Finite-Amplitude Stability of Rossby Wave Flow

ARTHUR Z. LOESCH

Atmospheric Science Department, State University of New York at Albany, Albany, N.Y. 12222

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ABSTRACT

Finite-amplitude stability characteristics of Rossby wave flow are investigated in the context of the inviscid barotropic model on a beta plane. It is shown that a superimposed disturbance, unstable in the linear sense, grows as long as it lags the basic Rossby wave. However, when the disturbance becomes sufficiently large, it alters the phase and the amplitude of the Rossby wave flow. The phase correction of the Rossby wave is to the west and, in time, large enough to reverse the phase relation between the disturbance and the basic wave. At the time when the two become in phase, the growth of the disturbance is halted and subsequently, when the disturbance leads the Rossby wave, the disturbance slowly decays out. The basic Rossby wave equilibrates with a phase and amplitude which differ from their initial values.

1. Introduction

The vast majority of theoretical investigations of stability of atmospheric flows has dealt with flows which are independent of time and longitude. Once the superimposed perturbations reach a finite-amplitude or a fully developed eddy stage by extracting a substantial amount of energy from the zonal flow via either the baroclinic or barotropic instability mechanisms, or a combination of both, the total flow pattern becomes nonsteady and zonally nonuniform. A question of fundamental importance arises: Are the fully developed zonally nonuniform flows stable with respect to further perturbations? For barotropic Rossby wave motion instability has been established by Lorenz (1972). By employing a linear perturbation analysis, Lorenz demonstrated that Rossby *et al.*'s (1939) original solution to the barotropic vorticity equation (representing an east–west propagating planetary wave embedded in a constant westerly flow) is unstable provided the wave amplitude is sufficiently large or its (zonal) wavenumber is sufficiently high. A generalization of Lorenz' study to that of a Rossby wave with wavenumber in an arbitrary direction was made by Gill (1974). For baroclinic wave motion instability has been established by Pedlosky (1975) and Yamagata (1976) in the context of the two-layer inviscid baroclinic model.

It is clearly of interest to determine how the nature of this instability is affected by the nonlinear processes, which become increasingly important as the perturbations superimposed upon the basic wave state reach a finite-amplitude stage. The effect of nonlinearity for small local Rossby number M, i.e., when the inertial effects associated with the basic wave are weak compared to the β-effect, have been discussed by Gill (1974). He showed that, in this limit, the basic Rossby wave and the perturbation waves may be regarded as a triad of resonantly interacting planetary waves, a situation studied by Longuet-Higgins and Gill (1967). Lorenz (1972) imposed cyclic continuity for boundary conditions which, in agreement with large-scale atmospheric flows, restricts all wavenumbers (zonal and meridional) to integers. Under this constraint a perturbation characterized by a given meridional wavenumber l, *does not* satisfy the conditions for resonance with a zonally propagating basic Rossby wave. Consequently, the perturbation becomes stable for M below a certain threshold value. For the atmosphere M is order unity (Lorenz chose $M=2$). In this region, as pointed out by Gill, the basic Rossby wave instability is a Rayleigh-type instability.

The present investigation represents an extension of the linear stability study of Lorenz (1972) into the nonlinear region, while the perturbation is *finite* but *small*. Under this assumption, the nonlinear analysis may be carried out using an asymptotic expansion for the perturbation flow and the procedure of multiple time scales. The reasons for choosing Lorenz' study as the basis for the finite-amplitude study are 1) closest correspondence with the atmosphere (the other studies are more applicable to the oceans), 2) absence of resonant interactions (to complicate the problem) and 3) relative mathematical simplicity (because the wavenumber of the basic Rossby wave is in the zonal direction).

