# On Spectral Energetics of Resonantly Interacting Waves in the Two-Layer Baroclinic Model

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## ABSTRACT

Spectral energetics associated with an $N$-wave system composed of a marginally unstable and $N/1$ neutral waves, interacting resonantly in triad configurations, are examined in the context of the two-layer baroclinic model. The unstable mode is found to dominate all allowable spectra at small initial energy levels. At larger initial energy levels this dominance persists only in the case of largest allowable spectra; otherwise, resonant interaction mechanism shifts the dominance to the neutral part of the spectrum. The shift occurs at lower initial energy levels and to a broader spectral range at higher values of internal rotational Froude number.

## 1. Introduction

This note is an extension of a study by Loesch and Domaracki (1977, hereafter referred to as DNR) into the spectral domain. In DNR, time-dependent dynamics of a finite discrete spectrum of resonantly interacting waves were examined within the context of the conventional two-layer baroclinic model on a $\beta$-plane. The resonant spectrum was restricted to $N$ waves consisting of (i) a marginally unstable wave near the minimum critical shear, (ii) primary neutral wave pair(s), interacting directly with the marginal wave to form the primary triad(s), and (iii) secondary neutral wave pairs, interacting with the waves in (ii) to form the secondary triads. This spectral configuration allowed the authors to study, in a simple context, the combined effects of instability and interaction on the evolution of the wave field. The present note examines the *time-averaged* energetics associated with the above $N$-wave system.

In the atmosphere, large-scale waves have their wavenumbers quantized. Under this constraint, conditions for the resonance can be satisfied only by a selective number of waves within the full spectrum. In DNR and here this part of the atmospheric spectrum is being approximated by the $N$-wave resonant configuration above. The motivation for studying the spectral energetics of the $N$-wave system is to determine the relative influence of instability and resonant interactions on the mean energy distribution within the spectrum.

The finite-amplitude analysis of the $N$-wave problem is presented in DNR and the reader is referred to that paper for details. In DNR, the equations governing the time evolution of the wave amplitudes $R_j(T)$ were solved numerically for a large set of initial conditions, at two values of the internal rotational Froude number, F = 8 and 12. As discussed in Loesch (1974), this choice of F assures investigation of the wave dynamics both for $\mathrm{F}<10.5$, where the finite-amplitude marginal wave is *stable* to small resonant perturbations, and for $\mathrm{F}>10.5$, where the marginal wave is *unstable* to such perturbations (see Loesch, 1974, Figs. 13 and 15 and p. 1199).

At the chosen values F = 8 and 12 (actually within a wide, geophysically relevant range $\pi^2<\sqrt{2}\mathrm{F}<4\pi^2$) the marginal wave, denoted $j=1$ in DNR, can interact resonantly with only *one* pair of neutral waves, denoted $j=2$ and 3 in DNR. Neutral mode $j=3$, like the marginal mode $j=1$, is involved in no additional interactions. Neutral mode $j=2$, however, can interact resonantly with up to three additional neutral wave pairs at F = 8 (denoted $j=4$ and 5, $j=6$ and 7, and $j=8$ and 9 in DNR) and with up to seven additional neutral wave pairs at F = 12 (denoted $j=4$ and 5, ..., $j=16$ and 17 in DNR). Consequently, the $N$-wave spectrum consists of $N\leqslant 9$ waves at F = 8 and $N\leqslant 17$ waves at F = 12, and the coupling between the primary triad and the secondary triads occurs via the $j=2$ member of the primary triad. The zonal and meridional wavenumbers ($\alpha_j$, $\pi m_j$) and frequencies $\sigma_j$ of the resonantly interacting waves at the two values of F are given in DNR in Table 1.

