Further Studies on a Spectral Model of the Global Barotropic Primitive Equations with Hough Harmonic Expansions

AKIRA KASAHARA

National Center for Atmospheric Research,[1] *Boulder, CO 80307*

(Manuscript received 7 March 1978, in final form 2 August 1978)

ABSTRACT

This paper describes further improvement in a new spectral model of the global barotropic primitive equations (Kasahara, 1977) which utilizes Hough harmonics as basis functions. A review is presented on a method of constructing Hough harmonics (normal modes of Laplace's tidal equations) with new results of the eigensolutions for the longitudinal wavenumber zero case.

Applying this complete set of orthonormal Hough harmonics, we formulate a spectral model of the nonlinear, barotropic primitive equations (shallow-water equations) over a sphere which eliminates separate treatment of the zonally averaged component equations in the previously proposed model by the author. An example of the model calculation with Haurwitz wavenumber 6 initial conditions is presented.

1. Introduction

Spherical harmonics have been used traditionally as basis functions to formulate spectral models of atmospheric prediction equations over a sphere. Because spherical harmonics are Haurwitz (1940) solutions to the barotropic nondivergent vorticity equation over a sphere (Craig, 1945; Neamtan, 1946), they are suitable basis functions for a spectral model of the global vorticity equation (Silberman, 1954; Platzman, 1960). They are also used as basis functions to formulate spectral models for the primitive equations over a sphere (Merilees, 1968b; Bourke, 1972, 1974). However, spherical harmonics are not eigensolutions to a linearized system of the barotropic primitive equations. We must use a series of spherical harmonics to construct their eigensolutions, and the wave components cannot be treated accurately unless a sufficient number of spherical harmonics are used to approximate their eigensolutions. Hence, the choice of expansion functions to formulate spectral models of the primitive equations is left open.

Kasahara (1977, hereafter referred to as K77) proposed a new spectral formulation for the nonlinear, barotropic primitive equations (shallow-water equations) over a sphere by applying Hough harmonics as expansion functions. Hough harmonics and associated eigenfrequencies are solutions of free oscillation (normal modes) to a linearized system of the global shallow-water equations. In this approach, the basic equations are split into two sets of equations for the zonally averaged and nonzonal components. The spectral equations for the nonzonal components are derived by expanding the dependent variables with Hough harmonics of the nonzonal equations. On the other hand, the zonally averaged component equations are solved by expanding the variables with Legendre polynomials (Eliasen *et al.*, 1970).

Since Legendre polynomials are not eigenfunctions of the zonally averaged basic equations, the separate treatment of the zonally averaged components is not only a deviation from the eigenfunction expansion approach, but it also prevents examination of the behavior of gravity waves in the zonally averaged motions.

The reason for this separate treatment is that the previously known eigensolutions of the zonally averaged equations were not in a form adaptable as expansion functions. As remarked in the footnote on p. 692 of K77, the author has now derived a complete set of orthonormal Hough vector functions for the zonally averaged equations.

This paper describes construction of such functions and application to solve the global shallow-water equations without distinguishing formally the treatment of the zonally averaged components from that of the nonzonal components in the spectral Hough harmonic expansions.

2. Basic equations

The nonlinear shallow-water equations over a sphere may be written in the form

$$\frac{\partial \mathbf{W}}{\partial t}+\mathbf{L}\mathbf{W}=i\mathbf{F}(\lambda,\phi,t), \tag{1}$$

[1] The National Center for Atmospheric Research is sponsored by the National Science Foundation.

0022-4928/78/2043-2051$05.00

by defining the vector dependent variable

$$\mathbf{W}=\begin{bmatrix} u \\ v \\ h \end{bmatrix}, \tag{2}$$

which is a function of the independent variables longitude λ, latitude ϕ and time t. Components u, v, h are, respectively, the eastward and northward velocity components and the height of free surface. They are made dimensionless in scaling by $(gh_e)^{\frac{1}{2}}$, $(gh_e)^{\frac{1}{2}}$ and h_e, respectively, where g denotes the acceleration of gravity (constant) and h_e a constant height. Time is also made dimensionless in multiplication by 2Ω, where Ω denotes the angular velocity of rotation.

In Eq. (1), $\mathbf{L}$ is the linear differential matrix operator and $\mathbf{F}$ is t e nonlinear vector function as given, respectively, by (2.7) and (2.8) of K77. Both $\mathbf{L}$ and $\mathbf{F}$ contain a single parameter

$$\gamma=(gh_e)^{\frac{1}{2}}/(2a\Omega) \tag{3}$$

in which a represents the radius of the globe (constant). The symbol i in (1) is introduced for convenience.

3. Construction of Hough harmonics

Eq. (1) with $\mathbf{F}=0$ is a linearized system of shallow-water equations in the basic state of no motion and constant h_e. Such equations are identical to Laplace's tidal equations (e.g., Lamb, 1932) without tide-generating functions. Various aspects of the nature of eigensolutions of the Laplace tidal equations including the method of solution, their asymptotic behavior and their numerical tables are presented by Margules (1893), Hough (1898), Dikii (1965), Flattery (1967) and Longuet-Higgins (1968). Here we briefly discuss a method of obtaining the normal mode solutions after Kasahara (1976, hereafter referred to as K76).