The basic questions to which this study addresses itself, which cannot be answered by the linear theory,

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are as follows: 1) What is the finite-amplitude evolution of the (linearly unstable) perturbation state? 2) How does the finite-amplitude perturbation state affect in time the basic Rossby wave flow? Answers to such questions exist for cases where the basic flows are zonally uniform. For example, in his finite-amplitude study of unstable baroclinic waves, Pedlosky (1970) found that in the absence of dissipation, the finite-amplitude state exhibits a long-period oscillation of both the basic zonal flow and the baroclinic wave. It will be of interest to see what similarities or differences arise in the finite-amplitude state when the basic state is zonally nonuniform.

2. Governing equations

We consider a shallow layer of incompressible inviscid homogeneous fluid of infinite horizontal extent and depth D rotating with angular velocity Ω. Following Rossby *et al.* (1939), the effects of the earth's sphericity on the motion can be accounted for by taking Ω as a linear function of the latitude y', i.e.,

$$2\Omega=f_0+\beta' y'.$$

With the scales $(L, D, L/U, U, DU/L)$ chosen to represent the horizontal coordinates, the vertical coordinate, time, horizontal velocity and vertical velocity, respectively, the equation which governs the fluid motion for small Rossby number $\mathrm{Ro}=U/f_0L$ is the quasi-geostrophic vorticity equation

$$\frac{\partial}{\partial t}\nabla^2\psi+J(\psi,\ \nabla^2\psi+\beta y)=0, \tag{2.1}$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$

$$J(f,g)=\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

In (2.1) x and y are the nondimensional longitude and latitude coordinates (scaled with L), ψ is the nondimensional geostrophic streamfunction (scaled with UL), and the parameter

$$\beta=\frac{\beta' L^2}{U} \tag{2.2}$$

is the planetary vorticity factor and is a reciprocal of the parameter M defined by Gill (1974).

Eq. (2.1) permits, as an exact solution, a wave in the form

$$\psi_0=-Uy+A\sin N(x-ct), \tag{2.3}$$

provided

$$c=U-\beta/N^2. \tag{2.4}$$

Eq. (2.4) is Rossby *et al.*'s (1939) expression for the phase speed c of a planetary wave of wavenumber N embedded in a constant westerly current U. For the atmosphere, on choosing $L=R\cos\phi$ (where R is the earth's radius and ϕ the latitude), N becomes an integer representing the number of waves around a given latitude circle.

The stability analysis of the basic wave flow (2.3) may be carried out by taking

$$\psi=\psi_0+\psi', \tag{2.5}$$

where ψ' represents a small (but not necessarily infinitesimal) perturbation of the wave flow. Substitution of (2.5) and (2.3) into (2.1), then yields an equation for ψ', i.e.,

$$\left[\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}+NA\cos N(x-ct)\frac{\partial}{\partial y}\right]\nabla^2\psi' +\left[\frac{\beta}{N^2}\frac{\partial}{\partial x}+NA\cos N(x-ct)\frac{\partial}{\partial y}\right]N^2\psi' +J(\psi',\nabla^2\psi')=0. \tag{2.6}$$

To eliminate the time dependence in coefficient $\cos N(x-ct)$ in (2.6) we transform coordinate x to $x_0=x-ct$, which moves with the basic wave flow. In the coordinate system (x_0,y,t), Eq. (2.6) becomes

$$\frac{\partial}{\partial t}\nabla^2\psi'+\left(\frac{\beta}{N^2}\frac{\partial}{\partial x_0}+NA\cos Nx_0\frac{\partial}{\partial y}\right)\times(\nabla^2\psi'+N^2\psi')+J(\psi',\nabla^2\psi')=0. \tag{2.7}$$

3. The linear stability analysis

The analysis of (2.7) when ψ' is infinitesimal, in which case the nonlinear Jacobian term $J(\psi',\nabla^2\psi')$ may be ignored with respect to the remaining linear terms, has been carried out in detail by Lorenz (1972). As an aid in the development of the finite-amplitude theory, this section recapitulates the important aspects of the linear theory.