Throughout this note we shall denote the $N$-wave spectrum by $S_N$. As in DNR, the nondimensional energy in the $j$th mode at time $T$ will be denoted by $E_j(T)$. It is related to the nondimensional wave amplitude

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$R_j(T)$ by

$$E_j(T)=\frac{1}{2}\left(\alpha_j^2+m_j^2\pi^2+\mathrm{F}-\frac{\mathrm{F}^2}{\alpha_j^2+m_j^2\pi^2+\mathrm{F}}\right)R_j^2(T). \quad (1)$$

The nondimensional spectral energy gain/loss associated with the $j$th mode in $S_N$ mode in $S_N$ is then

$$\Delta_j=\overline{E_j(T)}-E_j(0), \quad (2)$$

where the bar denotes a time average. Actual determination of the $\Delta_j$'s involves first numerically solving the amplitude equations [DNR, Eq. (4.1)] subject to given initial conditions, then, with the help of (1), obtaining the evolution of energy in each mode, $E_j(T)$, and finally averaging $E_j(T)$ over a complete oscillation period and employing (2).

To make relative comparisons between the $\Delta_j$'s for the different modes in $S_N$, unlike in DNR where amplitudes were taken initially of the same order, in this study *equal initial energies* $E_j(0)$ are chosen. Since total wavenumbers $K_j=(\alpha_j^2+m_j^2\pi^2)^{\frac{1}{2}}$ vary widely within $S_N$, we note from (1), that the assumption of equal initial energy yields widely varying initial amplitudes.

Our interest lies in determining the relative influence of instability and resonant interactions on spectral energy gains/losses $\Delta_j$, in $S_N$. Based on the previous studies, we expect this influence to depend on 1) the choice of the F-region ($\mathrm{F}<10.5$ or $\mathrm{F}>10.5$), 2) the size of the spectrum ($N\leqslant 9$ for $F<10.5$ and $N\leqslant 17$ for $F>10.5$) and 3) the initial energy levels $E_j(0)$. To maintain continuity from DNR, spectral calculations for different $S_N$ have been carried out at $\mathrm{F}=8$ and 12. Nondimensional initial energy level was allowed to vary in the range $10^{-3}\leqslant E_j(0)\leqslant 1$, which corresponds to nondimensional marginal wave amplitude range $0.01\leqslant A_1(0)\leqslant 0.3$. Discussion of the results follows.

## 2. Spectral energetics at F = 8

The discussion of spectral energy gains at $\mathrm{F}=8$ may conveniently be divided into small $[E_j(0)\leqslant 10^{-2}]$ and large $[E_j(0)\geqslant 10^{-1}]$ initial energy level situations.

At the small initial energy levels $[E_j(0)\leqslant 10^{-2}]$, irrespective of the size of the spectrum $S_N(3\leqslant N\leqslant 9)$, the marginally unstable mode $j=1$ is consistently characterized by the largest average energy gain. The neutral primary triad member $j=3$ undergoes the second largest gain. As the initial energy level $E_j(0)$ is increased the difference between $\Delta_1$ and $\Delta_3$ decreases. In proportion to $\Delta_1$, the remaining modes $j=2$ and $4\leqslant j\leqslant 9$ all undergo only slight energy changes. A typical situation, corresponding to $S_9$, is illustrated for $E_j(0)=10^{-3}$ in Fig. 1a and for $E_j(0)=10^{-2}$ in Fig. 1b.

At sufficiently large initial energy levels, $[E_j(0)\geqslant 10^{-1}]$, the unstable mode consistently dominates only the largest spectrum $S_9$. For other spectra, either $j=1$ or $j=3$ can play the dominant role, depending on $E_j(0)$. For example, when $E_j(0)=0.10$, $\Delta_1>\Delta_3$ for $S_3$ but $\Delta_3>\Delta_1$ for $S_5$ and $S_7$; when $E_j(0)=1.0$, $\Delta_3>\Delta_1$ for $S_3$, $S_5$ and $S_7$. Mode $j=2$, as in the lower energy situations, undergoes only small energy changes. The secondary waves $4\leqslant j\leqslant 9$ now undergo substantial energy changes in proportion to $\Delta_1$ or $\Delta_3$. Typical examples illustrating the large initial energy behavior at $S_7$ and $S_9$ are given in Figs. 2a and 2b, respectively.