To obtain normal model solutions, the dependent variables of the linearized system (1) with $\mathbf{F}=0$ are transformed using Helmholtz's relationships

$$u=\gamma\left(\frac{1}{\cos\phi}\frac{\partial\Phi}{\partial\lambda}-\frac{\partial\psi}{\partial\phi}\right),\qquad v=\gamma\left(\frac{\partial\Phi}{\partial\phi}+\frac{1}{\cos\phi}\frac{\partial\psi}{\partial\lambda}\right), \tag{4}$$

where Φ and ψ are the velocity potential and streamfunction, multiplied by $2\Omega/(gh_e)$ to make them dimensionless, and γ is defined by (3).

With transformation (4), the linearized system of (1) becomes the set of three equations governing the time rate of change of Φ', ψ' and free surface height h', where the prime indicates a solution of the linearized system. Such a system has constant coefficients for the derivatives in longitude λ and time t. Therefore, their solutions can be expressed by

$$\begin{bmatrix} \Phi' \\ \psi' \\ h' \end{bmatrix}=\begin{bmatrix} \hat{\Phi} \\ \hat{\psi} \\ \hat{Z} \end{bmatrix}\exp[i(s\lambda-\sigma t)], \tag{5}$$

where s denotes the longitudinal wavenumber and σ the dimensionless frequency scaled by 2Ω. Coefficients $\hat{\Phi}$, $\hat{\psi}$ and $\hat{Z}$ are functions of latitude only and must satisfy the system of equations

$$\left.\begin{aligned} (\sigma\nabla^2-s)(i\hat{\Phi})+(\mu\nabla^2+D)\hat{\psi}&=\nabla^2\hat{Z} \\ (\sigma\nabla^2-s)\hat{\psi}+(\mu\nabla^2+D)(i\hat{\Phi})&=0 \\ \sigma\hat{Z}&=-\gamma^2\nabla^2(i\hat{\Phi}) \end{aligned}\right\}, \tag{6}$$

with

$$\left.\begin{aligned} \nabla^2&=\frac{d}{d\mu}\left[(1-\mu^2)\frac{d}{d\mu}\right]-\frac{s^2}{1-\mu^2} \\ \mu&=\sin\phi \\ D&=(1-\mu^2)\frac{d}{d\mu} \end{aligned}\right\}. \tag{7}$$

No familiar functions satisfy (6) directly and their solutions must be constructed by a series of known functions. Because the associated Legendre functions $P_n^s(\mu)$ satisfy the Laplacian operator ∇^2, it is convenient to represent $\hat{\Phi}$, $\hat{\psi}$ and $\hat{Z}$ by

$$\begin{bmatrix} \hat{\Phi} \\ \hat{\psi} \\ \hat{Z} \end{bmatrix}=\sum_{n=s}^{\infty}\begin{bmatrix} iA_n^s \\ B_n^s \\ C_n^s \end{bmatrix}P_n^s(\mu), \tag{8}$$

where A_n^s, B_n^s and C_n^s are expansion coefficients.

Since the series (8) is an infinite series, we must truncate the series to determine the expansion coefficients A_n^s, B_n^s and C_n^s. Substituting (8) into (6), equating coefficients of P_n^s to zero and terminating the series at $n=s+2N+1$, where N is a natural number, we find that the system contains two independent cases. One consists of A_n^s and C_n^s for $n=s, s+2, \ldots,$ and B_n^s for $n=s+1$, s+3, ... In this case, if $\mathbf{X}$ is the column vector

$$\mathbf{X}=\text{col}(A_s^s, B_{s+1}^s, C_s^s, A_{s+2}^s, B_{s+3}^s, C_{s+2}^s, \ldots, A_{s+2N}^s, B_{s+2N+1}^s, C_{s+2N}^s), \tag{9}$$

then $\mathbf{X}$ is the solution of the homogeneous equation

$$\mathbf{AX}=\sigma\mathbf{X}. \tag{10}$$

Frequency σ as defined in (5) is the eigenvalue of matrix $\mathbf{A}$ of order $3N$. (The explicit form of matrix $\mathbf{A}$ and matrix $\mathbf{B}$ which appears subsequently is given in K76.) The other case consists of A_n^s and C_n^s for $n=s+1$, $s+3, \ldots,$ and B_n^s for $n=s, s+2, \ldots$. In this case, the column vector

$$\mathbf{Y}=\text{col}(B_s^s, A_{s+1}^s, C_{s+1}^s, B_{s+2}^s, A_{s+3}^s, C_{s+3}^s, \ldots, B_{s+2N}^s, A_{s+2N+1}^s, C_{s+2N+1}^s) \tag{11}$$

is the solution of

$$\mathbf{BY}=\sigma\mathbf{Y}. \tag{12}$$

The former case is *symmetric* because the height and zonal velocity are symmetric relative to the