Upon linearization, (2.7) becomes

$$\frac{\partial}{\partial t}\nabla^2\psi'+\left(\frac{\beta}{N^2}\frac{\partial}{\partial x_0}+NA\cos Nx_0\frac{\partial}{\partial y}\right)\times(\nabla^2\psi'+N^2\psi')=0, \tag{3.1}$$

whose simplest solution has the form

$$\psi'=\sum_{n=-\infty}^{\infty}X_n e^{i(nNx_0+ly-\sigma t)}+*. \tag{3.2}$$

Based on a result of Fjortøft (1953) for barotropic flow, the energy lost by a particular mode (in our case the Rossby wave of wavenumber N) must be transferred in part to a larger wavenumber and in part to a smaller wavenumber. Thus (3.2) can represent an amplifying solution of (3.1) *only* if $l<N$. For the atmosphere l, like N, must be an integer. A direct substitution of (3.2) into (3.1) yields an infinite set of homogeneous algebraic equations, each connecting three consecutive amplitudes, X_{n-1}, X_n, X_{n+1}. These may be written

$$Y_{n-1}+\frac{2}{NAl}\left[\left(\frac{n^2N^2+l^2}{N^2-n^2N^2-l^2}\right)\sigma+n\frac{\beta}{N}\right]\times Y_n+Y_{n+1}=0, \quad (3.3)$$

where

$$Y_n=(N^2-n^2N^2-l^2)X_n. \quad (3.4)$$

Lorenz obtained approximate solutions to (3.3) by a truncated expansion technique. He defined the Mth approximation to (3.3) as a system of $(2M+1)$ equations within (3.3) which retain at least two terms when Y_n is set equal to zero for $n>M$. Thus, to first approximation, (3.3) reduces to

$$\left.\begin{aligned} -\frac{2}{NAl}\left[\left(\frac{N^2+l^2}{l^2}\right)\sigma+\frac{\beta}{N}\right]Y_{-1}+Y_0=0\\ Y_{-1}+\frac{2}{NAl}\left(\frac{l^2}{N^2-l^2}\right)\sigma Y_0+Y_1=0\\ Y_0-\frac{2}{NAl}\left[\left(\frac{N^2+l^2}{l^2}\right)\sigma-\frac{\beta}{N}\right]Y_1=0 \end{aligned}\right\}. \quad (3.5)$$

Eqs. (3.5) have a nontrivial solution provided

$$\sigma=\pm\frac{\beta l^2}{N(N^2+l^2)}\left[1-\frac{N^4(N^4-l^4)}{2\beta^2l^2}A^2\right]^{\frac{1}{2}}. \quad (3.6)$$

Therefore, we have an amplifying perturbation, (i.e., σ becomes imaginary) only if $l<N$ and if the basic wave flow-amplitude A exceeds a certain critical value A_c given by

$$A_c=\frac{2^{\frac{1}{2}}\beta l}{N^2(N^4-l^4)^{\frac{1}{2}}}. \quad (3.7)$$

If the amplitude of the basic wave flow exceeds the critical value by a small increment Δ, i.e.,

$$A=A_c+\Delta, \quad \Delta\ll A_c, \quad (3.8)$$

we find from (3.6) that

$$\sigma=i\sigma_i=\pm i\frac{2^{\frac{3}{4}}\beta l^2}{NA_c^{\frac{1}{2}}(N^2+l^2)}|\Delta|^{\frac{1}{2}}. \quad (3.9)$$

We conclude that when the amplitude of the basic wave flow is Δ above the critical value, the resultant growth rate of the perturbation is $O(|\Delta|^{\frac{1}{2}})$. It is interesting to note that a similar result was obtained by Pedlosky (1970) in the context of inviscid baroclinic instability theory.

