac{\partial w'}{\partial z} = 0, \quad (6b)$$

so that the perturbation pressure $\bar{p}'$ in an individual layer is independent of height. On making use of this relation in (3) and (4) we then conclude that the horizontal velocities $u'$ and $v'$ in each layer are also independent of height, viz.,

$$\frac{\partial u'}{\partial x} = \frac{\partial v'}{\partial z} = 0. \quad (6c)$$

Thus, Eq. (5) shows that $w'$ varies linearly with $z$ within each layer and is given by

$$w'(z) = -\left[\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right](z - z_0), \quad (7)$$

where $z_0 = 0$ and $z_0 = H$. Here it is assumed that $w'$ vanishes both at the bottom and at the top.

At the interface $z = H_2$ we also have

$$w'(H_2) = \frac{dH_2}{dt} \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial z} \frac{dH_2}{dt} \frac{\partial w'}{\partial t} + \frac{\partial v'}{\partial y} \frac{\partial w'}{\partial t} = 0. \quad (8)$$

Substituting $w'(H_2) \approx w'(H_2)$ from (7) and $z'$ from (6a) into this equation we then find

$$\frac{1}{g(\rho_2 - \rho_1)} \left( \frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x} \right) (p_2 - p_1')$$

$$+ \frac{\partial H_2}{\partial y} + D \left[ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right] = 0. \quad (9)$$

where $D = H_3, D_1 = H_2 - H = H_1$.

For the stability problem on hand, we take the perturbation variables $u'$, $v'$ and $p'$ as represented by the products of their amplitudes and the wave factor $\exp[ik(x - ct)]$. Further, we use the horizontal scale of variation $L$ as the unit of $x$ and $y$, $U_0$ as the unit of $u'$, $v'$, $U_1$ and $c$, and $\rho U_0$ as the unit of $p'$, so that we have

$$u'(x, y, p') = U_0 \{u'(x, y), \rho U_0 \phi'(x, y)\} \times \exp[ik(x - ct)], \quad (10)$$

where the amplitude functions $u'(x, y)$ and $\phi'(x, y)$ are functions of $y$ only. Substituting these expressions in (3), (4) and (9) we then obtain the following equations for $u'(x, y)$ and $\phi'(x, y)$:

$$\sigma \mu \phi' - f_1 \phi' = -k \phi', \quad (11)$$

$$\sigma \phi' - f_1 \phi' = \phi'', \quad (12)$$

$$\lambda \phi'(x) - \phi'(x) + L c \phi'(x) + k u'(x) + v'(x) = 0, \quad (13)$$

where $\lambda, \sigma, \mu$ are defined below.
where the subscript \(y\) denotes differentiation and
\[
\begin{align*}
\sigma_{t} &= k(U_{1} - c), \quad \lambda_{1} = \rho U_{0} \frac{\partial}{\partial \rho_{0}} \left[ \frac{\partial p}{\partial \rho_{0}} \right] D_{1}, \quad L_{1} = f \lambda_{1}(U_{1} - U_{a1}), \quad f = \beta y \\
\beta &= \beta y L_{1}/U_{0}, \quad f' = f - U_{1} y \\
\text{Solving (11) and (12) for } u^{(1)} \text{ and } v^{(1)} \text{ in terms of } \phi^{(1)}, \text{ we find}
\begin{align}
\phi^{(1)} &= F_{1} \left[ k \sigma_{t} \phi^{(1)} - f' \phi_{y}^{(1)} \right], \\
v^{(1)} &= F_{1} \left[ k \sigma_{t} \phi^{(1)} - f' \phi_{y}^{(1)} \right].
\end{align}
\end{align*}
\]
where \(F_{1} = \frac{f' y}{\sigma_{t}}\). Substituting \(u^{(1)}\) and \(v^{(1)}\) from (15a,b) in (13) we then obtain
\[
\phi_{yy}^{(1)} + \left( L_{1} - \frac{F_{1} y}{F_{1}} \right) \phi_{y}^{(1)} - \left[ k^{2} + \frac{k}{\sigma_{t} \beta} + f' \left( L_{1} - \frac{F_{1} y}{F_{1}} \right) \right] \phi^{(1)} - \lambda_{1} F_{1} \left[ \phi^{(1)} - \phi^{(3-1)} \right] = 0. \tag{16}
\]

b. Two-level approximation of the continuous and compressible model

As has been mentioned before, Eqs. (11)–(16) can also be interpreted as the two-level approximation of the linearized primitive equations for the compressible and continuous model when the vertical shear term \(\psi U_{p}\) in the equation of motion in the \(x\) direction is neglected. When the \(\psi U_{p}\) term is included, the resulting equation differs from (16) somewhat but this difference is negligible when the Richardson number \(Ri = 1/(\lambda U_{p}^{2})\) is large. To demonstrate this statement, we first write down the corresponding equations of motion, the continuity and the heat equations by taking \(\xi = \rho/\Delta\rho\) as the vertical coordinate, where \(\Delta\rho\) is taken as equal to half of the surface pressure \(\rho_{0}\) so that \(\xi\) ranges from 0 to 2, and setting
\[
\begin{align*}
\frac{dp'}{dt'} &= \left[ i U_{0} \Delta \rho \psi(w/L) \right] \exp[ik(U_{1} - c)].
\end{align*}
\]
These equations then take the forms
\[
\begin{align}
\sigma_{x} u^{(1)} - f' \psi^{(1)} + U_{1} \psi^{(1)} &= -k \phi^{(1)}, \\
\sigma_{x} v^{(1)} - f \phi^{(1)} &= \psi_{y}^{(1)}, \\
k u^{(1)} + \psi_{y}^{(1)} + w^{(1)} &= 0, \\
k \phi^{(1)} - f U_{1} \phi^{(1)} + \frac{1}{\lambda^{*}_{0}} \psi^{(1)} &= 0,
\end{align}
\]
where \(\phi\) is the amplitude of the perturbation geopotential and \(\lambda^{*}_{0}\) the stability factor, which is defined in the continuous model by
\[
\begin{align*}
\lambda^{*}_{0} &= \frac{U_{0} \rho_{0} \phi_{0}}{(\Delta \rho)^{3} \partial \rho_{0} / \partial \rho}, \tag{17c}
\end{align*}
\]
where \(\theta_{0}\) is the potential temperature and \(\rho_{0}\) the density of the undisturbed state at the level \(\xi\). Substituting \(\omega^{(1)}\) from (17d) in (17a) and solving for \(u^{(1)}\) and \(\psi^{(1)}\) from the resulting equation and (17b) we then find
\[
\begin{align}
\phi^{(1)} &= \left( 1/f' \right) \left[ k \sigma_{t} \phi^{(1)} - f' \phi_{y}^{(1)} - \lambda_{1} \sigma_{t} U_{1} \psi^{(1)} \right], \\
v^{(1)} &= \left( 1/f' \right) \left[ k f \phi^{(1)} - \sigma_{t} \phi_{y}^{(1)} - \lambda_{1} \sigma_{t} f U_{1} \phi^{(1)} \right], \tag{18a}
\end{align}
\]
where
\[
\begin{align*}
\dot{\psi} &= f(1 - \lambda_{1} U_{1} \xi) - U_{1} y, \quad F_{1} = \dot{\psi} - \sigma^{2},
\end{align*}
\]
Under the normal conditions in the earth’s atmosphere, we have \(\lambda_{1} U_{1} \xi = Ri \xi < 1\) and hence \(\dot{\psi} \approx f'\), \(F_{1} = f\). Further, we also have the magnitude equality
\[
\sigma^{2} \phi^{(1)} \approx k \left[ U_{1} \phi^{(3)} - \phi^{(1)} \right].
\]
Therefore
\[
\begin{align*}
\sigma^{2} \phi^{(1)} &= \left( 1 \right) U_{1} \sigma \phi^{(1)} \approx k \left[ U_{1} \phi^{(3)} - \phi^{(1)} \right] \approx k \phi^{(1)}.
\end{align*}
\]
Thus the \(U_{1} \xi \psi^{(1)}\) term in (17a) and the \(U_{1} \xi \phi^{(1)}\) terms in (18a,b) can all be neglected so that these equations become identical with (11), (12) and (15a,b).

