

## The Vertical Scale of an Unstable Baroclinic Wave and Its Importance for Eddy Heat Flux Parameterizations

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### ABSTRACT

Linear, quasi-geostrophic waves destabilized by a surface temperature gradient produce eddy potential vorticity fluxes which characteristically extend above the surface to a height

$$h_0 \equiv f_0^2 \frac{\partial \bar{u}}{\partial z} / N^2 \frac{\partial \bar{q}}{\partial y},$$

where the vertical shear  $\partial \bar{u} / \partial z$ , static stability  $N^2$  and potential vorticity gradient  $\partial \bar{q} / \partial y$  of the zonal flow are evaluated at the surface. Utilizing this result and a simple scaling analysis, we argue that the time-averaged, vertically integrated, poleward eddy heat flux is proportional to the fifth power of the meridional temperature gradient when  $h_0$  is much less than the scale height of the atmosphere.

### 1. Introduction

An important parameter entering Charney's (1947) classic model of a baroclinically unstable flow on a mid-latitude beta-plane is the ratio of the height

$$h \equiv f_0^2 \frac{\partial \bar{u}}{\partial z} / \beta N^2$$

to the scale height  $H$  of the atmosphere. ( $\partial \bar{u} / \partial z$  and  $N$ , the vertical shear and Brunt-Väisälä frequency of the basic state, are independent of height;  $f_0 \equiv f_0 + \beta y$  is the Coriolis parameter.) When  $h \ll H$ , the horizontal wavelength and the vertical extent of the most unstable wave are both proportional to the horizontal temperature gradient. The amplitude of the most unstable wave decays more or less exponentially above the ground with an  $e$ -folding proportional to  $h$ , while the horizontal wavelength of the most unstable wave is proportional to

$$\lambda_h \equiv N h / f_0 = f_0 \frac{\partial \bar{u}}{\partial z} / \beta N.$$

When  $h \gg H$ , however,  $H$  replaces  $h$  as the relevant vertical scale, and the most unstable wavelength no longer increases with increasing shear but asymptotes to a value proportional to  $\lambda_H \equiv N H / f_0$ . We suggest that these simple properties of baroclinic waves are of fundamental importance for the dependence of the

time-averaged eddy heat fluxes on the mean flow gradients.

If we assume that the appropriate vertical scale for eddy motions is fixed (e.g., proportional to the scale height) then plausible scaling arguments lead to the result, discussed by Stone (1972) and Green (1970), that the time-averaged poleward eddy heat flux is proportional to the square of the horizontal temperature gradient. If, however, the relevant vertical scale is itself proportional to the horizontal gradient, similar scaling arguments lead to a vertically averaged heat flux proportional to the fifth power of the horizontal gradient. We suggest that the square law is appropriate when  $h \gg H$ , and the fifth power law when  $h \ll H$ . Such behavior of the eddy heat flux may provide an explanation for the relationship noted by Smagorinsky (1963), and again more recently by Moura and Stone (1976), between observed extratropical vertical shears and the critical vertical shears required for instability in the two-level model of a baroclinic atmosphere.

We begin in Section 2 by determining the height above the surface up to which linear quasi-geostrophic waves destabilized by a surface temperature gradient can produce substantial eddy fluxes. We then proceed in Section 3 to discuss scaling arguments utilizing this vertical scale.

### 2. The vertical scale

We consider the equation of motion for inviscid, adiabatic, quasi-geostrophic flow on a mid-latitude

beta-plane as presented by Charney and Stern (1962), We now define

$$\frac{\partial q}{\partial t} = -J(\psi, q),$$

$$q \equiv \nabla^2 \psi + \beta y + \frac{f_0^2}{\rho_0} \frac{\partial}{\partial z} \left( \frac{\rho_0}{N^2} \frac{\partial \psi}{\partial z} \right),$$

$$\mathcal{H} \equiv \frac{\int_0^\infty \overline{\rho_0 v' q'} dz}{\rho_0(0) \overline{v' q'}(0)}.$$

with the boundary condition

$$\frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial z} \right) = -J \left( \psi, \frac{\partial \psi}{\partial z} \right) \text{ at } z=0,$$

and suitable boundedness conditions as  $z \rightarrow \infty$ . Here  $q$  is the potential vorticity and  $\psi$  the streamfunction of the horizontal flow,  $J$  the horizontal Jacobian,  $\nabla^2$  the horizontal Laplacian, and  $\rho_0 \equiv \rho_{00} e^{-z/H}$ . Let  $u$  and  $v$  be the eastward and northward velocities,

$$u \equiv -\frac{\partial \psi}{\partial y}, \quad v \equiv \frac{\partial \psi}{\partial x}.$$