The first approximation to the solution (3.2) follows from (3.5), (3.6) and (3.4). On taking X_0, which is arbitrary, as real, we find, with the help of (3.6) and (3.7) that at $A=A_c$ we have a (marginally) neutral perturbation

$$\psi'=2X_0(\cos ly-2C_1\sin ly\sin Nx_0), \quad (3.10)$$

whereas at $A=A_c+\Delta$, we have an amplifying perturbation

$$\psi'=2X_0e^{\sigma_it}(\cos ly-2C_1\sin ly\sin Nx_0 - |\Delta|^{\frac{1}{2}}2S_1\sin ly\cos Nx_0), \quad (3.11)$$

where

$$C_1=\frac{N^2A_c}{2\beta l}(N^2-l^2), \quad (3.12)$$

$$S_1=\frac{A_c^{\frac{1}{2}}N^2}{2^{\frac{1}{4}}\beta l}(N^2-l^2). \quad (3.13)$$

Comparison between (3.10) and (3.11) shows that the $n=1$ component of the amplifying perturbation is phase shifted by arctan $(|\Delta|^{\frac{1}{2}}S_1/C_1)$ to the west of the marginal perturbation.

Higher approximations, by imposing a less severe truncation on (3.2) and (3.3), give more accurate estimates of σ_i and A_c. However, within the appropriate range of β and N for large-scale midlatitude disturbances, these estimates differ by less than 10% from those obtained from the first approximation. This fact is illustrated for A_c in Fig. 1, which is the plot of the neutral stability curve A_c versus the meridional wavenumber l (taken as a continuous variable) for $\beta=1$ and $N=6$. The solid line represents A_c based on the first approximation [Eq. (3.7)], whereas the dashed line represents A_c based on the third approximation. Table 1 compares the coefficients of $n=\pm1, \pm2, \pm3$ to $n=0$ in the solution (3.2) at $A=A_c$ for $\beta=1$, $N=6$, based on the first three approximations. As demonstrated by Lorenz (1972, p. 262), these results show that the first approximation gives excellent results for smaller values of l, whereas the third approximation would give results hardly distinguishable from correct values for all allowable values of l. In view of this, and the fact that the basic wave flow is *most* unstable to disturbances with *small* meridional wavenumber, the finite-amplitude analysis of (2.7), which follows, will also be truncated to the first approximation. This will immensely simplify the algebra associated with the finite-amplitude problem.

4. The finite-amplitude stability analysis

We consider the evolution of the perturbation flow in the vicinity of the neutral stability curve. Thus, the increment Δ by which the amplitude of the basic wave flow exceeds the critical value A_c is small, i.e.,

$$A-A_c=\Delta\ll A_c. \quad (4.1)$$

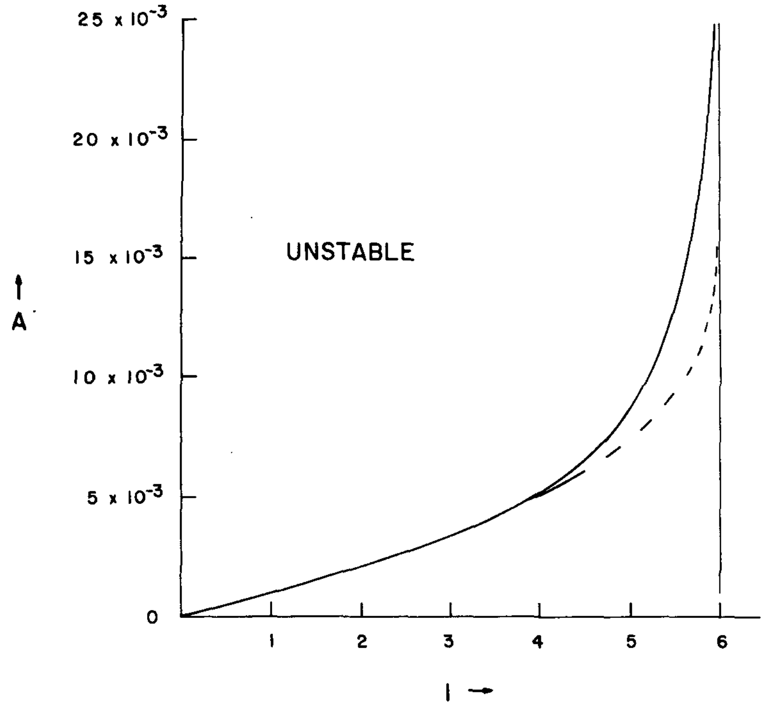

FIG. 1. Neutral stability curve $A=A_c$ as a function of the meridional wavenumber l based on the first approximation (solid line) and third approximation (dashed line).