The assumption that the vertical variations of \(u\), \(v\) and \(\phi\) are approximated by their values at two levels can be interpreted as though they vary linearly with \(\xi\). Therefore, we have
\[
\phi^{(1)} = \phi^{(2)} - \phi^{(1)}, \quad U_{1} \xi = U_{2} - U_{1}. \tag{18c,d}
\]
According to the continuity equation (17c), when \(u^{(1)}\) and \(v^{(1)}\) are linear, \(w\) will be quadratic in \(\xi\). Thus, when \(w\) is required to vanish both at \(\xi = 0\) and \(\xi = 2\), \(w^{(1)}\) and \(w^{(0)}\) are given by
\[
\begin{align*}
w^{(1)} &= \frac{\xi_{1}(2 - \xi)}{2} \psi, \quad w^{(1)} = (-1)^{i+1} \psi, \quad \xi_{1} = \frac{3}{2}, \quad \xi_{2} = \frac{3}{2} \text{ refer to the 250 and 750 mb surfaces.}
\end{align*}
\]
Substituting these relations in (17c) and (17d) and eliminating \(\psi\) from the two resulting equations we then obtain the following equation which is identical with (13):
\[
\begin{align*}
\lambda \sigma_{t} \left[ \phi^{(1)} - \phi^{(3-1)} \right] + L_{1} \psi^{(1)} + k u^{(1)} + \psi_{y}^{(1)} = 0, \tag{19}
\end{align*}
\]
where
\[
\begin{align*}
\lambda_{0} = \left[ 1/(\xi_{1}(2 - \xi_{1})) \right] \lambda_{0}^{*}, \quad L_{1} = -\lambda_{1} f(U_{1} - U_{1-1}),
\end{align*}
\]
and \(\lambda_{0}^{*}\) is defined by (17e). Notice that by using the linear relation (18c) for \(\dot{\psi}\) and the quadratic relation (18e) for \(\psi\), we can apply the heat equation (19) directly at the level \(\xi_{1}\) instead of the mid-level \(\xi = 1\). Since Eq. (19) is identical with (13), \(\phi^{(1)}\) is also given by Eq. (16) in the two-level approximation of the continuous and compressible system when the \(U_{1} \xi\) term in (17a) is neglected, which is valid when \(\lambda^{*}_{0} U_{1}^{2}\) is much smaller than unity.

When the \(U_{1} \xi\) terms are retained in (18a,b), the equation for \(\phi^{(1)}\) is given by
\[
\begin{align*}
\phi_{yy}^{(1)} + \left( L_{1} - \frac{F_{1} y}{F_{1}} \right) \phi_{y}^{(1)} - \left[ k^{2} + \frac{1}{\sigma_{t} \beta} \left( L_{1} - \frac{F_{1} y}{F_{1}} \right) \right] \phi^{(1)} - \lambda_{1} F_{1} \left[ \psi^{(1)} - \phi^{(3-1)} \right]
+ f_{1} \left( U_{1} \phi^{(1)} + \left[ U_{1} \psi^{(1)} + U_{1} U_{1} k / \sigma_{t} \phi^{(1)} \right] \right) = 0. \tag{20}
\end{align*}
\]
Since we have \( \lambda_t U^2 < 1 \) under normal atmospheric condition, we shall take (16) as the two-level approximation of the governing equation for atmospheric motion.

c. The \( \psi^{(i)} \) equation

For the stability problem on hand, it is often more convenient to solve the \( \psi^{(i)} \) equation instead of the \( \phi^{(i)} \) equation. Therefore, we shall obtain the \( \psi^{(i)} \) equation from (15a,b) and (13) in this subsection. To this end we first substitute \( \psi^{(i)} \) from (15a) in (13) or (19) and then substitute \( \phi^{(i)} \) from (15b) in the resulting equation, whereby we obtain

\[
\psi^{(i)} + \frac{k(f - U_{iy})}{\sigma_i} + L_i \psi^{(i)} = \left(\frac{k^2}{\sigma_i} - \lambda_{i}\sigma_i\right)\phi^{(i)} + \lambda_{i}\sigma_i\phi^{(i-1)} \tag{21}
\]

We differentiate this equation with respect to \( y \), use Eq. (15b) to eliminate \( \phi^{(i)} \) and finally use Eq. (21) to eliminate \( \phi^{(i)} \) to arrive at:

\[
\psi^{(i)} + C_{1i}\psi^{(i)} + C_{2i}\psi^{(i)} - C_{3i}\psi^{(i-1)} - C_{4i}\phi^{(i-1)} = 0 \tag{22}
\]

where the coefficients \( C_{lm} \) are given by

\[
C_{1i} = 2UJ_1 U_{iy}\lambda_i \\
+ L_i (\Delta - \lambda_{i}UJ_i^2 - \lambda_{i}U_{iy}UJ_i) / \Delta \tag{22a}
\]

\[
C_{2i} = (UJ_i) \left[ \beta - U_{iy} - L_i (f - U_{iy}) - k^2 - \lambda_{i} \right] \\
+ \left( \frac{f - U_{iy}}{UJ_i} + L_i \right) (C_{1i} - L_i) + L_{iy} \tag{22b}
\]

\[
C_{3i} = 2UJ_2 U_{iy}\lambda_i - (1 + \lambda_i UJ_i / f) L_i / \Delta \tag{22c}
\]

\[
C_{4i} = C_{1i} \left( \frac{f - U_{iy}}{UJ_{3i}} + L_{3i} \right) - \lambda_{i} UJ_i \cdot F_{3i} - \lambda_{i} UJ_2 \tag{22d}
\]

\[
UJ_1 = U_{iy} - \epsilon, \quad \Delta = 1 - \lambda_{i} UJ_i^2 - \lambda_{i} UJ_2 \tag{22e,f}
\]

3. The energy relations

The local kinetic and potential energy densities \( K_{ei} \) and \( P_{ei} \) of the perturbations in the two-layer system are given by

\[
K_{ei} = \frac{1}{2} \left[ (u^{(i)} + v^{(i)}) \cdot (u^{(i)} + v^{(i)}) \right], \tag{23a}
\]

\[
P_{ei} = \frac{1}{2} \left[ (\phi^{(i)} \phi^{(i)\ast} - 2 (\phi^{(i)} \phi^{(i-1)\ast})) \chi_{ei} \right], \tag{23b}
\]

where the asterisk represents the complex conjugate and the subscript \( \text{Re} \) denotes the real part of the quantity in question. On multiplying the equations of motion (11) and (12) by \( u^{(i)} \) and \( v^{(i)} \), averaging over a wavelength and adding the results for the two layers, we then find

\[
2k \epsilon_i \mathcal{E} = \sum_{l=1}^{2} \left\{ U_{1y} MT_l + WP_l \right\}, \tag{24}
\]

where \( KE = K_{e1} + K_{e2} \), \( MT_l \) represents the north-south momentum transport by the disturbance, and \( WP_l \) stands for the work done by the pressure force, which are, respectively, given by

\[
MT_l = \left[ u^{(i)} \phi^{(i)\ast} \right]_{1m}, \tag{24a}
\]

\[
WP_l = \left[ (\psi^{(i)} \phi^{(i)\ast} - k u^{(i)} \phi^{(i)\ast}) \right]_{1m}, \tag{24b}
\]

where the subscript \( \text{Im} \) denotes the imaginary part of the quantity. Similarly, multiplying (13) by \( \phi^{(i)} \) and rearranging, we then obtain the potential energy equation

\[
2k \epsilon_i \mathcal{E} = k (U_{2y} - U_{1y}) \phi^{\ast} \tag{25}
\]

\[
+ \sum_{l=1}^{2} \left[ \frac{\partial}{\partial y} (\psi^{(i)} \phi^{(i)\ast}) \right]_{1m} - WP_l \lambda_{i}^{-1}, \tag{25a}
\]

where

\[
PE = P_{e1} + P_{e2} \tag{25a}
\]

\[
P^\ast = \phi_{1} \phi_{2} - \phi_{1} \phi_{2} \tag{25a}
\]

Notice that since \( WP_l \) occurs with opposite signs in (24) and (25), it represents the conversion from potential energy to kinetic energy. When these equations are integrated over the entire range of \( y \), we then obtain the changes of the various kinds of energies for the entire region under consideration.

4. The basic flows and boundary conditions

The basic current \( U \) in the tropics, which we shall define as the prominent mean current in a region whose horizontal dimension is much larger than the wavelength of the disturbance under consideration, can be classified roughly into two different types, \( \text{viz} \), a shear-zone type current consisting of two parallel and nearly uniform but different currents on its two sides and a jet type current. The shear-zone type current is often representative of the basic flow in the lower and mid-latitude sphere in the ITCZ in many parts of the globe, and the origin of this type of current can be attributed directly to the mean convergence toward the ITCZ and deep cumulus convection (see Kuo, 1973b). On the other hand, the jet type current can occur anywhere over the globe, and it can also be representative of the easterly current on the pole side of the ITCZ, as in northern Africa. For simplicity, we shall represent these two types of mean current by the analytic expressions

\[
U_1 = U_{10} \tanh(y - y_0) + 0.5 U^*_1, \tag{26a}
\]

\[
U_1 = U_{10} \sech^2(y - y_0) + 0.5 U^*_1, \tag{26b}
\]

where \( U_{10} \), \( U^*_1 \) and \( y_0 \) are constants. Here the first term represents the shear profile in the \( i \)th layer in the \( y \) direction, while the second term represents either a constant current or a constant shear in the vertical direction.

For the velocity profiles situated sufficiently far away from the equator, we demand the vanishing of \( \psi^{(i)} \) at
The general properties of the solutions of Eq. (29) and the boundary conditions \( \psi = 0 \) at \( y = a \) and \( y = b \) have been discussed by the author (Kuo, 1949). According to these analyses, instability of the barotropic flow occurs only when the gradient of the basic absolute vorticity, \( Z_y \beta - U_{yy} \), vanishes and changes its sign in the region \((a, b)\) and, when this condition is satisfied, the transitional disturbance has its phase speed equal to the basic current velocity \( U \), at the critical latitude \( \gamma_c \) where \( Z_y \) vanishes.