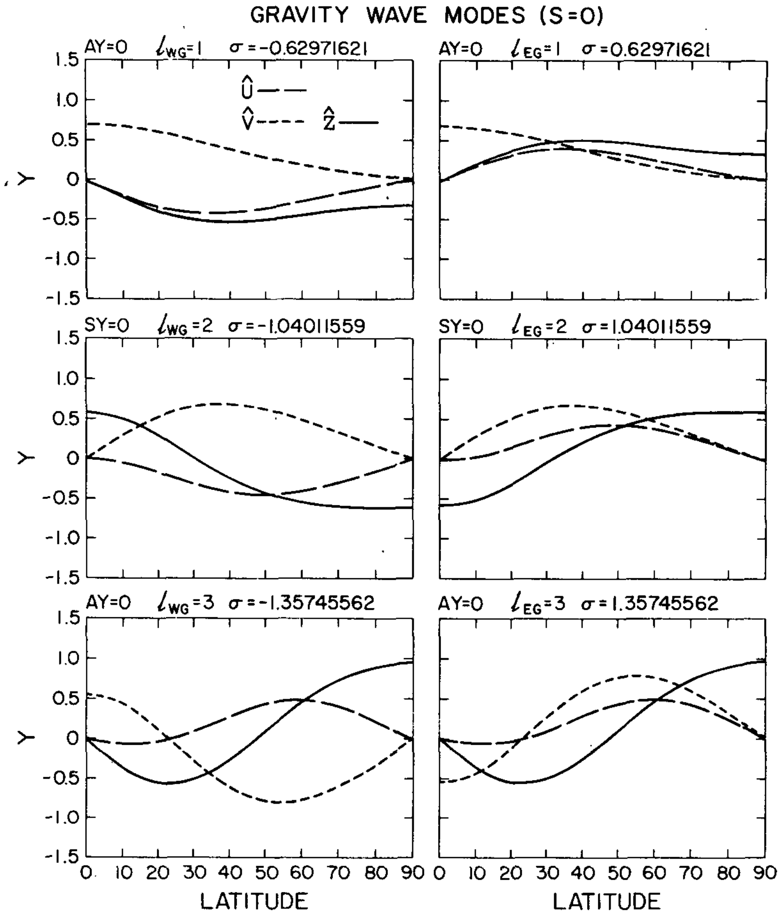

FIG. 1. Meridional structure of gravity wave modes for longitudinal wavenumber $s=0$. The meridional modal indices are denoted by l_{EG} for positive frequency and l_{WG} for negative frequency. AY represents the antisymmetric mode and SY the symmetric mode.

equator and the meridional velocity is antisymmetric. The latter case is *antisymmetric*, having antisymmetric height and zonal velocity and symmetric meridional velocity. The frequencies σ and associated vectors $\mathbf{X}$ and $\mathbf{Y}$ are obtained as eigenvalues and eigenvectors of matrices $\mathbf{A}$ and $\mathbf{B}$ using a standard computer routine as discussed in K76. The numerical values of frequency σ and associated vectors $\mathbf{X}$ and $\mathbf{Y}$ will converge as the number of expansion coefficients increases. For the equivalent height of $h_e=10$ km, sufficiently accurate solutions are obtained by using $N=20$ (see details in K76). The eigenfrequencies σ are real by nature of the linear operator L (Matsuno, 1966; Platzman, 1972; K77).

Once $\hat{\Phi}$, $\hat{\psi}$ and $\hat{Z}$ are calculated, the longitudinal and meridional velocity components are obtained from

$$\hat{U}=\gamma(is\hat{\Phi}-D\hat{\psi})/\cos\phi,\qquad \hat{V}=\gamma(is\hat{\psi}+D\hat{\Phi})/\cos\phi,\quad (13)$$

as derived from (4), where $\hat{U}$ and $\hat{V}$ are meridionally dependent parts of u and v.

The solution $\mathbf{W}'$ of the linearized equations of (1) with $\mathbf{F}=0$, in terms of velocity components u', v' and height h', are now expressed by

$$\mathbf{W}'(\lambda,\phi,t)=\mathbf{H}_l^s(\lambda,\phi)\exp(-i\sigma_l^s t),\qquad (14)$$

where

$$\mathbf{H}_l^s(\lambda,\phi)=\Theta_l^s(\phi)\,e^{is\lambda},\qquad (15)$$

$$\Theta_l^s(\phi)=\begin{pmatrix}\hat{U}_l^s(\phi)\\ -i\hat{V}_l^s(\phi)\\ \hat{Z}_l^s(\phi)\end{pmatrix}.\qquad (16)$$

The meridional structure function Θ_l^s and eigenfrequency σ_l^s are dependent on two indices. One is longitudinal wavenumber s and the other is meridional index l, which is related to the number of zero crossings between two poles in the meridional profiles of $\hat{U}_l^s$, $\hat{V}_l^s$ and $\hat{Z}_l^s$. We call $\Theta_l^s(\phi)$ the Hough vector function and $\mathbf{H}_l^s(\lambda,\phi)$ the Hough harmonic, after Hough (1898) who investigated the solutions of Laplace's tidal equations by expansions with spherical harmonics.

CHARACTERISTICS OF NORMAL MODES

In the case of $s\geqslant 1$, two different wave motions with distinct frequencies exist—eastward and westward-propagating gravity-inertia waves (first kind) and westward-propagating rotational waves of the Rossby/Haurwitz type (second kind). Fig. 1 of K77 shows the frequency σ (ordinate) as a function of longitudinal wavenumber s (abscissa) and meridional index l for the three different types in the case of $h_e=10$ km for the earth's atmosphere. We use l_{EG}, l_{WG} and l_{R} to distinguish meridional indices of eastward- and westward-propagating, gravity-inertia waves from westward-propagating rotational waves. Figs. 4–8 in K76 illustrate the meridional structures of Hough vector functions Θ_l^s for the three different wave types corresponding to $h_e=10$ km for the atmosphere.

The case of $s=0$ is unique in that the frequencies of gravity motions (first kind) appear as pairs of positive and negative values of the same magnitudes, and the frequencies of rotational motions (second kind) are all zero (Longuet-Higgins, 1968). The meaning of eastward and westward propagation is lost in this case, though the meridional index for positive frequency is denoted by l_{EG} and for negative frequency by l_{WG}. Fig. 1 shows the meridional structure of gravity wave modes for $s=0$ and $h_e=10$ km for the atmosphere. We see clearly that the meridional structures of Θ_l^0 vary as do the associated frequencies. The meridional structure of Θ_l^0 for positive frequency is different from that of negative frequency only in the sign of meridional velocity $\hat{V}$. Hence, the Hough vector function of the first kind, Θ_l^0 for negative frequency, is the complex conjugate of Θ_l^0 for positive frequency.