Based on the fact that under constraint (4.1) the linear theory predicts the real part of σ to vanish and the imaginary part of σ (i.e., the growth rate σ_i) of $O(|\Delta|^{\frac{1}{2}})$, we consider the finite-amplitude perturbation field to to be a function of x_0, y and a slow time T defined by

$$T=|\Delta|^{\frac{1}{2}}t. \tag{4.2}$$

Thus,

$$\psi'=\psi'(x_0,y,T). \tag{4.3}$$

Substituting (4.1), (4.2) and (4.3) into (2.7) yields

$$\left(\frac{\beta}{N^2}\frac{\partial}{\partial x_0}+NA_c\cos Nx_0\frac{\partial}{\partial y}\right)(\nabla^2\psi'+N^2\psi')$$
$$=-|\Delta|^{\frac{1}{2}}\frac{\partial}{\partial T}\nabla^2\psi'-\Delta N\cos Nx_0\frac{\partial}{\partial y}(\nabla^2\psi'+N^2\psi')$$
$$-J(\psi',\nabla^2\psi'). \tag{4.4}$$

The appropriate scale for the perturbation amplitude in terms of the parameter Δ, obtained by demanding that the linear destabilization mechanism be of the same order as the nonlinear effects in (4.4), is $O(|\Delta|^{\frac{1}{2}})$, i.e., of the same order as the long time scale. Therefore, the anticipated asymptotic expansion for the perturbation field ψ' has the form

$$\psi'=|\Delta|^{\frac{1}{2}}\psi^{(1)}+|\Delta|\psi^{(2)}+|\Delta|^{\frac{3}{2}}\psi^{(3)}+\cdots. \tag{4.5}$$

Substituting (4.5) into (4.4) yields a sequence of linear problems for $\psi^{(j)}$, which we consider in detail next.

The lowest order, i.e., $O(|\Delta|^{\frac{1}{2}})$, problem, given by

$$\left(\frac{\beta}{N^2}\frac{\partial}{\partial x_0}+NA_c\cos Nx_0\frac{\partial}{\partial y}\right)(\nabla^2\psi^{(1)}+N^2\psi^{(1)})=0, \tag{4.6}$$

leads to the specification of the perturbation on the marginal stability curve $A=A_c$. In general, the form of perturbation is as given by (3.2) with $\sigma=0$, i.e.,

$$\psi_h^{(1)}=e^{ily}\sum_n X_n(T)e^{inNx_0}+*. \tag{4.7}$$

TABLE 1. Comparison between X_m and X_0 for the first three approximations, $M=1$, 2 and 3 at $A=A_c$, $\beta=1$, $N=6$ and $l=1$, 3 or 5.

l	1			3			5		
M	1	2	3	1	2	3	1	2	3
X_1/X_0	0.687	0.687	0.687	0.548	0.546	0.547	0.300	0.294	0.294
X_2/X_0	0	0.619×10^{-4}	0.619×10^{-4}	0	0.376×10^{-2}	0.378×10^{-2}	0	0.156×10^{-1}	0.162×10^{-1}
X_3/X_0	0	0	0.152×10^{-6}	0	0	0.89×10^{-4}	0	0	0.128×10^{-2}