The stability of the hyperbolic-tangent profile \( U = \pm \tanh y \) for purely two-dimensional nondivergent flows and \( \psi = 0 \) has been investigated by many authors in the past. For example, Michalke (1964) and Betchov and Criminale (1967) have determined the unstable eigenvalues for the non-rotating system \( (\beta = 0) \), and Howard and Drazin (1964) and Lipps (1965) have discussed the influence of the Rossby parameter for the near marginal solutions while the author has determined the eigenvalues and the eigenfunctions for the entire range of instability \( 0 \leq \beta \leq 0.7698 \), when \( U_0 \) is taken as \( \frac{1}{2} \) of the total range of \( U \), with the results so obtained summarized in the semi-review paper by the author (Kuo, 1973a). This problem has also been treated by Dickinson and Clare (1973). Since the results we obtain from the two-levels primitive equations have to be compared with that given by the nondivergent barotropic model, we shall present the barotropic results obtained by the author earlier in some detail here.

As can be seen from Fig. 1 that, for the hyperbolic tangent basic flow profile \( U = \pm \tanh y \), the critical value of \( |\beta| \) below which unstable solution exists is \( \beta_c = (\frac{1}{2})^{1} = 0.7698 \) and, when \( |\beta| \) is smaller than \( \beta_c \), the critical latitude \( y = y_c \) is given by the cubic equation for \( z_c (= \tanh y_c) \):

\[
2z_c^3 - 2z_c + \beta = 0.
\]  

The roots of this equation are given by

\[
z_{cj} = (\frac{1}{2})^{1} \cos[((\pi + 2\pi j)/3)], \quad j = 1, 2, 3.
\]  

\[
\theta = \cos^{-1}(-2^{1/2} / 4) - \frac{1}{2} \pi \leq \theta \leq \frac{1}{2} \pi.
\]  

For \( \beta = 0 \), we have \( z_{c1} = -1, z_{c2} = 0, z_{c3} = 1 \), and therefore there exists three critical values of \( y_c \). Here, \( y_{c1} = -\infty \), \( y_{c2} = 0, y_{c3} = \infty \). On the other hand, for \( \beta > 0 \) only \( z_c \) and \( y_c \) lies within the range of \( |U| \), while \( |z_{c1}| \) becomes greater than 1 and hence must be excluded. Thus, for \( 0 < |\beta| < \beta_c \), there exist only two critical latitudes \( y_{c2} \) and \( y_{c3} \). Therefore, the neutral solutions corresponding to \( \epsilon_j = -z_{cj}, j = 2, 3 \) are given by

\[
\psi_j(y) = (1 - z)^{(1 + z_{cj})/2}(1 + z)^{(1 - z_{cj})/2},
\]  

where \( z = \tanh y \), while the wavenumber \( k_j \) of these neutral disturbances is given by

\[
k_j = (1 - z_{cj})^{1} = \text{sech} \ y_{cj}.
\]
Table 1. Some numerical values of $c_r$ and $c_i$ for barotropic $U = \tan y$ flow and various values of $k$ and $\beta$.

<table>
<thead>
<tr>
<th></th>
<th>$k$</th>
<th>$\beta=0$</th>
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This solution was obtained by Draizin (1958) for $\beta=0$ and by Lipps (1965) for $\beta \geq 0$ with the sign of $c_i$ reversed. It can be shown that the solution $(\psi_3,k_3,c_3)$ represents the short-wave cutoff of the amplifying waves and the solution $(\psi_3,k_3,c_3)$ represents the transition from the amplifying to the stable waves for $|\beta| > 0.17$, while for $|\beta| < 0.17$ the amplifying waves occupy the entire range $0 \leq k \leq k_{0}$. The solutions of Eq. (29) for the hyperbolic tangent velocity profile $U = -\tanh y$ and the radiation type boundary conditions are

$$
\psi \propto \begin{cases} 
\exp \left( \frac{k^2 - \beta}{1 - \epsilon} \right)^i y & \text{as } y \to -\infty \\
\exp \left( -\frac{k^2 + \beta}{1 + \epsilon} \right)^i y & \text{as } y \to \infty
\end{cases} \quad (32a)
$$

which correspond to $\psi(\mp \infty) = 0$, have been obtained by the author by a number of different integration schemes, including the very efficient reduction method developed in the Appendix and the Runge-Kutta method, for the whole range of instability $0 \leq \beta \leq 0.7698$, $k_{0} \leq k \leq k_{\infty}$, and the results given by the different methods are in good agreement except in the range $0 \leq \beta \leq 0.15$, $0 \leq k \leq k_{\infty}$, where the value of $c_i$, the imaginary part of $c_i$, obtained from Eq. (29) is somewhat lower than that given by the corresponding equation for the function $\phi = \psi/\psi'$. These results are represented in Figs. 2-4. Some numerical values of $c_r$ and $c_i$ for this problem are given in Table 1.

From Fig. 2a we see that the dimensionless relative phase velocity $c_r$ is negative for all $k$ and $\beta$, which means that the waves propagate toward west relative to the mean current $U$. Further, the absolute value of $c_r$ increases with $\beta$ for a fixed $k$ and decreases with $k$ for a fixed $\beta$ in such a way that the value of $-c_r$ is less than $1$ in the whole $(k,\beta)$ domain of the unstable disturbances, but has a maximum value of about $0.9$ near the lower neutral solution $(\psi_3,k_3,c_3)$ at $k=0.35$. From Fig. 2b we see that for $\beta = 0$, the maximum $c_i(=1)$ occurs at $k=0$, but it shifts toward a larger $k$ as $\beta$ increases. For a fixed $k$, $c_i$ decreases with increasing $\beta$, showing the stabilizing influence of $\beta$. One interesting and unexpected result of the integration is that it showed that for $\beta < 0.170$, the whole region $0 \leq k \leq k_{\infty}$ is unstable and hence the lower neutral solution given by (31) is not a transition from stability to instability. This result is in agreement with the conclusion reached by Howard and Drazin (1964) that the disturbances close to $k=0$ are unstable for sufficiently small $\beta$. Notice from Figs. 2a and 2b that, in the region $0 \leq k \leq k_{\infty} \approx 0.30$, $0 \leq \beta \leq 0.17$, $c_r$ and $c_i$ vary only slightly with $k$ but with $c_i$ decreasing toward zero as $\beta$ approaches $0.17$. Notice also the rapid change of the slope of the $c_r$ and $c_i$ profiles across the boundary between this region and the region $k > k_{\infty}$, indicating a change of the nature of the solution. This feature is also revealed by the results obtained by Dickinson and Clare (1973), who also found that two different modes are present in this region. The upper limit of $\beta$ obtained by these authors appears also to be close to the value $0.17$ as their $\beta$ is defined in terms of the total range of shear $U_m$ and hence is equivalent to one-half the $\beta$-parameter used here, even though they did not locate this upper limit of $\beta$ exactly for all $k$.

The dimensionless growth rate $\delta = k c_i$ is shown in
Fig. 2c as a function of $\beta$ and $k$. It is seen that $\delta$ has a maximum at $k = k_c$ and $k_c$ increases gradually with increasing $\beta$, from $k_c = 0.4449$ at $\beta = 0$ to $k_c = 0.817$ at $\beta = 0.7698$, while the maximum value of $\delta$ decreases from 0.188 to zero. Thus increasing $\beta$ causes the most favored disturbance to shift to a larger $k$. However, if the increase in $\beta$ is due to an increase in the value of the half-width $L$, then the wavelength of the most favored disturbance is increased.

From the observed wind distributions in the tropics we find that $U_0$, the velocity difference on the two sides of the shear zone, is usually about 10 m s$^{-1}$ and $L$ is of the order of 250 km, corresponding to $\beta \approx 0.14$, $k_m = 0.50$, $\omega_m = -0.24$, $\delta_{\text{max}} = 0.175$. Thus the wavelength of the most unstable disturbance is about 3000 km, and this disturbance moves westward relative to the mean current at about 2.4 m s$^{-1}$ and its amplitude will magnify an $\epsilon$-fold in about 1.6 days. These results are in rough agreement with the available observations on the properties of the disturbances in the tropical region (see Wallace et al., 1970), showing that barotropic instability is most likely the primary cause of these disturbances.

The variations of the amplitude $|\psi|$ and the phase $\alpha$ of $\psi$ of the nearly most unstable disturbances with $y$ and $\beta$ are illustrated in Fig. 3. It is seen that, for $\beta = 0$, the amplitude is symmetric about the center of the shear zone $y = 0$ with two maxima at $y = \pm 0.65$ and a relative minimum at $y = 0$, and the phase angle $\alpha$ is antisymmetric about $y = 0$. On the other hand, for $|\beta| > 0$, the central position of the disturbance is shifted toward the equator and the magnitude of the maximum of $|\psi|$ on the equator side (i.e., $y < 0$) is greatly increased while that on the pole side is reduced; and for $\beta > 0.2$, this latter maximum can no longer be recognized. However, for the unstable disturbances in the region $0 \leq \beta \leq 0.170$, $0 \leq k \leq k_{\text{cc}}$, the two-maxima structure is retained, but the value of the maximum on the equator side of $y = 0$
is much larger than that on the pole side, and the amplitudes decay only slowly with $y$ beyond the equator side maximum.