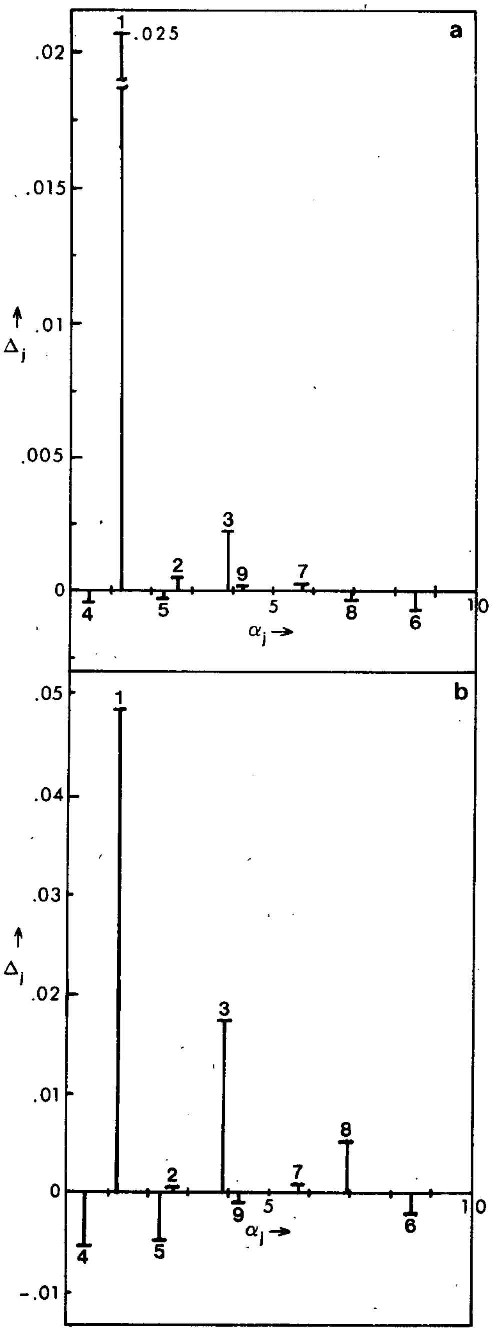

FIG. 1. Spectral energy changes $\Delta_j$ as a function of the zonal wavenumber $\alpha_j$ at $\mathrm{F}=8$ for (a) $S_9$, $E_j(0)=10^{-3}$ and (b) $S_9$, $E_j(0)=10^{-2}$. Integers identify the individual modes $j=1, 2, \ldots, N$ in $S_N$.

Overall we observe that the instability mechanism, which manifests itself through the marginal mode $j=1$, strongly dominates the energetics in $S_N$ at smaller initial energy levels. At larger energy levels, resonant interactions become increasingly important, especially through the primary triad. Energy gains/losses in the secondary pairs are in conformity with the integral constants derived in DNR [see DNR, Eq. (3.41), p. 28]. Mode $j=2$, which is involved in all triads, acts mainly as a catalyst for energy transfer between the marginal mode and the rest of $S_N$.

## 3. Spectral energetics at F = 12

At F = 12 it proves convenient to divide the discussion of spectral energy gains to very small $[E_j(0) \leqslant 10^{-3}]$, intermediate $[E_j(0) \approx 10^{-2}]$ and large $[E_j(0) \geqslant 10^{-1}]$ initial energy level situations.

At very small initial energy levels, $[E_j(0) \leqslant 10^{-3}]$, irrespective of the size of $S_N$, $N \leqslant 17$, the marginally unstable mode $j=1$ is characterized by the largest average energy gains. For spectra $S_N$, $N \leqslant 13$, energy gain in $j=3$ member of the primary triad is also substantial (several times greater than that of the other modes in $S_N$). For $S_{15}$ and $S_{17}$, however, $\Delta_3$ reduces to a value comparable in magnitude with that of the other modes, which undergo only small energy changes, and mode $j=1$ alone strongly dominates the spectrum. Examples depicting the situation when $E_j(0)=10^{-3}$ for $S_{11}$ and $S_{17}$ are given in Figs. 3a and 3b, respectively.