Linearize about a zonal flow, denote zonal means by an overbar and deviations from zonal means by a prime, and define

$$M \equiv \int_0^\infty \overline{\rho_0 u' v'} dz.$$

The well-known identity

$$\overline{v' q'} = -\frac{\partial}{\partial y} (\overline{u' v'}) + \frac{f_0^2}{\rho_0} \frac{\partial}{\partial z} \left( \frac{\rho_0}{N^2} \overline{v' \frac{\partial \psi'}{\partial z}} \right)$$

then leads to the relation

$$\frac{\partial M}{\partial y} = -\int_0^\infty \overline{\rho_0 v' q'} dz - \frac{f_0^2 \rho_0}{N^2} \overline{v' \frac{\partial \psi'}{\partial z}} \Big|_{z=0},$$

assuming that

$$\frac{\rho_0}{N^2} \overline{v' \frac{\partial \psi'}{\partial z}} \rightarrow 0 \text{ as } z \rightarrow \infty.$$

Utilizing the linearized boundary condition and equation of motion at  $z=0$ , one can show that

$$\left. \frac{\overline{v' q'}}{\partial \bar{q} / \partial y} \right|_{z=0} = - \left. \frac{\overline{v' \frac{\partial \psi'}{\partial z}}}{\partial \bar{u} / \partial z} \right|_{z=0}.$$

We can therefore write

$$\frac{\partial M}{\partial y} = -\rho_0(0) \overline{v' q'}(0) \left[ \frac{\int_0^\infty \overline{\rho_0 v' q'} dz}{\rho_0(0) \overline{v' q'}(0)} - \frac{f_0^2 \frac{\partial \bar{u}}{\partial z}(0)}{N^2(0) \frac{\partial \bar{q}}{\partial y}(0)} \right].$$

We would like to argue that  $\mathcal{H}$  is a measure of the vertical extent of the eddy fluxes. For this identification to make any sense, we must first of all restrict our attention to zonal flows for which

$$\frac{\partial \bar{q}}{\partial y} \equiv \beta - \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{f_0^2}{\rho_0} \frac{\partial}{\partial z} \left( \frac{\rho_0}{N^2} \frac{\partial \bar{u}}{\partial z} \right)$$

is everywhere positive definite, so that (according to Charney and Stern) unstable waves exist only if  $\partial \bar{u} / \partial z|_{z=0} > 0$  at some latitude. For an unstable wave,  $\overline{v' q'}$  must then be negative definite everywhere since the eddy potential vorticity flux must be down-gradient (see, e.g., Held, 1975). In this case, the integrand in our definition of  $\mathcal{H}$  cannot change sign. We must assume, in addition, that  $\overline{v' q'}$  does not have a great deal of structure as a function of height. This is not generally true for all unstable waves. If, for example, the wave is only very weakly unstable,  $\overline{v' q'}$  will have a great deal of structure near the steering level. The ultralong wave instabilities displayed by Geisler and Garcia (1977) also have considerable vertical structure. But the most strongly unstable waves on Charney's idealized flow and on more realistic atmospheric flows (Gall, 1976; Simmons and Hoskins, 1976) have little vertical structure. For these strongly unstable waves, at least,  $\mathcal{H}$  is a reasonable measure of the vertical extent of the eddy potential vorticity flux.

For the separable problem in which  $\bar{u}$  and  $N$  are independent of  $y$  (so that  $\partial M / \partial y \equiv 0$ ),  $\mathcal{H}$  is also independent of  $y$  and equal to

$$h_0 \equiv \frac{f_0^2 \frac{\partial \bar{u}}{\partial z}(0)}{N^2(0) \frac{\partial \bar{q}}{\partial y}(0)}. \tag{1}$$

If, in addition,  $\partial \bar{u} / \partial z$  and  $N$  are independent of  $z$ , then

$$\frac{\partial \bar{q}}{\partial y} = \beta + \frac{f_0^2}{N^2 H} \frac{\partial \bar{u}}{\partial z} = \beta \left( 1 + \frac{h}{H} \right)$$

and

$$h_0 = \frac{h}{1 + h/H},$$

where, as before,

$$h \equiv f_0^2 \frac{\partial \bar{u}}{\partial z} / \beta N^2.$$

If  $h/H \gg 1$  we have  $h_0 \approx H$ , and the eddy fluxes due to the most unstable waves extend throughout the bulk of the atmosphere. If  $h/H \ll 1$ , we have instead  $h_0 \approx h$ , and the vertical extent of the eddy fluxes is proportional to the horizontal temperature gradient.