The amplitudes X_n can be determined in terms of X_0 with the help of (3.3) and (3.4). We can write

$$X_n(T)=C_nX_0(T), \tag{4.8}$$

where the exact form of C_n in terms of β, l and N depends on the degree of approximation M, and where it can be shown that $C_n^*=C_n$ and $C_{-n}=(-1)^nC_n$. Relation (4.8) connecting the amplitudes is valid in the finite-amplitude analysis *only* in the absence of resonant interactions between the perturbation modes in (4.7) and the basic wave flow [otherwise each $X_n(T)$ must be treated independently of $X_0(T)$]. As discussed by Gill (1974), this indeed is the case for the perturbation flow (3.2) and we have

$$\psi_h^{(1)}=X_0(T)e^{ily}\sum_n C_ne^{inNx_0}+*, \tag{4.9}$$

with

$$C_{-n}=(-1)^nC_n \quad \text{and} \quad C_n^*=C_n. \tag{4.10}$$

To first approximation, with X_0 real, (4.9) together with (4.10) reduces to (3.10). Eq. (4.9), however, is *not* the complete $O(|\Delta|^{\frac{1}{2}})$ solution. We shall, through the Jacobian inhomogeneity in (4.4), produce at higher orders terms proportional to $e^{\pm iNx_0}$. As this structure satisfies the homogeneous problem trivially, such terms would, unless removed, produce a secularity and invalidate asymptotic expansion (4.5). Their removal can be accomplished by adding, at each order, to the perturbation solution a solution

$$\psi_R^{(n)}=A^{(n)}(T)e^{iNx_0}+*, \tag{4.11}$$

which physically represents the correction to the basic wave flow (2.3), produced by the finite-amplitude perturbations. Thus the complete solution, to $O(|\Delta|^{\frac{1}{2}})$, is

$$\begin{aligned}\psi^{(1)}&=\psi_R^{(1)}+\psi_h^{(1)}\\&=A^{(1)}(T)e^{iNx_0}+X_0(T)e^{ily}\sum_n C_ne^{inNx_0}+*.\end{aligned} \tag{4.12}$$

From (4.4) and (4.5), the equation which governs $O(|\Delta|)$ problem is

$$\left(\frac{\beta}{N^2}\frac{\partial}{\partial x_0}+NA_c\cos Nx_0\frac{\partial}{\partial y}\right)(\nabla^2\psi^{(2)}+N^2\psi^{(2)}) =-\frac{\partial}{\partial T}\nabla^2\psi^{(1)}-J(\psi^{(1)},\nabla^2\psi^{(1)}). \tag{4.13}$$

The right-hand side of (4.13) may be evaluated with the help of (4.12). The result is

$$\begin{aligned}&\left(\frac{\beta}{N^2}\frac{\partial}{\partial x_0}+NA_c\cos Nx_0\frac{\partial}{\partial y}\right)(\nabla^2\psi^{(2)}+N^2\psi^{(2)})=\eta^{(2)}\\
&=N^2\frac{dA^{(1)}}{dT}e^{iNx_0}+\frac{dX_0}{dT}e^{ily}\sum_n(l^2+n^2N^2)C_ne^{inNx_0}\\
&-lNX_0e^{ily}\sum_m(l^2+n^2N^2-N^2)C_m[A^{(1)}e^{i(m+1)Nx_0}-A^{(1)*}e^{i(m-1)Nx_0}]\\
&-lNX_0^2e^{2ily}\sum_m[\sum_p(m-p)(l^2+p^2N^2)C_mC_pe^{i(m+p)Nx_0}]\\
&+lN|X_0|^2\sum_m[\sum_p(m-p)(l^2+p^2N^2)C_mC_pe^{i(m-p)Nx_0}]+*.\end{aligned} \tag{4.14}$$

We note that the last summation term in (4.14) yields terms proportional to $e^{\pm iNx_0}$ whenever $p=m\mp1$, necessitating the inclusion of $\psi_R^{(1)}$ in (4.12).