The average momentum transport $\overline{u'v'}$ produced by the pure barotropic disturbance is given by (see Kuo, 1949):

$$\overline{u'v'} = \frac{k}{2} \left( \psi_x \frac{d\psi_y}{dy} - \psi_y \frac{d\psi_x}{dy} \right),$$

$$= -\frac{k}{2} |\psi|^2 \frac{d\alpha}{dy}, \quad (33)$$

where $\alpha$ is the phase angle of the trough line. From Fig. 3 we see that $d\alpha/dy$ is mainly positive for the $U = -\tanh y$ profile under consideration except for $y > 1.5, \beta > 0$, and the maximum momentum transports by the most unstable waves occur close to $y = 0$. The rate of change of the mean zonal current produced by the disturbance is given by $-d(\overline{u'v'})/dy$.

The variations of the quantities $\overline{u'v'}$ and $d(\overline{u'v'})/dy$ with $y$ for $\beta = 0.20, k = 0.50$ are illustrated in Fig. 4a. It is seen that this momentum transport tends to reduce the shear in the shear zone, and convert part of the kinetic energy of the mean flow into the perturbation kinetic energy. Consequently, the half-width $L$ of the mean flow under disturbed condition will be much larger than that in the undisturbed condition. The total streamfunction

$$\psi^*(x,y) = \psi'(x,y) - A \int_0^y \tanh \eta d\eta$$

of this unstable wave with $|\psi|_{\text{max}} = 1, A = 5$ is shown in Fig. 4b. Here the coordinates cover one wavelength in $x$ and about 5.3 times the half-width $L$ of the shear zone in $y$. It is seen that the troughs and ridges tilt in the southwest-northeast direction, indicating that a northward transport of momentum is produced by the disturbance and the kinetic energy of the mean flow is being converted into the kinetic energy of the perturba-

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**Fig. 4.** (a) Momentum transport $\overline{u'v'}$ and its gradient $d(\overline{u'v'})/dy$; (b) composite flow pattern of the most unstable barotropic non-divergent disturbance for $\beta = 0.2, k = 0.5$.

**Fig. 5.** (a) Eigenvalues of divergent and ageostrophic barotropic flow, top left; (b) corresponding eigenfunctions of most unstable mode for $\beta = 0.022, k = 0.5$, right; (c) energy conversions, bottom left.
tion. Notice that, because of the large positive vorticity in the basic flow, only closed cyclonic vortices form in the shear zone.

We mention that the basic current \( U_0 \) on the pole side of the ITCZ is usually larger than the current \( U_e \) on the equator side, which tends to make the streamlines more crowded on the pole side and less crowded or even open on the equator side.

2) THREE-DIMENSIONAL AND AGEOSTROPHIC DISTURBANCES IN BAROTROPIC BASIC FLOW

When the basic flow is barotropic but the perturbation possesses horizontal divergence and is ageostrophic and three-dimensional, we find that the disturbances in the two layers are nearly 180° out of phase but their amplitudes are almost the same. Further, these disturbances are influenced very little by the change of the mean stability of the atmosphere, in contrast with that under baroclinic conditions. The eigenvalues \( \omega \) and \( k_e \) for the case with \( U_0 = 10 \text{ m s}^{-1}, L = 1 \times 10^6 \text{ m} \) are presented in the top left of Fig. 5a and the corresponding eigenfunctions and the divergence field are represented in Fig. 5b, right, while the energy conversions are represented in Fig. 5c, bottom left. Notice that work is being done by the pressure force for this case and this work is contributing to both the kinetic and the potential energies of the perturbation.

b. Combined barotropic and baroclinic effects of the hyperbolic-tangent flow in the two-layer model

The basic flow for this problem is represented by (26a) with \( U_{10} \neq U_{20} \) or \( U_1^* \neq 0, U_2^* \neq 0 \) or both, while the perturbations are represented by Eq. (16) and the boundary conditions (27a,b) or (28a,b). The influence of the stable stratification is investigated by assigning five different values to the Richardson number \( R_i \), ranging from 7.2 to 72 for the upper layer and about a third of these values for the lower layer. The eigenvalue problem represented by these systems of equations are solved mainly by the reduction method described in the Appendix, with some of the results checked by a more elaborate Runge-Kutta shooting method. It is found that the results obtained by these two methods are in good agreement. Here we shall present the results of these calculations only, leaving the description of the very efficient reduction method of solution to the Appendix.

1) QUASI-BAROTROPIC FLOW, WITH ZERO MEAN VERTICAL SHEAR BUT REVERSED HORIZONTAL SHEARS IN THE TWO LAYERS, CHARACTERIZED BY \( U_1^* = U_2^* = 0, U_{10} = 1, U_{20} = -1 \)

The eigenvalues and eigenfunctions in the upper layer are nearly the same as those for the pure barotropic case in Figs. 5a and 5b, while the amplitudes of the eigenfunctions in the lower layer vary from 1.1 to 0.8 times that in the upper layer as \( R_i \) varies from 7.2 to 72, and \( U_2 \) is nearly 180° out of phase with \( U_1 \) and \( U_2 \) is nearly in phase with \( U_1 \).

2) TRULY BAROCLINIC SHEARING FLOW, WITH BOTH MEAN VERTICAL SHEAR AND REVERSED HORIZONTAL SHEARS IN THE TWO LAYERS, CHARACTERIZED BY \( U_1^* = -U_2^* = U_{10} = -U_{20} = 1 \)

We use this case to illustrate the combined barotropic and baroclinic influences of the mean flow. The results show that, when the shear zone is centered less than 15° from the equator, the mean vertical shear \( U^* \) has little effect on the degree of instability of the system. However, it has a very important influence on the vertical

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**Fig. 6. Eigenvalues of ageostrophic and baroclinic disturbances: (a) westward propagating disturbances; (b) eastward propagating disturbances.**
structure of the disturbances in that it tends to separate the disturbances into two distinct groups, namely, an eastward propagating group confined mainly in the upper layer and a westward propagating group confined mainly in the lower layer. The variations of the growth rate $k \epsilon$, and the phase speed $\epsilon$, with the wavenumber $k$ and the upper layer Richardson number $Ri$ for the westward propagating group are represented in Fig. 6a, while that for the eastward propagating group are represented in Fig. 6b. It is seen that, in the presence of the mean vertical shear, the growth rate is no longer independent of the static stability parameter $Ri$ but increases with $Ri$ when $Ri$ is smaller than 40, and this influence is especially large for the westward propagating disturbances. Thus, even though the growth rates of both the eastward and the westward propagating disturbances in the presence of $U^*$ are nearly the same as that of the purely barotropic flow when $Ri$ is larger than 40, the westward propagating waves are much more stable than the eastward moving waves when $Ri$ is lower than 10. This situation is reversed when the sign of the vertical shear is reversed, namely, the westward propagating disturbances are concentrated in the upper layer while the eastward propagating disturbances are concentrated in the lower layer, and the former are more unstable than the latter when $Ri$ is smaller than 40. Notice also from Fig. 6 that these unstable disturbances move slowly toward west relative to the mean velocities $\pm 0.5 \ U^*$ of their respective layer of confinement.

The eigenfunctions of the most unstable eastward propagating disturbance in the two layers for $k=0.5$, 

![Fig. 7. Eigenfunctions of most unstable eastward propagating ageostrophic disturbances for $\beta=0.022$, $k=0.5$, $U^*=1$: (a) $Ri=72$, (b) $Ri=7.2$.](image-url)
Fig. 8. Growth rate $k c_i$ of unstable disturbances in shear flow centered at $20^\circ$; 1, baroclinic; 2, quasi-barotropic, with $U_0 = 0$.

Ri = 72 and Ri = 7.2 obtained under the condition $v_1(2y_0) = v_2(2y_0) = 0$ are represented in Figs. 7a and 7b, respectively. It is seen that the eigenfunctions in the upper layer are very close to that of the barotropic case in Fig. 5 for both values of Ri, while the disturbance in the lower layer is very weak under the more stable condition Ri = 72, but is comparable with that in the upper layer under the less stable condition Ri = 7.2, especially for the eastward propagating waves and on the pole side of the shear zone, as can be seen from Fig. 7b. On the other hand, for the westward propagating disturbances, more perturbation energy is contained in the lower layer than in the upper layer, especially under a more stable condition such as Ri $>10$.

We point out that, even though the growth rates of the disturbances in the baroclinic case are almost the same as that for barotropic flow under normal stratification Ri $>50$, there is a fundamental difference between the disturbances in these two cases, namely, the work done by the pressure force in the presence of mean vertical shear is essentially in the same direction as the transformation of the kinetic energy of the mean flow. Consequently, both of these two processes contribute to the production of the eddy kinetic energy for the baroclinic flow. However, relatively little eddy potential energy is produced under stable conditions since the perturbation is essentially confined in one layer. On the other hand, a significant amount of eddy potential energy is produced under less stable stratification, especially on the pole side of the shear zone.