We now discuss new results on the normal mode solutions for the second kind in the case of $s=0$. Remembering that $\sigma=0$ and $s=0$, we reduce Eqs. (6) to

$$D(i\hat{\Phi})=0 \quad \text{and} \quad (\mu\nabla^2+D)\hat{\psi}=\nabla^2\hat{Z}. \tag{17}$$

The first equation implies that the meridional motion is absent; the second equation is the linear balance equation for zonally symmetric flow on the sphere. A similar equation appears in Merilees (1968a). To construct a complete set of expansion functions for $\hat{\psi}$ and $\hat{Z}$, we can assume polynomial functions of various degrees for $\hat{\psi}$ and calculate for $\hat{Z}$, or vice versa. To generate such an algorithm parallel with that for the case of $s\geqslant 1$, we expand $\hat{\psi}$ and $\hat{Z}$ in terms of a finite series of Legendre polynomials $P_n(\mu)$:

$$\begin{pmatrix}\hat{\psi}\\ \hat{Z}\end{pmatrix}=\sum_{n=0}^{2N+1}\begin{pmatrix}B_n\\ C_n\end{pmatrix}P_n(\mu). \tag{18}$$

Substitution of the above expressions into the second equation of (17) and equating the coefficients of P_n to zero yield

$$B_{n+1}\left(\frac{n+2}{2n+3}\right)+B_{n-1}\left(\frac{n-1}{2n-1}\right)=C_n. \tag{19}$$

Corresponding to (9), the coefficient vector for the symmetric case is expressed by

$$\mathrm{col}(B_1, C_0, B_3, C_2, \ldots, B_{2N+1}, C_{2N}).$$

Here N independent symmetric solutions can be obtained by choosing a single coefficient B_{n+1}, where n is an even integer in $[0, 2N-2]$, to be unity and all other B's are zero. In this case, (19) yields

$$\left.\begin{aligned}B_{n+1}&=1\\ C_n&=\frac{n+2}{2n+3}\\ C_{n+2}&=\frac{n+1}{2n+3}\end{aligned}\right\}, \tag{20}$$

for $n=0, 2, 4, \ldots, 2N-2$.

Similarly, corresponding to (11), the coefficient vector for the antisymmetric case is expressed by

$$\mathrm{col}(B_0, C_1, B_2, C_3, \ldots, B_{2N}, C_{2N+1}).$$

Here N independent antisymmetric solutions can be obtained by choosing a single coefficient B_{n+1}, where n is an odd integer in $[1, 2N-1]$, to be unity and all other B's are zero. In this case, (19) yields

$$\left.\begin{aligned}B_{n+1}&=1\\ C_n&=\frac{n+2}{2n+3}\\ C_{n+2}&=\frac{n+1}{2n+3}\end{aligned}\right\}, \tag{21}$$

for $n=1, 3, 5, \ldots, 2N-1$.

Once the coefficients B_n, C_n are determined, zonal velocity $\hat{U}$ and height $\hat{Z}$ are calculated from the equation of (13) with $\hat{\Phi}=0$ and from (18). The meridional velocity $\hat{V}$ is zero in this case. The proportional factor for $\hat{U}$ and $\hat{Z}$ is still arbitrary and will be determined by the normalization

$$\int_{-1}^{1}(\hat{U}^2+\hat{Z}^2)d\mu=1. \tag{22}$$

Since the total of N modes of $(\hat{U},\hat{Z})$ so obtained are solutions of the geostrophic equation (17), it may be appropriate to call them *geostrophic modes*. Calculations show that they are not orthogonal, i.e.,

$$\int_{-1}^{1}(\hat{U}_j\hat{U}_k+\hat{Z}_j\hat{Z}_k)d\mu\neq 0,$$

when j and k represent different geostrophic modes.

Because the corresponding eigenfrequencies of the second kind are all zero, we can transform these solutions linearly to create another set of geostrophic