As the initial energy is increased to an intermediate level, $10^{-2} \leqslant E_j(0) < 10^{-1}$, the marginal mode $j=1$ dominates only the largest spectra $S_{15}$ and $S_{17}$. For $S_N$, $N \leqslant 13$, mode $j=3$ dominates, while $j=1$ shows comparatively insignificant gains. Spectra $S_{11}$ and $S_{13}$, which include the neutral secondary pair (10, 11), are characterized by a second strong peak associated with the $j=10$ mode. The remaining modes remain energetically relatively inactive. Behavior for $S_{11}$ and $S_{17}$, when $E_j(0)=10^{-2}$, is presented in Fig. 4a and 4b, respectively.

At large initial energy levels, $[E_j(0) \geqslant 10^{-1}]$, as at the intermediate levels, the marginal mode dominates only the largest spectra $S_{15}$ and $S_{17}$. Primary triad member $j=3$ still dominates $S_N$ for $N \leqslant 9$, but spectra $S_{11}$ and $S_{13}$ are now dominated by the secondary triad member $j=10$. Within each $S_N$, energy changes $\Delta_j$ associated with all the remaining modes are now quite substantial and comparable in magnitude. In $S_{15}$ and $S_{17}$, dominated by $j=1$, $j=3$ and 10 suffer energy losses, i.e., $\Delta_3<0$ and $\Delta_{10}<0$; in $S_N$, $N \leqslant 9$, dominated by $j=3$, $\Delta_1>0$ for $S_7$ and $S_9$ but $\Delta_1<0$ for $S_3$ and $S_5$; in $S_{11}$ and $S_{13}$, dominated by $j=10$, $\Delta_1>0$ and $\Delta_3>0$. Examples illustrating the large energy behavior, when $E_j(0)=1.2$, are given for $S_7$, $S_{11}$ and $S_{17}$ in Figs. 5a, 5b and 5c, respectively.

Overall we observe that at F = 12 the instability mechanism is capable of strongly dominating only the largest spectra, $S_{15}$ and $S_{17}$. For other spectra $S_N$, $N \leqslant 13$, other than at very small initial energy levels, the resonant interaction mechanism shifts the dominance to a neutral part of the $N$-wave spectrum. For $S_N$, $N \leqslant 9$, the dominant peak is associated with a neutral member of the primary triad (the $j=3$ mode). For $S_{11}$ and $S_{13}$ it remains there at smaller energy levels, but at larger energy levels it shifts to a member of a secondary triad (the $j=10$ mode). As at F=8,