In this simplest of problems, with  $\partial \bar{u} / \partial z$  and  $N$  independent of  $y$  and  $z$  and with  $h/H \ll 1$ , one can see more precisely how  $h$  controls the vertical scale. The perturbation equation of motion and boundary condition become

$$\frac{f_0^2}{N^2} \frac{\partial^2 \bar{\psi}}{\partial z^2} = \left( k^2 - \frac{\beta}{\bar{u} - c} \right) \bar{\psi},$$

$$\frac{\partial \bar{\psi}}{\partial z} = \frac{1}{\bar{u} - c} \frac{\partial \bar{u}}{\partial z} \bar{\psi} \quad \text{at } z=0,$$

where

$$\bar{\psi}' = \text{Re} \bar{\psi} e^{ik(x-ct)}.$$

By defining

$$z^* \equiv z/h, \quad k^* \equiv k \frac{Nh}{f_0} \quad \text{and} \quad c^* \equiv \frac{c}{h \partial \bar{u} / \partial z},$$

we can eliminate all reference to the parameters describing the mean flow:

$$\frac{\partial^2 \bar{\psi}}{\partial z^{*2}} = \left( k^{*2} - \frac{1}{z^* - c^*} \right) \bar{\psi},$$

$$\frac{\partial \bar{\psi}}{\partial z^*} = \frac{1}{z^* - c^*} \bar{\psi} \quad \text{at } z=0.$$

If  $k^* = K^*$  is the nondimensional wavenumber of the most unstable normal mode in this reduced problem and  $\bar{\psi} = g(z^*)$  is the corresponding eigenfunction, then, in terms of dimensional variables, the most unstable wavelength is

$$\frac{2\pi}{K^*} \left( \frac{Nh}{f_0} \right)$$

and the vertical structure of the corresponding eigenfunction is  $g(z/h)$ . In this simplest case, therefore, the vertical extent of the most unstable wave is indeed proportional to  $h$ , and its horizontal scale proportional to  $Nh/f_0$ .

In the general nonseparable problem, vertically integrated momentum fluxes exist to the extent that the eddy scale  $\mathcal{H}(y)$  is unable to adjust to the imposed vertical scale  $h_0(y)$ . [If, as seems plausible,

$\mathcal{H}(y)$  is a smoother function of  $y$  than is  $h_0(y)$ , the vertically integrated flux of westerly momentum will be directed into regions of large  $h_0$ .] However, momentum conservation implies that a positively weighted average of  $\mathcal{H}(y) - h_0(y)$  must vanish, i.e.,

$$\int dy [-\overline{v'q'}(0)] (\mathcal{H} - h_0) = 0,$$

so we again feel justified in thinking of  $h_0$  averaged over the region of substantial eddy activity as the mean vertical extent of the eddy fluxes in that region.

There is an intimate connection between the vertical scale  $h_0$  and the critical shear for baroclinic instability in the quasi-geostrophic two-layer model. Using the definition (1) of  $h_0$ , we can write

$$h_0 \frac{\partial \bar{u}}{\partial z}(0) = \frac{N^2(0)h_0^2}{f_0^2} \frac{\partial \bar{q}}{\partial y}(0). \quad (2)$$

The critical westerly shear for baroclinic instability in the two-layer model (Pedlosky, 1964) is

$$\bar{u}_1 - \bar{u}_2 = \frac{g(\bar{\rho}_1 - \bar{\rho}_2)}{\bar{\rho}_2} \frac{D_2}{f_0^2} \left( \beta - \frac{\partial^2 \bar{u}_2}{\partial y^2} \right). \quad (3)$$

(The subscripts 1 and 2 refer to the upper and lower layers respectively.  $D_i$  and  $\bar{\rho}_i$  are the depth and density of the  $i$ th layer.) The analogy with (2) is evident. The two-layer model predicts instability when vertical shears are sufficient to produce unstable waves which extend into the model's upper layer. More generally, the critical westerly vertical shear for instability in a multi-layer model will be proportional to the depth of the lowest model layer.