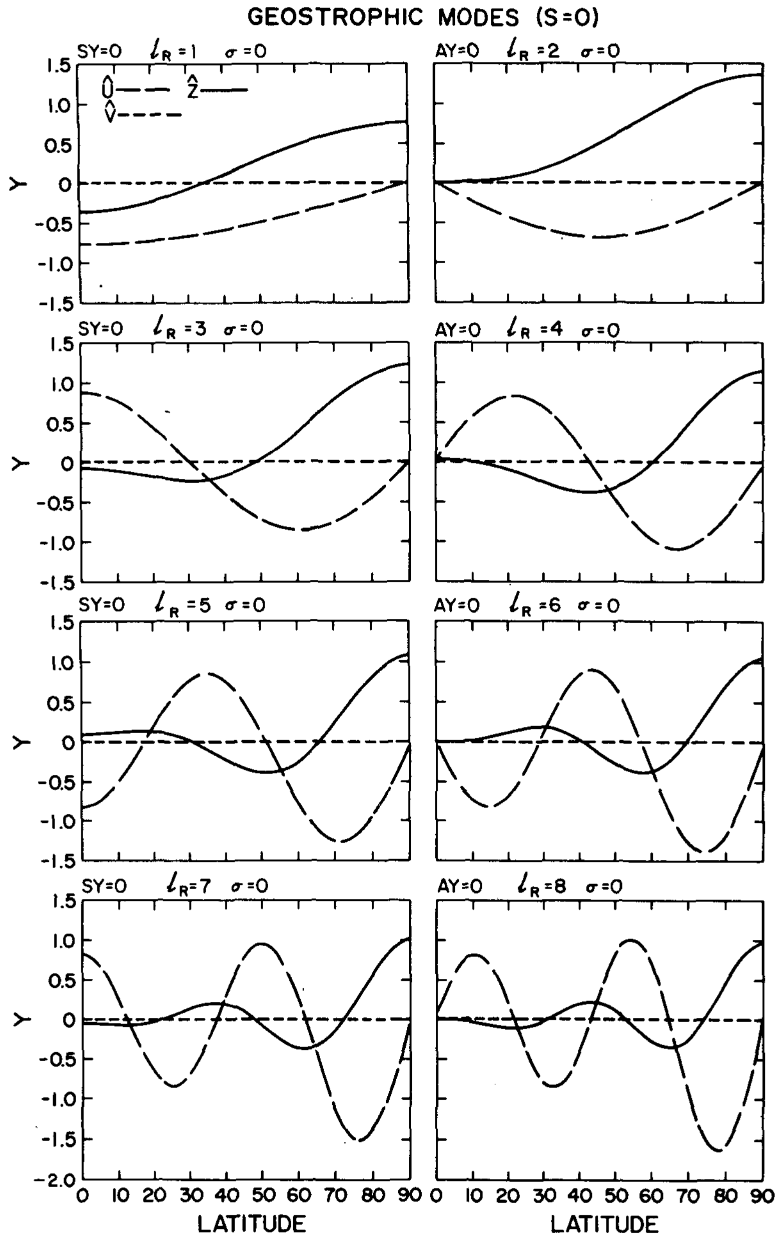

FIG. 2. Meridional structure of geostrophic modes ($s=0$). The meridional modal index is shown by l_R. Frequencies are all zero.

modes which are orthogonal to themselves. A vector orthogonalization routine based on the Gram–Schmidt method (e.g., Lanczos, 1961) works well for this purpose.

The geostrophic modes are still not complete unless we add a vector corresponding to $\hat{U}=0$, $\hat{V}=0$ and $\hat{Z}=$constant to represent a constant height. With the same normalization condition (22), $\hat{Z}$ should be equal to $1/\sqrt{2}$. Using this vector as the lowest-order Hough vector function ($l_R=0$) of the second kind, we orthonormalize the remaining Hough vector functions of the second kind to satisfy

$$\int_{-1}^{1} (\hat{U}_l^0\hat{U}_{l'}^0+\hat{Z}_l^0\hat{Z}_{l'}^0)d\mu=\delta_{ll'}, \tag{23}$$

where $\delta_{ll'}=1$ if $l=l'$ and zero otherwise.

Fig. 2 illustrates the meridional structures of orthonormalized geostrophic modes ($s=0$) for $l_R \geqslant 1$. The values of the parameters used are the same as for Fig. 1. The lowest mode $l_R=0$ consists of $\hat{U}_0^0=0$, $\hat{V}_0^0=0$ and $\hat{Z}_0^0=1/\sqrt{2}$, which is not shown.

4. Orthogonality of Hough harmonics

As shown in K77, the eigensolutions of Laplace's tidal equations are orthogonal in the following sense:

$$(\sigma_j-\sigma_k)\int_0^{2\pi}\int_{-1}^{1}\mathbf{H}_j\cdot\mathbf{H}_k^*d\lambda d\mu=0,\quad j\neq k,\qquad(24)$$

where $\mathbf{H}_j$ is a Hough harmonic corresponding to a single (two-dimensional) modal index combining longitudinal wavenumber s_j and meridional index l_j and the asterisk denotes the complex conjugate. As noted earlier, frequency σ_j is real by nature of the linear operator $\mathbf{L}$.

Eq. (24) implies that $\mathbf{H}_j$ associated with σ_j must be orthogonal to the $\mathbf{H}_k$ associated with σ_k, as long as $\sigma_j\neq\sigma_k$ regardless of whether either σ_j or σ_k is zero. If both σ_j and σ_k are zero in motions of the second kind for $s=0$, Eq. (24) does not provide any information as to the orthogonality of the particular modes. However, as discussed in Section 3, we can construct an orthonormal set of geostrophic modes ($s=0$). Thus, we now have a complete set of orthonormal Hough harmonics.

Because the meridional index l commonly represents three different modes l_{EG}, l_{WG} and l_{R}, the use of l without a proper subscript is confusing. To avoid this ambiguity, we tag serial number r to a complete set of Hough vector functions for a specific longitudinal wavenumber s. One way to do so is to place serial number r on Hough vector functions in order of decreasing frequency from the largest positive value to the largest negative value in the set. For $s\geqslant 1$, it is no problem to arrange the order of r for the functions, since the corresponding frequencies are all distinct. In this arrangement, the modes for l_{EG} appear first, the modes for l_{R} come next, and the modes for l_{WG} line up last. Let us designate the first one in the serial number as $r=1$ and the last one as $r=R$. For $s=0$, this ordering creates a problem, because the frequencies corresponding to the geostrophic modes are all zero. However, we can specify serial numbers in order of increasing l_{R}, for example.

We now state the orthonormality of Hough harmonics using the new serial number r as follows:

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{-1}^{1}\mathbf{H}_r^s\cdot\mathbf{H}_{r'}^{s'*}d\mu d\lambda=\delta_{rr'}\delta_{ss'},\qquad(25)$$

including $s=0$, where $\delta_{rr'}=1$ if $r=r'$ and zero otherwise, and similarly for $\delta_{ss'}$.