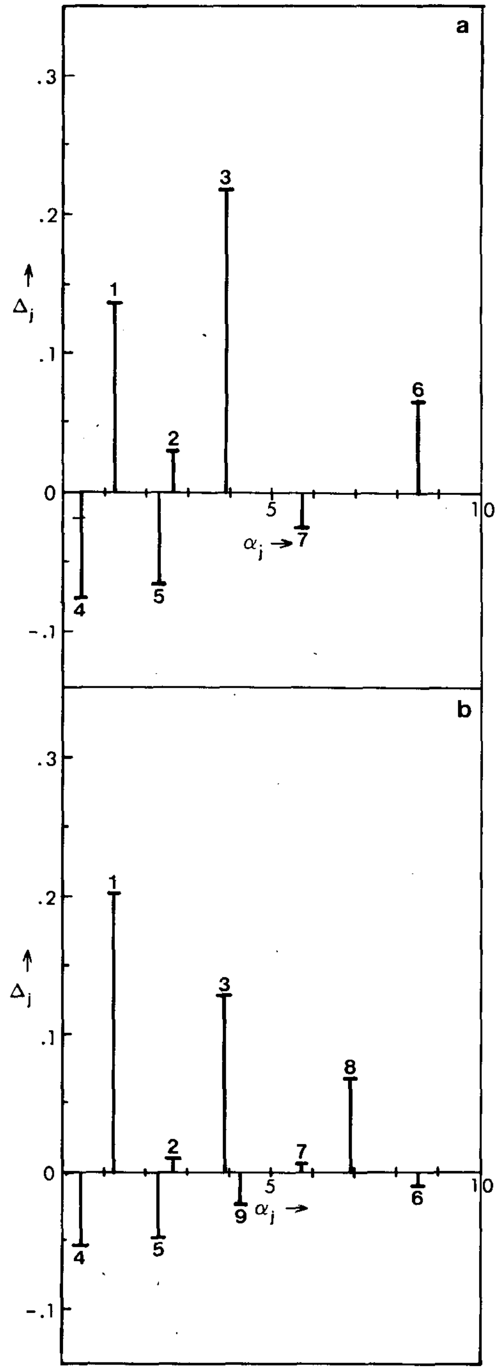

FIG. 2. As in Fig. 1 except for (a) $S_7$, $E_j(0)=10^{-1}$ and (b) $S_9$, $E_j(0)=10^{-1}$.