The influences of varying the value of the mean vertical shear and relative values of $U_{10}$ and $U_{30}$ have also been investigated. It is found that reducing the value of $U_{10}^*$ tends to make the solutions approach the barotropic solution, while reducing $U_{10}$ or $U_{30}$ tends to make the disturbance concentrate in one layer, as is to be expected.

c. Influences of baroclinicity and position of shear zone

The calculations presented above used $U_0 = 10$ m s$^{-1}$, $L = N \times 10^8$ m and $N = 1$, $y_0 = 5$, which corresponds to a shear zone centered at $5^\circ$ and $\beta = 0.022$. Calculations have also been made for $N = 2, 3$ and $4$, $\beta = 0.022 N^2$ and $y_0 = 2, 3$ to examine the influences of the position of the shear zone center and the value of $\beta$ on the stability. It is found that, so long as $y_0$ is less than $15^\circ$ from the equator, the stability is influenced by $L$ only through the increase of $\beta$ with $L^2$, which tends to make the flow more stable and to shift the most unstable mode toward a higher wave number with increasing $L$, just as for barotropic nondivergent flow. The influence of the mean vertical shear when $y_0$ is less than $15^\circ$ is merely to make the disturbance essentially confined within one layer. On the other hand, when $y_0$ is beyond $20^\circ$, a baroclinically unstable mode also comes into existence, whose energy is derived mainly from the conversion of the potential energy of the mean flow beyond $y_0$. The variation of the growth rates of this disturbance and that of the essentially barotropic disturbance with $k$ are illustrated in Fig. 8 for $U_{10} = U_{30} = U_0 = 1$, Ri = 20. It is seen that baroclinic instability is becoming more important at higher latitude.

d. Influence of latent heat released by deep and shallow cumulus

It is well known that the lowest 500–700 m of the atmosphere in the tropics is usually well mixed by the convection created by the static instability of the surface layer, and shallow cumulus are also usually present under all weather conditions in the tropics. As has been stressed by the author in two previous papers (Kuo, 1974; 1975a), under these circumstances it is rather natural to expect that additional latent heat will be released by the large-scale disturbance in proportion to the boundary layer suction velocity $w_b$ created by the perturbation. Thus a latent heat term $-RQ_l/c_p T_l$ is added to the right-hand side of the heat equation (17d) and consequently a term

$$(-1)\lambda_t(F_i Q_l/c_p T_l)$$

is also added to the right side of Eq. (16). According to the author's (1975b) analysis, the boundary layer suction $w_b$ and the additional latent heat $Q_l$ released in Fig. 9. Eigenvalues of equatorial disturbances in baroclinic hyperbolic-tangent shear flow under CISK influence.
the equatorial waves in which the frequency $\omega$ becomes equal to $\pm f$ at the critical latitude can be taken as given by

$$w_b = \left( \frac{\nu}{2f_0} \right) \left[ \frac{(D+\zeta)}{(1+i)(f+\omega)^4} + \frac{(D-\zeta)}{(1-i)(f-\omega)^4} \right], \quad (34a)$$

$$Q_1 = w_b \eta_1, \quad (34b)$$

where $D = \nu^{(2)} + k \nu^{(3)}$ is the divergence and $\zeta = -\left[ \nu^{(2)} + k \nu^{(3)} \right]$ is the vorticity at the lower level 2, and the minus and plus signs in the second term of (34a) are for $f > \omega$ and $f < \omega$, respectively, while $\eta_1$ and $\eta_2$ are two constants with their sum equal to the latent heat released by a unit suction velocity. When these relations for $Q_1$ are used in (16), we find two rather complicated equations in $\phi^{(1)}$ and $\phi^{(2)}$ and hence they will not be given here. But some calculations have been made from these equations and the variations of the phase velocity $c_\sigma$ and the growth rate $k \epsilon_c$ with $k$ for $U_{10} = U_{20} = U^* = 1$ and two different values of $\text{Re}$ under this CISK effect with a neutral surface layer are illustrated in Fig. 9. It is seen that the growth rate of the most unstable disturbance under CISK is almost double that when no latent heat is involved and the short waves ($k > 1.0$) are also made unstable, but its distribution with $k$ is still determined by the shear instability. However, when the surface layer \( 0 < z < z_s \) \((\sim 1 \text{ km})\) is stable and the lower boundary condition \( w = w_b \) is applied at \( z = z_b \) with an adiabatically cooled temperature, then the influence of the latent heat is almost cancelled by the additional adiabatic cooling associated with \( w_b \). These results indicate that the CISK effect is very sensitive to the conditions in the environment. Further, it is also sensitive to the boundary layer formula for \( w_b \). It is also found that the influence of the added latent heat is slight for the slowly eastward moving waves concentrated in the upper layer under a positive $U^*$. 

6. Disturbances created by instability of a jet current

As has been mentioned in Section 4, the basic flow on the north side of the ITCZ over Africa often takes the form of an easterly jet instead of a nearly uniform current, and such jets also occur in the equatorial region. Since both the barotropic and the baroclinic jets can be represented adequately by Eq. (26b), we shall investigate the instability of such a jet in this section.

a. Two-dimensional nondivergent disturbance on barotropic jet

The stability of the barotropic jet characterized by \( U_{10} = U_{20}, U^* = 0 \) for two-dimensional and nondivergent flows has been investigated by the author and the results have been reported briefly in the review paper cited before (Kuo, 1973a). To facilitate our discussions on the ageostrophic and baroclinic effects on this problem, we shall summarize some of these results here again. As has been mentioned already in the preceding section, the two-dimensional and nondivergent disturbance is represented by Eq. (29) and the boundary conditions (32a,b) or the second set of (28a) and (28b). These solutions are independent of $\nu_{0\theta}$ and the necessary requirement for instability is the change of the sign of the absolute vorticity gradient $(\beta - U_{yy})$ of the basic flow. From the variation of $U_{yy}$ in Fig. 10 it is readily seen that the range of instability of the parameter $\beta$ is $-2 < \beta < 0$, where negative values of $\beta$ correspond to negative $U$. The transition from stability to instability for $\beta > 0$ has been found by Lippa (1962) and is represented by the neutral solution

$$\psi = \psi_\sigma = \text{sech}^2 y, \quad k^2 = k^2 = 2\left[ 1 \pm \left( 1 - \frac{3\beta}{2} \right)^{1/2} \right],$$

$$c^2_\sigma = k^2 / 6. \quad (35)$$

This same solution but only with the positive sign before the radical in $k^2$ also represents the upper transition from stability to instability for negative $\beta$.

The variations of $\epsilon_c, c_\sigma$ and the growth rate $k \epsilon_c$ of the unstable nondivergent disturbances with the wave-number $k$ and the dimensionless Rossby parameter $\beta$ are represented in Figs. 10a–10c. Notice that the easterly jet is made more unstable by $\beta$ within the range $-0.84 < \beta < 0$, while the westerly jet is made more stable by $\beta$, and the maximum growth rate is at $\beta = -0.4$. Notice also that the phase velocity $c_\sigma$ of the unstable disturbance is always within the range of the basic current and it increases with $k$ for positive $\beta$ and decreases with $k$ for negative $\beta$. Notice also that $c_\sigma$ has a secondary maximum at $k = 0$ and a minimum between $k = 0$ and the main maximum for $-0.8 < \beta < -0.2$.

b. Stability of equatorial jet under symmetry condition at the equator

For the more general disturbances governed by Eq. (16), the degree of the stability of the jet depends not only on the position of the jet but also on the side conditions imposed, especially when the jet is situated close to the equator. To illustrate this effect, we first consider the jet centered at the equator and impose the symmetry condition $\nu^{(2)}(0) = 0$, which corresponds to $\phi^{(1)}(0) = 0$ according to Eq. (21). We shall take $U_1 = U_2 = -1$ for the barotropic easterly jet and $U_{10} = 1, U_{20} = -1$ for the baroclinic jet, and use $U^* = 0, \beta = -1$. The solutions for both cases. Further, for the barotropic jet we shall restrict this study to either the purely barotropic and nondivergent in-phase solution characterized by $\nu^{(2)} = 0$ or to the purely out-of-phase divergent solution characterized by $\text{Re} \nu^{(2)} = -\text{Re} \nu^{(1)}$. In the former case the two governing equations (22) reduce to a single equation identical to Eq. (29) with $\nu = k \psi$, while for the latter case Eq. (22) also reduces to a single equation, viz.,

$$\nu_{yy} + (\lambda_1 + \lambda_2) \nu \psi = -k^2 - (U - c_\sigma)^{-1} (\beta - U_{yy}) - (\lambda_1 + \lambda_2) c_{\sigma} \nu = 0, \quad (36)$$
where \( c_1 \) and \( c_4 \) are given by (22a) and (22d) with \( L_1=0 \). These equations are solved by the method described in the Appendix and the eigenvalues are obtained from the simple formula (A6). On the other hand, for the baroclinic case the coupled equations (16) or (22) must be solved in their entirety.