By substituting (15) into (25), we have

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{-1}^{1}\Theta_r^s\cdot\Theta_{r'}^{s'*}e^{i(s-s')\lambda}d\mu d\lambda=\delta_{rr'}\delta_{ss'}.$$

Hence, with definition (16) and for $s=s'$,

$$\int_{-1}^{1}\Theta_r^s\cdot\Theta_{r'}^{s*}d\mu=\int_{-1}^{1}(\hat{U}_r^s\hat{U}_{r'}^s+\hat{V}_r^s\hat{V}_{r'}^s+\hat{Z}_r^s\hat{Z}_{r'}^s)d\mu\qquad(26)$$

$$=\delta_{rr'}.$$

This is the orthonormality of Hough vector functions.

The difference between the Hough harmonics derived presently and those used in K77 is that the present set contains the case of $s=0$ whereas the previous set did not. This is why in K77 we had to deal separately with solution of the zonally averaged component equations.

5. Spectral form of nonlinear shallow-water equations

We assume that the solution of nonlinear equations (1) can be expressed by a series of Hough harmonics

$$\mathbf{W}(\lambda,\phi,t)=\sum_{r=1}^{R}\sum_{s=-M}^{M}W_r^s(t)\mathbf{H}_r^s(\lambda,\phi),\qquad(27)$$

where $W_r^s(t)$ denotes the expansion coefficients that are functions of time only. Note that longitudinal wavenumber s runs from a negative integer $-M$ to a positive integer M including zero. The summation for serial number r from 1 to R should contain all meridional modes l_{EG}, l_{WG} and l_{R} as mentioned in Section 4. We should emphasize that expansion (27) applies to the total field of $\mathbf{W}$. This is the important distinction from the previous spectral formulation in K77.

Substituting (27) into (1), integrating the resulting equation over the entire globe after multiplication by the complex conjugate of Hough harmonics, and utilizing the fact that $\mathbf{H}_r^s$ and σ_r^s satisfy the relationship

$$\mathbf{L}[\mathbf{H}_r^s(\lambda,\phi)]=i\sigma_r^s\mathbf{H}_r^s(\lambda,\phi)\qquad(28)$$

and the orthogonality condition (25), we obtain the spectral equation

$$\frac{dW_r^s(t)}{dt}+i\sigma_r^sW_r^s(t)=iF_r^s(t),\qquad(29)$$

where

$$F_r^s(t)=\int_{-1}^{1}\mathbf{F}_s(\phi,t)\cdot\Theta_r^{s*}d\mu,\qquad(30)$$

$$\mathbf{F}_s(\phi,t)=\frac{1}{2\pi}\int_0^{2\pi}\mathbf{F}(\lambda,\phi,t)e^{-is\lambda}d\lambda.\qquad(31)$$

Eq. (30) is the Hough transform and Eq. (31) is the Fourier transform of the nonlinear term.

As in K77, we apply the transform method (Eliasen *et al.*, 1970; Orszag, 1970) to evaluate the right-hand side of (29) and to calculate the integrals (30) and (31) exactly within the accuracy of the truncated repre-

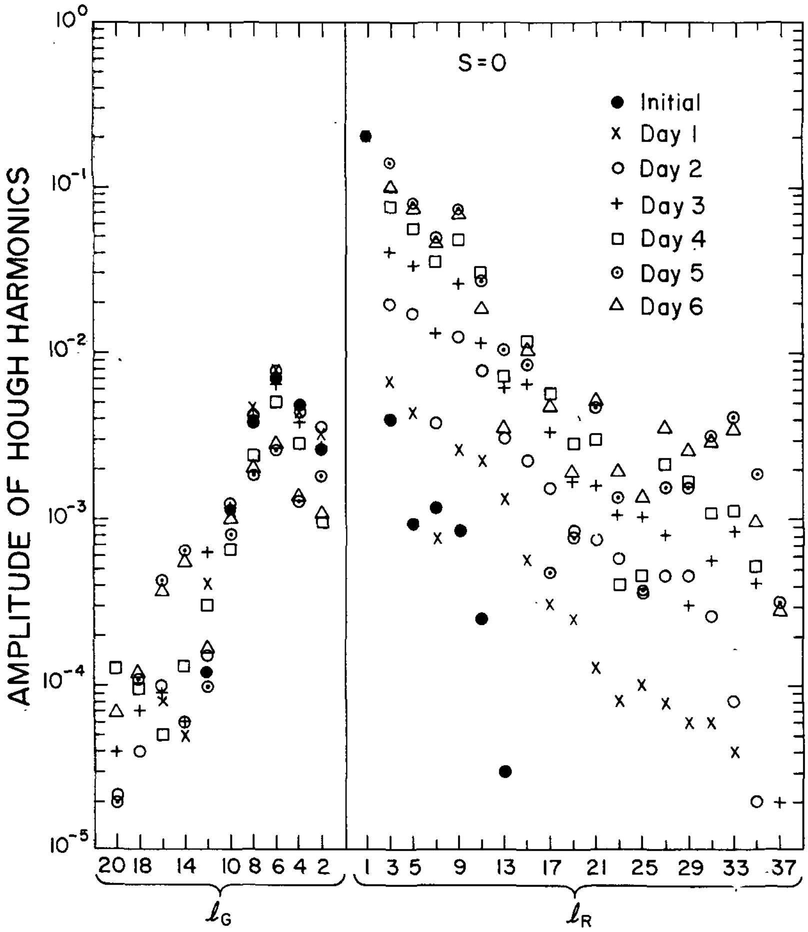

FIG. 3. Spectral distributions of Hough harmonic amplitudes for $s=0$ with respect to meridional index l and time in days, classified into gravity waves modes l_G (l_{EG} or l_{WG}) and rotational modes l_R.

sentation of $\mathbf{F}(\lambda,\phi,t)$. To calculate the nonlinear term $\mathbf{F}(\lambda,\phi,t)$, we evaluate u, v and h and their derivatives at longitude-latitude grid points. Their derivatives can be evaluated analytically, since u, v and h are represented by the series of Hough harmonics which are differentiable.