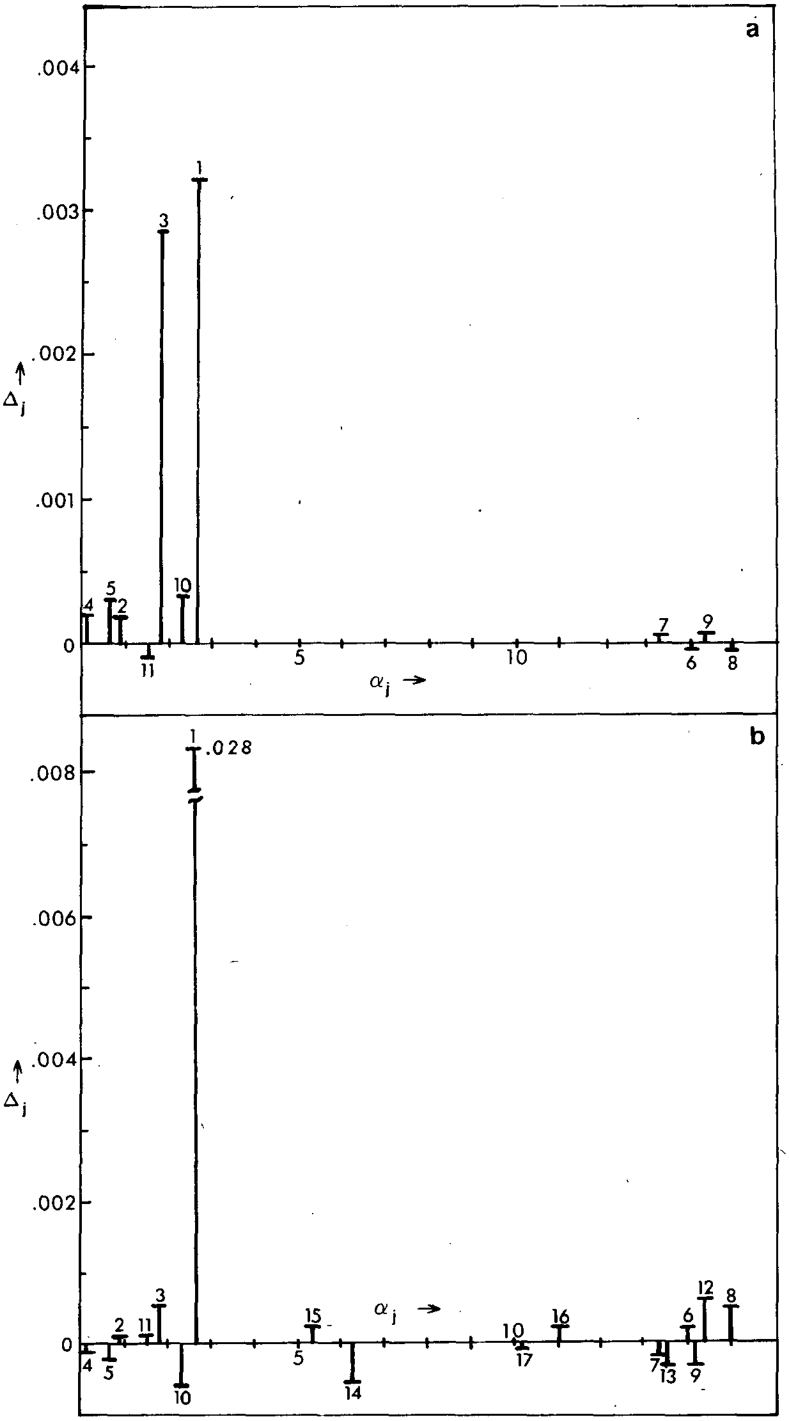

FIG. 3. Spectral energy changes $\Delta_j$ as a function of the zonal wavenumber $\alpha_j$ at F=12 for (a) $S_{11}$, $E_j(0)=10^{-3}$ and (b) $S_{17}$, $E_j(0)=10^{-3}$.

energy gains/losses in the secondary pairs are in conformity with the integral constraints in DNR, and mode $j=2$ appears mainly to play a catalytic role in transferring energy between the marginally unstable mode and the neutral modes.

## 4. Summary and remarks

The discussion in the preceding two sections may be summarized as follows: The instability mechanism, which manifests itself through the marginally unstable mode, dominates the spectral energetics, for all $S_N$, in