If we take a particular solution at $O(|\Delta|)$ to be

$$\psi_h^{(2)}=e^{ily}\sum_n X_n^{(2)}(T)e^{inNx_0}+* \tag{4.15}$$

and substitute it into (4.14), we obtain

$$\sum_n\left[Y_{n-1}^{(2)}+\frac{2\beta}{N^2A_cl}nY_n^{(2)}+Y_{n+1}^{(2)}\right]e^{i(ly+nNx_0)}+*=-\frac{2i}{NA_cl}\eta^{(2)}, \tag{4.16}$$

where

$$Y_n^{(2)}=(N^2-n^2N^2-l^2)X_n^{(2)}. \tag{4.17}$$

On extracting from $\eta^{(2)}$ in (4.14) all terms of $e^{\pm i(ly+nNx_0)}$ structure, (4.16) leads to an infinite set of inhomogeneous algebraic relations for the $Y_n^{(2)}$'s. From the discussion in Section 3 pertaining to the accuracy of the first approximation, we perform the analysis of (4.16) retaining only the terms corresponding to $n=0$, and ±1. We then have three equations

$$\left.\begin{aligned}
-\frac{2\beta}{N^2A_cl}Y_{-1}^{(2)}+Y_0^{(2)}&=+\frac{2i}{NA_cl}\left[(l^2+N^2)C_1\frac{dX_0}{dT}+lN(N^2-l^2)X_0A^{(1)*}\right]\\
Y_{-1}^{(2)}+Y_1^{(2)}&=-\frac{2i}{NA_cl}\left[l^2\frac{dX_0}{dT}+l^3NC_1X_0(A^{(1)}+A^{(1)*})\right]\\
\frac{2\beta}{N^2A_cl}Y_1^{(2)}+Y_0^{(2)}&=-\frac{2i}{NA_cl}\left[(l^2+N^2)C_1\frac{dX_0}{dT}+lN(N^2-l^2)X_0A^{(1)}\right]
\end{aligned}\right\}. \tag{4.18}$$

Since the determinant of the coefficients of $Y_{-1}^{(2)}$, $Y_0^{(2)}$ and $Y_1^{(2)}$ vanishes, a solution is possible only if

$$\frac{dX_0}{dT}\left[2(l^2+N^2)C_1-\frac{2\beta l}{N^2A_c}\right]+(A^{(1)}+A^{(1)*})X_0\left[lN(N^2-l^2)-\frac{2\beta l^2}{NA_c}C_1\right]=0. \tag{4.19}$$

Using the definition of A_c and C_1 from (3.7) and (3.12), respectively, we find that both square bracket terms in (4.19) vanish. Hence, dX_0/dT remains thus far undetermined, and (4.18) together with (4.17) yields

$$\left.\begin{aligned}
X_{-1}^{(2)}&=-C_1X_0^{(2)}+i\left[\frac{1}{NA_cl}\frac{dX_0}{dT}+\frac{N^2(N^2-l^2)}{\beta l}X_0A^{(1)*}\right]\\
X_1^{(2)}&=C_1X_0^{(2)}+i\left[\frac{1}{NA_cl}\frac{dX_0}{dT}+\frac{N^2(N^2-l^2)}{\beta l}X_0A^{(1)}\right]
\end{aligned}\right\}. \tag{4.20}$$

Substitution of (4.20) into (4.15) shows that the terms containing $X_0^{(2)}$ yield an $O(|\Delta|)$ solution whose (x_0,y) structure is identical to the $O(|\Delta|^{\frac{1}{2}})$ solution on the marginal curve $A=A_c$ [given by (3.10)]. Thus, if we assume that *all* of the structure of the (marginally) neutral perturbation is specified by the $O(|\Delta|^{\frac{1}{2}})$ problem we can set $X_0^{(2)}$, which is arbitrary, equal to zero. The terms in square brackets, on the other hand, yield a phase shift between the $O(|\Delta|^{\frac{1}{2}})$ and the $O(|\Delta|)$ solutions. Later discussion [see Eq. (4.29)] will show that $A^{(1)}$ in (4.20) may be taken as real. We then have

$$X_1^{(2)}=X_{-1}^{(2)}=i\left[\frac{1}{NA_cl}\frac{dX_0}{dT}+\frac{N^2(N^2-l^2)}{\beta l}X_0A^{(1)}\right]. \tag{4.21}$$

For X_0 real, (4.15) together with (4.21) yields an $O(|\Delta|)$ solution

$$\psi_h^{(2)}=-4\left[\frac{1}{NA_cl}\frac{dX_0}{dT}+\frac{N^2(N^2-l^2)}{\beta l}X_0A^{(1)}\right]\sin ly\cos Nx_0, \tag{4.22}$$

which should be compared to the last term in (3.11). [We note that if $X_0(T)=X_0(0)e^{\sigma_i t}$, i.e., X_0 is as given by the linear theory, and if the nonlinear effects are neglected, the two become identical.]