The methods of evaluating integrals (30) and (31), as well as calculation of the derivatives of Hough harmonics, are the same as those described in K77.

6. Test calculations

We integrated the spectral equation (29) using a Haurwitz wave as initial conditions (Phillips, 1959) and compared the result with that in K77. The initial velocity and geopotential height fields are given by (9.1) and (9.2) of K77 with the same numerical values for the constants and parameters. For the Haurwitz wave initial conditions, the flow pattern is symmetric with respect to the equator. We used 20 symmetric modes ($l_{EG}=0, 2, 4, 6$; $l_{WG}=0, 2, 4, 6$; $l_R=1, 3, 5, \ldots, 23$) of Hough vector functions for each longitudinal wavenumber s ($\geqslant 1$) up to $M=18$. For $s=0$, to obtain the accuracy comparable to that in K77, we adopted a total of 30 Hough vector functions consisting of 10 gravity wave modes ($l_{WG}=2, 4, 6, \ldots, 20$) and 20 geostrophic modes ($l_R=0, 1, 3, \ldots, 37$). For gravity wave modes in the case of $s=0$, we need to carry only those of either positive or negative frequency as mentioned in Section 3.

The present integration was performed using the leapfrog time extrapolation scheme with a time step of 6 min starting from the wavenumber 6 initial conditions as in the K77 calculation. For 6 days of integration, the results are almost identical to those of the K77 calculation, giving only insignificant difference in detail.

The present calculation enables us to examine the behavior of gravity wave components in the zonally averaged motions ($s=0$). This examination was not possible in K77. Fig. 3 shows the spectral distributions of Hough harmonic amplitudes for $s=0$ with respect to meridional index and time in days, classified into gravity wave modes l_G (l_{WG} or l_{EG}) and geostrophic modes l_R. (A constant component corresponding to $l_R=0$ is not included in the presentation.) Initially, the spectra have the maximum at $l_R=1$ and small amplitudes are seen in higher geostrophic modes and also in gravity wave modes. The amplitudes of geostrophic modes grow with time except for the initial peak at $l_R=1$, but their growth slows down and the distribution reaches a quasi-stationary state in 5 days or so. We see some excessive growth of amplitudes in higher meridional components as the result of blocking or damming-up. As explained by Orszag (1971), this type of error occurs in any nondissipative quadratic conserving scheme, such as the present spectral method, by disallowing all interactions with wave components beyond the cutoff. Blocking is a less serious error than those suffered by other numerical schemes and it is relatively easy to eliminate by adding a small amount of dissipation in higher modal components.

A noteworthy point in Fig. 3 is that the amplitudes of gravity wave modes are not growing, except for the higher meridional modes. In fact, the amplitudes of the lower meridional modes decrease with time. This implies that the interactions between gravity waves and geostrophic modes are much less than the interactions between different meridional modes of geostrophic motions. This is in agreement with the conclusion borne out from the previous calculation of K77 concerning the spectral behavior of nonzonal motions, as seen from Fig. 8 of K77. Of course, this conclusion is based on this particular experiment which may be too simple for a typical large-scale atmospheric motion. In the real atmosphere, gravity waves may be excited by forcings. Also, in a baroclinic atmosphere there is likely to be somewhat more significant interactions between internal gravity waves and rotational modes. The point of this demonstration is nothing more than to show that the normal mode decomposition is a useful means to investigate the nature of interactions between gravity waves and rotational modes and the present spectral formulation is far more suitable than the conventional approach in this respect.

7. Remarks

A unified treatment of zonally averaged and nonzonal components of the shallow-water equations with Hough harmonic expansions has the following advantage. The spectral equation now becomes formally similar to that of the nondivergent vorticity equation (e.g., Platzman, 1960). Moreover, the present formulation will enable us to investigate the behavior of gravity waves and their interactions with the meteorologically significant slow motions with greater flexibility than the use of spherical harmonics spectral models.

One important area of application of the present approach is the initialization of the primitive equations model. When the fields of pressure and horizontal velocity are not suitably adjusted initially, large-amplitude gravity waves may become visible during numerical integrations of the primitive equations model. Dickinson and Williamson (1972) proposed a method to initialize data by expanding the data into normal modes of a primitive equations model. Once the data are expanded into the normal modes, the gravity wave mode amplitudes may be set to zero. Williamson (1976) tested this procedure with the shallow-water equations. As noted by Machenhauer (1977), however, the initial elimination of all the gravity wave modes still leads to generation of gravity waves—though their amplitudes are small—due to nonlinear interactions of the remaining modes. The procedure of Machenhauer (1977) is to leave small-amplitude gravity waves initially to offset further growth of the gravity waves resulting from the nonlinear interactions of wave components including the zonal flow. Baer (1977) proposed a different approach using a Rossby number expansion, but he arrived at essentially the same initialization procedure incorporating the nonlinear interactions into the initialization scheme so that the high-frequency oscillations are suppressed. This type of initialization procedure is now referred to as the *nonlinear normal mode initialization* and is further investigated by Baer and Tribbia (1977), Daley (1978) and Leith (1978). The present modal decomposition is directly applicable to the fields of mass and horizontal velocity. This may be an advantage over other modal decomposition schemes that are applicable to the fields of streamfunction, velocity potential and mass. The same principle is applicable to initialize data for multilevel primitive equation prediction models based on spherical harmonics. In this case, the normal modes for values of the equivalent height corresponding to internal modes must be constructed based on the vertical structure equation as well as the horizontal structure equation. However, detailed discussions should be the subject of a separate report.