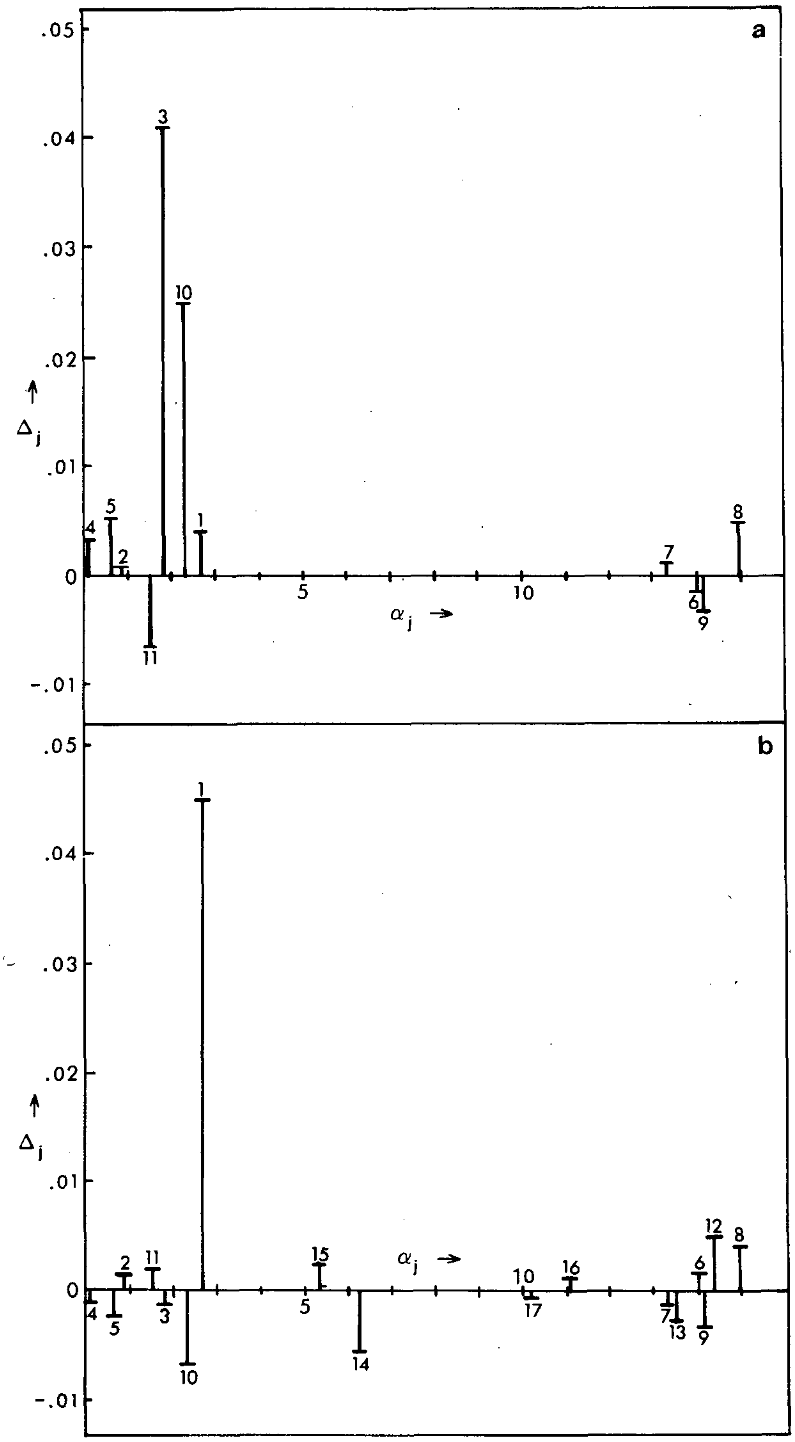

FIG. 4. As in Fig. 3 except for (a) $S_{11}$, $E_j(0)=10^{-2}$ and (b) $S_{17}$, $E_j(0)=10^{-2}$.

both F regions at small initial energy levels. This dominance is *stronger* for F<10.5, i.e., in gravitationally more stable situations, and persists there to initial energy levels an order of magnitude greater than for F>10.5 (cf. Figs. 1b and 4a). At sufficiently large initial energy levels [$E_j(0)\gtrsim 10^{-1}$ at F=8; $E_j(0)\gtrsim 10^{-2}$ at F=12] the resonant interaction mechanism shifts the dominance to a neutral part of the spectrum for all but the largest spectra ($S_9$ at F=8; $S_{15}$ and $S_{17}$ at F=12). For F<10.5 the shift is always to a neutral member of the primary triad (the $j=3$ mode). The same is true for F>10.5 at intermediate initial energy