The values of the phase velocity \( c_\phi \) and the growth rate \( k_c \); for these cases are illustrated in Fig. 11 as functions of the wavenumber \( k \). Here the curve 1a represents the barotropic and nondivergent solution obtained under the asymptotic northern boundary condition \( V=V_N \exp[a(y_N-y)] \) for \( y \geq y_N \), and the curve 1b is given by the condition \( V(y_N)=0 \), while the curves 2 and 3 are for the divergent barotropic and the baroclinic flows under the symmetry condition at the equator. Observe that the long waves are also unstable under the asymptotic condition at \( y_N \) and the growth rate represented by 1a has a secondary maximum at \( k=0.85 \). From the eigenfunctions of these long-wave solutions we find that they are higher mode perturbations and extend far beyond the limit \( y_N=4 \) used in the integration, which are permitted by the asymptotic boundary condition for barotropic flow. Notice that the growth rate of these disturbances is comparable to that of the barotropic and nondivergent solutions in Fig. 10, but the maximum growth rate becomes larger and is shifted toward a lower \( k \) by the divergence and the ageostrophy. Notice also that the influence of the baroclinity on the growth rate is relatively slight, except that it tends to make the disturbance more concentrated in the layer of maximum easterly flow.

From the eigenfunctions in Fig. 11b we see that, under the symmetry condition \( \phi^{(1)}=0 \) at the equator, \( v^{(1)} \) has a minor maximum at the equator and a larger maximum away from the equator both for the barotropic and for the baroclinic flows in the lower layer, while the maximum of \( v_1 \) in the upper layer is at the equator for the baroclinic flow.
c. Influences of the position of the jet and the asymmetry at the equator on the instability of the easterly jet

Since the Coriolis parameter $f$ is involved in the general equations (16) and (22), we expect that the degree of instability of the jet will depend on the position of the jet center and therefore have assigned six different values to $\gamma_0$ to investigate this effect, ranging from $0^\circ$ to $15^\circ$. Further, we expect the flow to contain a quasi-geostrophic component such that the geopotential $\phi$ will be proportional to the streamfunction $\psi$, and hence must not use the condition $\phi^{(0)}=0$ at the equator. Instead, we shall impose the condition (27b), viz., $\phi^{(0)}=-a\phi^{(i)}$, both at $y=y_0+y_00$ and at $y=y_0-y_00$, and use $\gamma_0=5L$, where $L$ is the scale length of the jet profile, which is taken as equal to $3\times10^6$ m or $3^\circ$ latitude. Further, we take $U_0=10$ m s$^{-1}$ as the maximum velocity of the jet, $\beta'=2.2\times10^{-11}$ m$^{-1}$ s$^{-1}$ as the Rossby parameter $d\beta'/dy$, and $R_1=3R_2$. We then have $\beta=0.20$. Here we also define the barotropic easterly jet by $U_{10}=U_{30}=-1$. On the other hand, we shall define the baroclinic jet by $2U_{10}=U_{30}=-1$ so that the basic current will be easterly at both levels. It is found that for these basic currents there exist two different sets of disturbances, one is more unstable and the second set is less unstable.

The variations of the growth rates $kc_i$ of the high $c_i$ disturbances for the barotropic and baroclinic flows with the wavenumber $k$ and the latitude of the jet center are illustrated Figs. 12a and 12b, respectively. In Fig. 12a, the dashed curves on the top are for the in-phase barotropic solutions while the continuous curves are for the out-of-phase barotropic solutions. It is seen that $c_i$ increases with $R_1$ for both the out-of-phase barotropic solutions and for the baroclinic solutions. It is also seen that for the same jet profile, the maximum growth rate varies with the latitudinal position of the jet and the absolute maximum occurs between $\gamma_0=6^\circ$ and $\gamma_0=9^\circ$. Further, the wavenumber $k_m$ of the most unstable disturbance also increases with the distance $\gamma_0$ of the jet center from the equator. These variations of $k_{cm}$ and $k_m$ of the high $c_i$ disturbances with $\gamma_0$ are demonstrated more clearly by the curves $H$ in Fig. 14.

The variations of the phase speed $c_p$ of these disturbances with $k$ and $\gamma_0$ are illustrated in Fig. 12c. These results indicate that the very long waves and the shorter waves are of different characters.

The structures of the nearly most unstable barotropic and baroclinic disturbances are illustrated by the variations of the eigenfunctions $\phi$, $u$, and $v$ of these disturbances with the distance $(y-\gamma_0)$ from the jet center in Fig. 12d. A very prominent feature of these disturbances is that the maximum amplitude of the disturbance always occurs on the pole side of the jet, especially for the velocities $v$ and $u$. This feature of the disturbance seems to result from the influence of the horizontal shear $U_y$, which reduces the value of the quantity $F_1$ in
Fig. 12. Variations of the characteristics of high $c_i$ disturbances in easterly jet with the position of the jet $y_0$ and with Richardson number $R_i$.

a. Growth rate of disturbances in barotropic flow, $U_{10} = U_{20} = -1$. Dashed curves on top, in-phase solutions; continuous curves, out-of-phase solutions for various values of $R_i$, in the upper layer.

(5a,b) on the pole side of the easterly jet. It should be mentioned that for the baroclinic flow the perturbation is concentrated at the lower level for $R_i > 10$.

On taking the maximum value of $kc_i$ in Fig. 12a as 0.5 and using $U_0 = 10$ m s$^{-1}$, $L = 3 \times 10^5$ m we find that the $e$-folding time of the most unstable disturbance is about 16.6 h.

As has been mentioned before, for the easterly jet there also exist westward propagating disturbances whose growth rates are significantly lower than that of the high $c_i$ set discussed above. The variations of $kc_i$ and $c_i$ of these disturbances with $k$ and $y_0$ for barotropic flow are illustrated in Fig. 13, and those for the baroclinic flow are of comparable magnitude. It is seen that the growth rate of the most unstable disturbance of this set has its maximum around $y_0 = 6^\circ$, while the corresponding wavenumber also increases with $y_0$. These latitudinal dependencies of $kc_{im}$ and $k_m$ are

b. Growth rate of disturbances in baroclinic flow, $2U_{10} = U_{20} = -1$. 
illustrated by the L curves in Fig. 14. For the barotropic flow at the equator, the structure of the most unstable disturbance of this set is similar to that given in Fig. 11b, but for the baroclinic case and barotropic flow with $\gamma_0 \geq 3^\circ$, the maximum amplitude of the disturbance also occurs on the pole side of the jet.

d. Variations of the amplitudes and phases (a) of the eigenfunctions $\phi$ and $\psi$ of the nearly most unstable disturbances with the distance $(y - \gamma_0)$ from the jet center in units of $L = 3^\circ$ latitude. (a) barotropic, $\gamma_0 = 0$, $k = 1.1$; (b) barotropic, $\gamma_0 = 9^\circ$, $k = 1.1$; (c) baroclinic $\nu = 0$, $k = 0.7$; (d) baroclinic, $\gamma_0 = 9^\circ$, $k = 1.3$.

7. Summary and further discussions

To examine whether the asymmetry of the disturbances can actually be attributed to the influence of $U_\nu$ in the factor $F_t$, we have also considered the barotropic westerly jet defined by $U_{10} = U_{30} = 1$ and the baroclinic jet $U_{10} = -U_{30} = 1$. The variations of the growth rate $\kappa\xi$ and the phase speed $c_p$ of these disturbances for $\gamma_0 = 0$ and $\gamma_0 = 6^\circ$ are illustrated in Fig. 15.1. It is seen that the growth rates of these disturbances are close to that in Figs. 10 and 11 except for the long waves for $\gamma_0 = 0$. Notice also that the growth rate is larger for $\gamma_0 = 6^\circ$ than for $\gamma_0 = 0$. The eigenfunctions of the most unstable in-phase barotropic disturbances with $k = 0.9$ for $\gamma_0 = 0$ and $\gamma_0 = 6^\circ$ are illustrated on the left side of Fig. 15.2, while that of the out-of-phase barotropic and baroclinic disturbances for $\gamma_0 = 0$ are represented on the right side. It is seen that, for the major most unstable disturbance $k = 0.9$, the maximum amplitude of the disturbance lies on the equator side of the jet, that is, on the side of positive $U_\nu$ of the jet.
lower layer, thereby permitting two separate systems to exist simultaneously. Since this effect increases with the stability of the stratification, the growth rate of the unstable mode also increases. When the basic flow is of the shear zone type, such as that which occurs in the ITCZ region which we have taken as a hyperbolic-tangent velocity profile, it is found that pure barotropic instability extends to $k = 0$ for $\beta \leq 0.17$. The degree of instability is not altered significantly when the horizontal shears in the two layers are of opposite signs, nor when a constant mean vertical shear is present, except for the effect on the vertical separation mentioned above. On the other hand, when the surface layer is thoroughly mixed and latent heat is released by the deep cumulus in the system through boundary layer convergence, the growth rate of the most unstable disturbance in the shear zone is greatly increased by the CISK effect. The $e$-folding time of such disturbances can be less than 1 day.

The stability of the sech²($y - y_0$) jet profile centered at different latitudes has also been investigated. It is found that the $\beta$-effect stabilizes the westerly jet and destabilizes the easterly jet. For the easterly jet and for three-dimensional ageostrophic disturbances, there exist two different sets of perturbations, one being more unstable than the other. One of the most interesting results we found for this case is that the growth rate of the most unstable disturbance has its maximum

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**Fig. 13.** Growth rate $k^c$ (full curves) and the phase velocity $c^r$ (dashed curves) of low $c^r$ disturbances in barotropic easterly flow $U_{10} = U_{20} = -1$. Scale of $-c^r$ on right margin.