### 3. The scaling analysis

We now consider the eddy fluxes in the statistically steady state of a baroclinically unstable atmosphere on a beta-plane. The following scaling analysis is basic to Stone's (1972) parameterization of the eddy heat flux, and suggests the modifications required for consistency with our discussion of the vertical extent of the eddy fluxes. (Throughout this section, overbars will refer to time averages and primes, deviations from time averages.)

An eddy diffusivity  $\mathfrak{D}$  must have the dimensions of a velocity  $V$  multiplied by a length  $L$ . The relevant velocity scale is taken to be that determined by the vertical shear of the zonal wind and a vertical scale  $d$  (as yet unspecified), i.e.,

$$V \equiv d \partial \bar{u} / \partial z.$$

The characteristic length is taken to be the radius

of deformation corresponding to this vertical scale,

$$L \equiv L_d = \frac{Nd}{f_0}$$

so that the characteristic time scale

$$\tau \equiv \left(\frac{Nd}{f_0}\right) / \left(\frac{\partial \bar{u}}{\partial z}\right) = N / \left(f_0 \frac{\partial \bar{u}}{\partial z}\right)$$

is inversely proportional to the growth rate of the most unstable waves on the time averaged zonal flow. Therefore,

$$\mathcal{D} \propto Nd^2 \frac{\partial \bar{u}}{\partial z} / f_0,$$

and

$$\frac{g}{\Theta_0} \overline{v'\Theta'} \propto \mathcal{D} \frac{g}{\Theta_0} \frac{\partial \bar{\Theta}}{\partial y} \propto Nd^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2$$

using the fact that the time-averaged flow is in geostrophic balance, i.e.,

$$\frac{g}{\Theta_0} \frac{\partial \bar{\Theta}}{\partial y} = -f_0 \frac{\partial \bar{u}}{\partial z},$$

where  $g$  is the gravitational acceleration and  $\Theta_0$  some mean potential temperature.

Alternatively, we can argue that

$$\Theta' \propto L_d \frac{\partial \bar{\Theta}}{\partial y} \quad \text{and} \quad v' \propto \frac{g}{N} \frac{\Theta'}{\Theta_0},$$

so that, once again,

$$\frac{g}{\Theta_0} \overline{v'\Theta'} \propto \frac{1}{N} \left(\frac{g}{\Theta_0} \Theta'\right)^2 \propto Nd^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2.$$

That is, we require that the kinetic energy in the relevant eddies be proportional to their available potential energy, and that this eddy available potential energy, in turn, be proportional to the zonal available potential energy within a latitudinal span of width  $L_d$ .

In Stone's analysis,  $d=H$ , and as a result the horizontal potential temperature flux is proportional to

$$\left(\frac{\partial \bar{\Theta}}{\partial y}\right)^2 \left(\frac{\partial \bar{\Theta}}{\partial z}\right)^{\frac{1}{2}}.$$

We suggest that this parameterization is appropriate only when  $h \gg H$ , that is, when the stabilizing effect of  $\beta$  is negligible. When  $h \ll H$ , the previous analysis suggests that the eddy fluxes will be confined to a region above the surface of depth proportional to  $h$ .

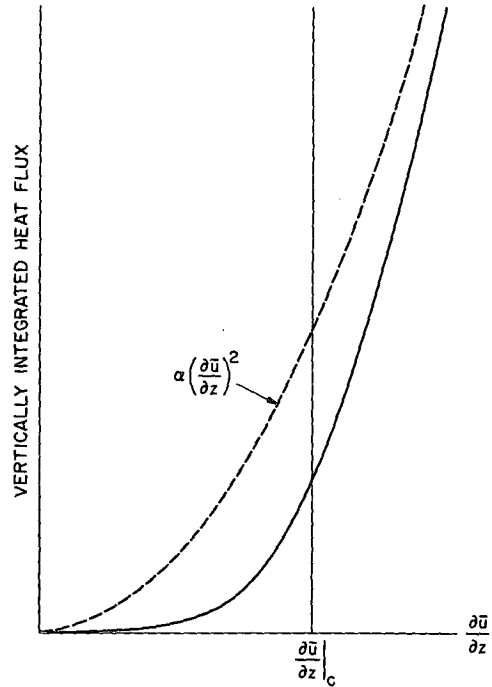


FIG. 1. The vertically averaged heat flux as a function of the vertical shear of the zonal wind. The solid line is the behavior of the heat flux suggested by the scaling arguments discussed in the text. The dotted line is the parabola to which the solid line asymptotes for large vertical shears.