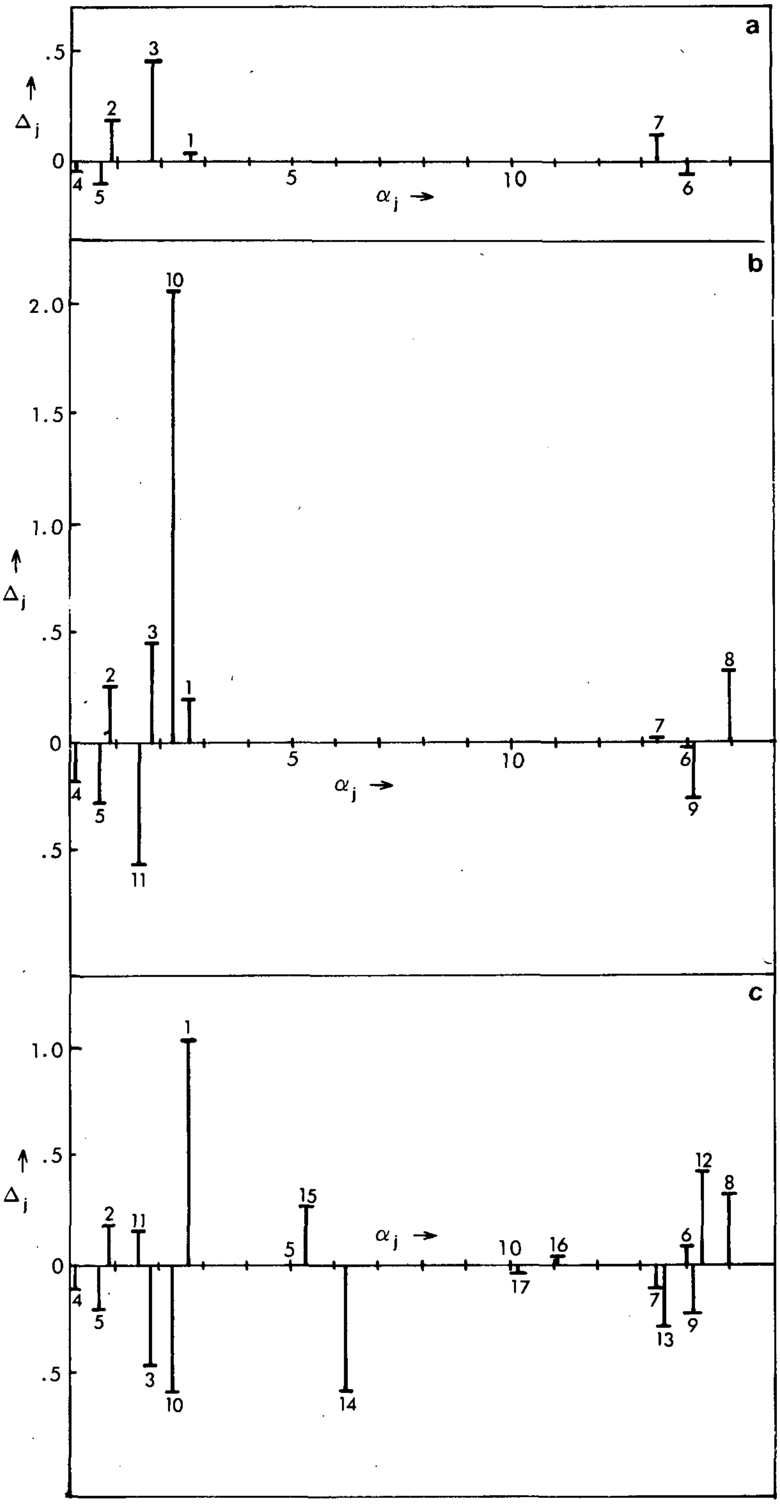

FIG. 5. As in Fig. 3 except for (a) $S_7$, $E_j(0)=1.2$, (b) $S_{11}$, $E_j(0)=1.2$ and (c) $S_{17}$, $E_j(0)=1.2$.

levels $[E_j(0)\sim10^{-2}]$ but at high initial energy levels $[E_j(0)\gtrsim10^{-1}]$ in case of larger spectra ($S_{11}$ and $S_{13}$) the shift is to a neutral member of a secondary triad (the $j=10$ mode). In both F regions, the primary triad member which couples the primary and secondary triads (the $j=2$ mode) undergoes comparatively inignificant energy changes; it's main role is that of a atalyst for energy transfer between the marginal mode nd the rest of the spectrum.

The above results are based on solutions to the $N$-wave problem assuming equal initial energy in all modes. We have also examined spectral energetics in situations in which the marginal mode is initially at energy levels an order of magnitude higher than that of the neutral modes. We found that spectral energy gains/losses in such situations maintained the same qualitative characteristics as those found for equal initial energy levels.

The model employed here is admittedly far too simple to correctly model large-scale atmospheric energetics. Certain gross features generated by the model may, however, have analogues in the atmosphere. For example, the dominant peaks in large-scale spectra may not necessarily be associated with the unstable modes, especially since wavenumber quantization in the atmosphere will restrict the number of resonant triads to relatively few. The shift of dominance to the neutral part of the spectrum is more likely to occur at larger values of internal rotational Fronde number, i.e., under gravitationally less stable conditions, such as would exist in the atmosphere during the warmer months. Validity of such conclusions must, of course, be checked against observations.

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