**Fig. 14.** Variations of the maximum growth rate $k_{c_{lm}}$ and the corresponding wavenumber $k_m$ of the high (H) and low (L) $c^r$ disturbances in an easterly jet with the position $y^o$ of the jet center: (a) left, barotropic flow; (b) right, baroclinic flow.

**Fig. 15.1.** Growth rate $k^c$ (top) and phase velocity $c^r$ (bottom) of disturbances in a westerly jet: (a) $y_0 = 0$. Curves 1a and 2a: main short-wavelength disturbances in barotropic ($U_{10} = U_{20} = 1$) and baroclinic ($U_{10} = -U_{20} = 1$) jets. (b) $y_0 = 6^\circ$. Curves 1b and 2b: long-wavelength disturbances in barotropic and baroclinic jets.
when the jet center is located between $6^\circ$ and $10^\circ$ from the equator, as is demonstrated very clearly in Fig. 14. This result is of paramount importance for the mean position of the ITCZ on account of the feedback effect of the latent heat released by the disturbances. If we take the maximum velocity of the jet as $U_0 = 10 \text{ m s}^{-1}$ and the scale length $L$ of the jet as $3 \times 10^6 \text{ m}$, and use the maximum value of $k_c = 0.5$ given in Figs. 12a and 12b, we then find that the e-folding time of the most unstable disturbance is only about 16.6 h.

Another important conclusion is that the maximum intensity of the disturbance occurs on that side of the jet where the absolute vorticity $(f - U_0)$ is small.

Many of these results appear to be unexpected and surprising, such as the increase of the growth rate of the baroclinic disturbance with increasing static stability and the existence of a maximum growth rate with regard to the latitude of the jet center for the jet-type mean flow. The physical reason for the first effect can be attributed to the predominance of the barotropic effect itself, which requires the uncoupling of the upper and the lower systems in baroclinic flow to realize the barotropic instability fully at the level of maximum intensity. This is possible only under more stable conditions, since under nearly neutral stratifications the disturbance will extend to all levels.

The lack of significance of the vertical shear for the growth rate of large-scale disturbances in tropics can also be seen directly from the well-known result obtained from the two-level quasi-geostrophic $\beta$-plane, $f = f_0$ model for baroclinic waves. According to this model [see Eq. (8.18a,b) in Kuo (1973a), and other references], the minimum vertical shear $|\Delta U|_{\text{min}} = \left| U_\text{top} - U_\text{bottom} \right| = 2 |U*|_{\text{min}}$ required for baroclinic instability is given by

$$\Delta U_{\text{min}} = -\frac{\beta'}{f^2} \frac{\partial^2 (\rho \partial \theta)}{\partial \phi \partial \partial \theta}.$$

With the normal stratification of the atmosphere we have $\{-[\partial (\Delta p)/\partial \theta_0] \partial \theta_0/\partial \phi \} = 6 \times 10^3 \text{ m}^2 \text{ s}^{-2}$ and $\beta' = 2.2 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$; therefore, the minimum $|\Delta U|$ required at $5^\circ$, $10^\circ$, $15^\circ$ latitude for baroclinic instability are about 880, 203 and 91 m s$^{-1}$, respectively. These values are far above the actual vertical shears present in the atmosphere and hence the mean flow is baroclinically stable below about $20^\circ$ latitude. This inference is also in agreement with the well-known result that the linear vertical shear is stable in a nonrotating system when the vertical velocity is required to vanish both at the bottom and at the top, which is close to the situation in the vicinity of the equator. Thus, below $15^\circ$ or $20^\circ$ latitude the vertical shear is actually exerting an adverse effect on the growth of the disturbances and decoupling under more stable stratification tends to minimize this adverse effect.

No simple physical explanation of the latitudinal preference of the easterly jet has yet been found.

Fig. 15.2. Variations of the amplitude $\phi$ and phase $\alpha$ of the geopotential $\phi$ of the nearly most unstable disturbances: (a) short in-phase barotropic solutions for $k=0.9$, $\gamma_0=0$, and $\gamma_0=6^\circ$; (b) long-wave disturbances $k=0.3$, $\gamma_0=0$: curves 1, barotropic and out of phase solution; curve 2, baroclinic flow.

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APPENDIX

A Simple Reduction Method for Solving Eigenvalue Problems

1. The eigenvalue problem represented by a single second-order equation

Let us consider that the eigenfunction $f$ is governed by the following second-order homogeneous differential equation and boundary conditions in finite-difference forms:

$$A_j f_{j+1} - B_j f_j + C_j f_{j-1} = 0, \quad (A1)$$

$$f_0 = f_1 + b_1 f_1, \quad f_{J+1} = f_{J-1} - b_2 f_J. \quad (A1a,b)$$

Here the two boundaries are at $j=1$ and $j=J$, respectively, and the coefficients $A_j$, $B_j$, $C_j$, $b_1$ and $b_2$ may all involve the complex eigenvalue $c$. We write the solution of (A1) as

$$f_j = E_j f_{j+1}, \quad (A2)$$

where the coefficient $E_j$ is to be determined in terms of $A_j$, $B_j$, $C_j$, $b_1$ and $b_2$. Consequently, we also have $f_{j-1} = E_{j-1} f_j$. Substituting this relation in (A1) we then find

$$E_j = A_j/(B_j - C_j E_{j-1}), \quad (A3)$$

which holds for all the interior points and it can be used to determine $E_j$ consecutively for increasing $j$ in $2 \leq j \leq J-1$ when $E_1$ is known. For $E_1$ we substitute $f_0$ from (A1a) in (A1) for $j=1$ and thereby obtain

$$E_1 = \frac{2}{f_2 - B_1 C_1 b_1}. \quad (A4)$$

Similarly, on substituting $f_{j+1}$ from (A1b) in (A1) for
\( j = J \) we find
\[
f_{j-1}/f_j = (B_j + A_j b_2)/2. \tag{A5}
\]
Now, according to (A2), the ratio \( f_{j-1}/f_j \) is also equal to \( E_{j-1} \), which is obtained from (A3) and (A4) consecutively for increasing \( j \). Therefore the compatibility equation for the eigenvalue \( c \) is
\[
G(c) = E_{j-1} - \frac{1}{2}(B_j + A_j b_2) = 0. \tag{A6}
\]

The scheme given above works well when \( f_j \) differs from zero, and especially when the maximum of \( f \) occurs at or close to \( j = J \). In case \( f \) has its maximum somewhere in the middle of the region, it is advantageous to integrate (A1) from the two ends and write the respective solutions as
\[
f_j^{(1)} = E_j f_j^{(1)} - 1, \quad f_j^{(2)} = E_j f_j^{(2)} - 1, \tag{A7a,b}
\]
where \( j \) and \( k \) increase away from the respective boundary. The continuity of \( f \) at the junction point \( j = J, k = K \) then gives
\[
G(c) = E_{j-1} - 1/E_{K-1} = 0. \tag{A8}
\]
The eigenvalue \( c \) for a given wavenumber \( k \) can be extracted from (A6) or (A8) by an iterative procedure which is to start with a first guess \( c_1 \) for \( c \) and then find the successive corrections. If we treat \( G(c) \) as a quadratic equation in \( c \) in the neighborhood of \( c_n \), we then have
\[
G(c_n + \Delta c) = (\Delta c)^2[G_0 + (\Delta c)^2 + G'/\Delta c + \frac{1}{2} G''],
\]
where \( G_0 = G(c_n) \) and
\[
G' = (1/2\delta c)[G(c_n + \delta c) - G(c_n - \delta c)],
\]
\[
G'' = \frac{1}{2}\Delta c[G(c_n + \delta c) + G(c_n - \delta c) - 2G_0],
\]
and \( \delta c \) is a conveniently chosen increment of \( c \). Thus the correction \( \Delta c \) is given by
\[
\Delta c = \frac{-2G_0}{[G' + (G'^2 - 2G_0 G'')^{1/2}]}, \tag{A9}
\]
where the sign of the radical is taken to make the denominator have the larger absolute value. Instead of this strictly quadratic approximation, Laguerre's formula (see Wilkinson, 1965)
\[
\Delta c = -\frac{NG_0}{G' + [(N-1)G'^2 - N(N-1)G_0 G'']^{1/2}}, \tag{A9a}
\]
may also be adopted. Here \( N \) is an arbitrarily selected large integer and the sign of the radical is also taken as that which yields the larger denominator in absolute value. Notice that (A9a) reduces to (A9) for \( N = 2 \).