Acknowledgments. The author thanks Taroh Matsuno for his useful comments on an earlier version of the manuscript and Ann Modahl for her assistance in preparation of this manuscript. The calculations were performed on the Control Data Corporation 7600 at the NCAR Computing Facility. The figures were drafted by the NCAR Graphics Department.

REFERENCES

Baer, F., 1977: Adjustment of initial conditions required to suppress gravity oscillations in nonlinear flows. *Beitr. Phys. Atmos.*, **50**, 350–366.

——, and J. J. Tribbia, 1977: On complete filtering of gravity modes through nonlinear initialization. *Mon. Wea. Rev.*, **105**, 1536–1539.

Bourke, W., 1972: An efficient, one level primitive equations spectral model. *Mon. Wea. Rev.*, **100**, 683–689.

——, 1974: A multi-level spectral model I. Formulation and hemispheric integrations. *Mon. Wea. Rev.*, **102**, 687–701.

Craig, R., 1945: A solution of the nonlinear vorticity equations for atmospheric motion. *J. Meteor.*, **2**, 175–178.

Daley, R., 1978: Variational non-linear normal mode initialization. *Tellus*, **30**, 201–218.

Dickinson, R. E., and D. L. Williamson, 1972: Free oscillations of a discrete stratified fluid with application to numerical weather prediction. *J. Atmos. Sci.*, **29**, 623–640.

Dikii, L. A., 1965: The terrestrial atmosphere as an oscillating system. *Izv. Atmos. Ocean. Phys.*, **1**, 275–286.

Eliasen, E., B. Machenhauer and E. Rasmusson, 1970: On a numerical method for integration of the hydrodynamical equations with a spectral representation of the horizontal fields. Rep. No. 2, Dept. Meteor., Copenhagen University, 35 pp.

Flattery, T. W., 1967: Hough functions. Tech. Rept. No. 21, Dept. Geophys. Sci., The University of Chicago, 175 pp.

Haurwitz, B., 1940: The motion of atmospheric disturbances on the spherical earth. *J. Mar. Res.*, **3**, 254–267.

Hough, S. S., 1898: On the application of harmonic analysis to the dynamical theory of the tides—Part II. On the general integration of Laplace's dynamical equations. *Phil. Trans. Roy. Soc. London*, **A191**, 139–185.

Kasahara, A., 1976: Normal modes of ultralong waves in the atmosphere. *Mon. Wea. Rev.*, **104**, 669–690.

——, 1977: Numerical integration of the global barotropic primitive equations with Hough harmonic expansions. *J. Atmos. Sci.*, **34**, 687–701.

Lamb, H., 1932: *Hydrodynamics*, 6th ed. Dover, 738 pp.

Lanczos, C., 1961: *Applied Analysis*. Prentice Hall, 539 pp.

Leith, C. E., 1978: Objective methods for weather prediction. *Annual Reviews of Fluid Mechanics*, Vol. 10, Annual Reviews, Inc., 107–128.

Longuet-Higgins, M. S., 1968: The eigenfunctions of Laplace's tidal equations over a sphere. *Phil. Trans. Roy. Soc. London*, **A262**, 511–607.

Machenhauer, B., 1977: On the dynamics of gravity oscillations in a shallow water model, with applications to normal mode initialization. *Beit. Phys. Atmos.*, **50**, 253–271.

Margules, M., 1893: Luftbewegungen in einer rotierenden Sphäroidschale, Teil II. *S.-B. Akad. Wiss. Wien, Math.-Naturwiss. I*, Abt. IIa, **102**, 11–56.

Matsuno, T., 1966: Quasi-geostrophic motions in the equatorial area. *J. Meteor. Soc. Japan*, Ser. II, **44**, 25–43.

Merilees, P. E., 1968a: On the linear balance equation in terms of spherical harmonics. *Tellus*, **20**, 200–202.

——, 1968b: The equations of motion in spectral form. *J. Atmos. Sci.*, **25**, 736–743.

Neamtan, S. M., 1946: The motion of harmonic waves in the atmosphere. *J. Meteor.*, **3**, 53–56.

Orszag, S. A., 1970: Transform method for the calculation of vector-coupled sums: Application to the spectral form of the vorticity equation. *J. Atmos. Sci.*, **27**, 890–895.

——, 1971: Numerical simulation of incompressible flows within simple boundaries: accuracy. *J. Fluid Mech.*, **49**, 75–112.

Phillips, N. A., 1959: Numerical integration of the primitive equations on the hemisphere. *Mon. Wea. Rev.*, **87**, 333–345.

Platzman, G. W., 1960: The spectral form of the vorticity equation. *J. Meteor.*, **17**, 635–644.

——, 1972: Two-dimensional free oscillations in natural basins. *J. Phys. Oceanogr.*, **2**, 117–138.

Silberman, I., 1954: Planetary waves in the atmosphere. *J. Meteor.*, **11**, 27–34.

Williamson, D. L., 1976: Normal mode initialization procedure applied to forecasts with the global shallow water equations. *Mon. Wea. Rev.*, **104**, 195–206.