We therefore choose  $d=h$  in this limit, so that

$$\frac{g}{\Theta_0} \overline{v'\Theta'} \propto f_0^4 \left(\frac{\partial \bar{u}}{\partial z}\right)^4 / N^3 \beta^2.$$

But in this case the eddy fluxes occupy a region which contains only a fraction  $h/H$  of the mass of the atmosphere. The vertically integrated heat flux will therefore be proportional to

$$\frac{Nh^3}{H} \left(\frac{\partial \bar{u}}{\partial z}\right)^2 = f_0^6 \left(\frac{\partial \bar{u}}{\partial z}\right)^5 / N^5 \beta^3,$$

that is, proportional to

$$\left(\frac{\partial \bar{\Theta}}{\partial y}\right)^5 \left(\frac{\partial \bar{\Theta}}{\partial z}\right)^{-\frac{1}{2}}.$$

In Fig. 1 we plot the dependence of the poleward eddy heat flux on  $\partial \bar{u} / \partial z$  suggested by these scaling arguments, holding all other parameters fixed. Also plotted is the square law to which the heat flux asymptotes for

$$\frac{\partial \bar{u}}{\partial z} \gg \left.\frac{\partial \bar{u}}{\partial z}\right|_c \equiv \beta N^2 H / f_0^2.$$

Compared with this square law, the flux drops off

much more rapidly as  $\partial\bar{u}/\partial z$  drops below  $\partial\bar{u}/\partial z|_c$ , as the eddies begin to shrink in both vertical and horizontal extent and thereby become less efficient in transporting heat. (Just how rapidly this dropoff occurs depends on the details of the transition from the square law to fifth power law asymptotic regimes.) As a result, one might expect  $\partial\bar{u}/\partial z \approx \partial\bar{u}/\partial z|_c$  for a rather wide range of radiative forcing of the mean state. It has been noted by Smagorinsky (1963) and by Moura and Stone (1976) that the latitudinal structure of the observed time-averaged extratropical vertical shears is well fit by a function proportional to  $\cos\theta/\sin^2\theta$ . Ignoring latitudinal variations in  $N$  and  $H$ , this is the latitudinal dependence of  $\partial\bar{u}/\partial z|_c$ , or, equivalently, the latitudinal dependence of the critical shear required for instability in the two-layer model [setting  $\partial^2\bar{u}/\partial y^2=0$  in (3)]. A local relationship between vertical shear and heat flux of the form displayed in Fig. 1 might, therefore, help explain these observations.

These scaling arguments are evidently best suited to a system in which the eddy statistics have negligible variations over length scales comparable to  $NH/f_0$ . In effect, we have assumed that the eddy heat flux is independent of all other length scales characterizing the system when  $NH/f_0$  is much smaller than these other length scales. Similarly, we have assumed that the heat flux is independent of all explicit radiative and dissipative time scales when  $\tau \equiv N/(f_0\partial\bar{u}/\partial z)$  is much smaller than these other time scales, and that the heat flux is independent of other vertical scales in the problem (in particular, independent of the scale height  $H$ ) when  $h$  is much smaller than these vertical scales, but that the heat flux is independent of  $h$  (i.e., independent of  $\beta$ ) when  $h$  is much larger than other relevant vertical scales.

The validity of these assumptions is not self-evident. One can, for example, question the choice of the radius of deformation as the relevant eddy meridional scale and "mixing length." After all, the most unstable wave on a zonal flow that is independent of  $y$  has the largest possible meridional scale. Pedlosky (1975), however, has shown how such a wave with no meridional structure is itself unstable to waves with meridional structure on the order of the radius of deformation—lending some support to the assumption that the radius of deformation asserts itself as

the characteristic meridional as well as zonal eddy scale in the statistically steady state.

Even if these assumptions are valid, in trying to understand eddy heat fluxes in the earth's atmosphere we have the additional problem that the radius of deformation is not, in fact, very much smaller than length scales characterizing the time-averaged flow. One cannot, therefore, expect to find a local relationship between time-averaged heat fluxes and temperature gradients. Along with the complexities introduced by latent heat release, this problem prevents any meaningful detailed comparisons between this theory and observations.

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