2. Eigenvalue problems represented by two coupled second-order equations

In the two-level primitive equation model, the eigenvalue problem is represented by the two equations (16) or (22) and the appropriate boundary conditions. To find the solutions of this problem, we first write the differential equations (16) or (22) in the following finite difference form in terms of the grid point index \( j \) defined either by \( y_j = (j-2)\delta \) or by \( \eta_j = y_N - y(j) = (j-2)\delta \):
\[
A_j^{(1)} Z_{j+1} - B_j^{(1)} Z_j^{(1)} + C_j^{(1)} Z_{j-1}^{(1)} + \mu_j^{(1)} Z_{j+1}^{(3-0)} - \beta_j^{(1)} Z_{j-1}^{(3-0)} = 0, \tag{A10}
\]
where \( Z_j^{(1)} \) stands for either \( \phi_j^{(1)} \) or for \( \psi_j^{(1)} \) according to whether (16) or (22) is being used, and the coefficients \( A_j^{(1)}, B_j^{(1)}, C_j^{(1)}, \mu_j^{(1)} \) and \( \beta_j^{(1)} \) are given by the values of the coefficients of these equations. For example, for Eq. (22) in \( x \), we have \( A_j^{(1)} = 1 + C_{11} b_2/2, C_j^{(1)} = 1 - C_{11} b_2/2, \) etc.

Next we write the solutions of (A10) formally as
\[
Z_j^{(1)} = E_j^{(1)} Z_j^{(0)} + \alpha_j^{(1)} Z_j^{(3-0)}, \tag{A11}
\]
where the coefficients \( E_j^{(1)} \) and \( \alpha_j^{(1)} \) are determined so that (A10) is satisfied by this function. According to (A11), we also have
\[
Z_j^{(0)} = E_j^{(0)} Z_j^{(0)} + \alpha_j^{(0)} Z_j^{(3-0)}. \tag{A11a}
\]
Substituting this expression of \( Z_j^{(0)} \) in (A10) for \( l = 1 \) and \( l = 2 \) separately and solving for \( Z_j^{(1)} \) in terms of \( Z_j^{(0)} \) and \( Z_{j+1}^{(1)} \) and comparing the result with (A11) we then find
\[
E_j^{(1)} = [A_j^{(1)} B_j^{(1)} - \mu_j^{(1)}]/M_j, \tag{A12a}
\]
\[
\alpha_j^{(1)} = [\mu_j^{(1)} B_j^{(1)} - A_j^{(1)}]/M_j, \tag{A12b}
\]
where
\[
\beta_j^{(1)} = B_j^{(1)} - C_j^{(1)} E_i^{(1)} - \mu_j^{(1)} \alpha_j^{(1)}, \tag{A12c}
\]
\[
\alpha_j^{(1)} = B_j^{(1)} - C_j^{(1)} E_i^{(1)} - \mu_j^{(1)} \alpha_j^{(1)}, \tag{A12d}
\]
\[
M_j = B_j^{(1)} - B_j^{(1)} - \beta_j^{(1)}. \tag{A12e}
\]
Since the right hand sides of (A12c) and (A12d) contain only \( E_j^{(1)} \) and \( \alpha_j^{(1)} \), the parameters \( B_j^{(1)}, C_j^{(1)} \) and \( \mu_j^{(1)} \), the coefficients \( B_j^{(1)} \) and \( \alpha_j^{(1)} \) can be obtained consecutively from (A12a) and (A12b) provided the values of \( E_j^{(1)} \) and \( \alpha_j^{(1)} \) are known for both \( l = 1 \) and \( l = 2 \). These starting values of \( E_j^{(1)} \) and \( \alpha_j^{(1)} \) are determined by using the boundary conditions at the boundary \( j = 2 \) in (A10). For example, for the boundary condition \( Z_j^{(1)} = 0 \) at \( y = 0 \), we simply have
\[
E_j^{(1)} = 0, \quad \alpha_j^{(1)} = 0. \tag{A13}
\]
On the other hand, for the condition \( dZ_j^{(1)}/dy = 0 \) at \( y = 0 \), we have \( Z_j^{(1)} = Z_j^{(0)} \). Thus, on setting \( j = 2 \) in (A10), making use of this relation and solving for \( Z_j^{(1)} \) and \( Z_j^{(2)} \) in terms of \( Z_j^{(1)} \) and \( Z_j^{(3-0)} \) in the form of (A11) we then find
\[
E_j^{(0)} = 2B_2^{(3-0)} M_j, \quad \alpha_j^{(0)} = -2\beta_j^{(3-0)} M_j. \tag{A14}
\]
The values of \( E_j^{(0)} \) and \( \alpha_j^{(0)} \) for the asymptotic boundary conditions (27b) or (27b') can be obtained in a similar manner. Thus, with these values of \( E_j^{(0)} \) and \( \alpha_j^{(0)} \), we can
obtain \( E_j^\theta \) and \( \alpha_j^\theta \) from (A12a,b) for \( j=3, 4, \ldots, J \) consecutively.

Finally, we find the relation satisfied by the eigenvalue from the boundary conditions at \( j=J \). In case this condition is of the form (28a) or (27a') and is at \( y=0 \), we then make the integration of (A10) in terms of \( \eta=y_N-y \) so that \( y=0 \) corresponds to \( \eta=y_N \), \( j=N \). The boundary conditions are then

\[
Z_N^{(j)} = Z_{N-1}^{(j)}.
\]

Substituting these relations in (A10) for \( j=N \) and solving for \( Z_N^{(1)} \) and \( Z_N^{(2)} \), we then find

\[
2Z_N^{(j)} = B_N^{(j)} Z_N^{(1)} + \beta_N^{(j)} Z_N^{(2)},
\]

We also have

\[
Z_N^{(1)} = E_N^{(1)} Z_N^{(1)} + \alpha_N^{(1)} Z_N^{(2)},
\]

Substituting \( Z_N^{(1)} \) and \( Z_N^{(2)} \) from (A16a,b) into (A15a,b) we obtain the following two expressions for the amplitude ratio \( Z^{(2)}/Z^{(1)} \):

\[
\frac{Z^{(2)}}{Z^{(1)}} = \frac{B_N^{(2)} - 2E_N^{(2)}}{B_N^{(1)} - 2E_N^{(1)}},
\]

Dividing (A17b) by (A17a) we then find the relation

\[
G(c) = \frac{\beta_N^{(1)} - 2\alpha_N^{(1)} - \beta_N^{(2)} - 2\alpha_N^{(2)}}{B_N^{(1)} - 2E_N^{(1)} - B_N^{(2)} - 2E_N^{(2)}} - 1 = 0.
\]

This equation determines the eigenvalue \( c \) when the boundary conditions are of the type (28a,b) or (27a,b), which are for the two-layer problem.

When \( Z^{(1)} \) and \( Z^{(2)} \) vanish at both ends of the domain of \( y \) or when the maxima of \( Z^{(1)} \) occur in the middle part of the region, it is often necessary to integrate Eqs. (A10) from the two end points toward a point \( y=y_M \) somewhere in the middle. Let us use (A11) to represent the solution obtained by integrating (A10) from left to right, that is, from \( y=0 \) to \( y=y_M \) where \( j=J_M \), and write the solution obtained from integrating from the other end \( y=y_M \) to \( y=y_N \) as

\[
Z_0^{(j)} = E_0^{(j)} Z_{j+1}^{(j)} + \alpha_0^{(j)} Z_{j+1}^{(j-1)}.
\]

Suppose the junction point is at \( j=J \), \( k=N \). Then the point \( j=J+1 \) corresponds to \( k=N-1 \) and hence the continuity requirements for \( Z^{(1)} \) and its derivative at \( y=y_M \) are

\[
Z_N^{(1)} = Z_J^{(1)}, \quad Z_{N-1}^{(2)} = Z_{J+1}^{(2)}.
\]

Thus, on making use of these relations in (A19) for \( k=N-1 \) we then obtain

\[
Z_{j+1}^{(j)} = E_j^{(j)} Z_j^{(j)} + \alpha_j^{(j)} Z_{j+1}^{(j-1)}.
\]

We now set \( j=J \) in (A11) and substitute \( Z_{j+1}^{(j)} \) from (A20a,b) in this equation, thereby obtaining the following equations for \( Z^{(1)} \) and \( Z^{(2)} \):

\[
[1 - E_j^{(j)} \alpha_j^{(j)} - \alpha_j^{(j)} Z_j^{(j)}] = \frac{E_j^{(j)} \alpha_j^{(j)} Z_j^{(j)}}{E_j^{(j)} Z_j^{(j)}},
\]

Dividing the amplitude ratio \( Z^{(2)}/Z^{(1)} \) given by (A22a) by that given by (A22b) and taking the ratio as equal to 1 we then obtain for the eigenvalue \( c \)

\[
G(c) = \frac{[1 - E_j^{(j)} \alpha_j^{(j)} Z_j^{(j)}] [1 - E_j^{(j)} \alpha_j^{(j)} Z_j^{(j)}]}{E_j^{(j)} \alpha_j^{(j)} Z_j^{(j)} + \alpha_j^{(j)} Z_j^{(j)}} - 1 = 0.
\]

The eigenvalue \( c \) for this problem can also be obtained by the iterative method represented by (A9) or (A9a).

Notice that, in this method, which represents a generalization of the method for solving boundary value problems for nonhomogeneous equations described by Richtmyer and Morton (1965) and Lindzen and Kuo (1969) to eigenvalues problems represented by homogeneous equations, the amplitude ratio \( Z^{(1)}/Z^{(2)} \) at a reference point \( y=y \) is determined automatically, in contrast to the direct shooting method where this ratio must be determined by another iterative scheme (see Fox, 1960).

REFERENCES


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