In addition to the phase-shifted solution above, the inhomogeneous terms in (4.14) give rise to a forced $O(|\Delta|)$ solution of the form

$$\psi_f^{(2)}=e^{i2ly}\sum_n F_n^{(2)}e^{inNx_0}+*. \tag{4.23}$$

Substitution of (4.23) into (4.14) yields

$$\sum_n\{[N^2-4l^2-(n-1)^2N^2]F_{n-1}^{(2)}+(\beta n/N^2A_cl)(N^2-4l^2-n^2N^2)F_n^{(2)}+[N^2-4l^2-(n+1)^2N^2]F_{n+1}^{(2)}\}\times e^{i(2ly+nNx_0)}=-(i/NA_cl)\eta^{(2)}. \tag{4.24}$$

Evaluated to the first approximation (i.e., retaining only terms $n=0, \pm 1$), Eq. (4.24) with the help of (4.14) yields

$$\left.\begin{aligned}(4\beta l/N^2A_c)F_{-1}^{(2)}+(N^2-4l^2)F_0^{(2)}&=-(iN^2C_1/A_c)X_0^2\\ F_{-1}^{(2)}+F_1^{(2)}&=0\\ -(4\beta l/N^2A_c)F_1^{(2)}+(N^2-4l^2)F_0^{(2)}&=-(iN^2C_1/A_c)X_0^2\end{aligned}\right\}. \tag{4.25}$$

Since $e^{\pm i2ly}$ is absent among the forced terms, $\eta^{(2)}$, we can set $F_0^{(2)}$ arbitrarily to zero. We find then that

$$F_1^{(2)}=-F_{-1}^{(2)}=[iN^2/4(l^2+N^2)A_c]X_0^2 \tag{4.26}$$

and that for X_0 real, to first approximation, (4.23) reduces to

$$\psi_f^{(2)}=-[N^2/(l^2+N^2)A_c]\cos 2ly \sin Nx_0. \tag{4.27}$$

We have thus far dealt with all inhomogeneities in (4.14), except that proportional to $e^{\pm iNx_0}$. As stated earlier in this section, to ensure the validity of the asymptotic expansion (4.5), the coefficient of this inhomogeneity must be set identically to zero. Retaining in the coefficient only terms arising at the first approximation, we obtain

$$\frac{dA^{(1)}}{dT}=\frac{N^3A_c}{\beta}(N^2-l^2)|X_0|^2 \tag{4.28}$$

which is an equation for the long-time evolution of the amplitude of $\psi_R^{(1)}$ in (4.12). We note that (4.28) implies

$$A^{(1)}(T)-A(T)^*=A^{(1)}(0)-A^{(1)}(0)^*.$$

But physically, at $T=0$, we expect all of the $e^{\pm iNx_0}$ structure to be confined to the basic wave flow (2.3). It follows that we can set $A^{(1)}(0)=A^{(1)*}(0)=0$. Then

$$A^{(1)}(T)=A^{(1)}(T)^* \tag{4.29}$$

and $\psi_R^{(1)}$ in (4.11), which represents the $O(|\Delta|^{\frac{1}{2}})$ correction to the basic wave flow (2.3), becomes

$$\psi_R^{(1)}=2A^{(1)}\cos Nx_0. \tag{4.30}$$

A comparison between the $O(1)$ basic wave flow ψ_0 and the corrected basic wave flow $\psi_0+|\Delta|^{\frac{1}{2}}\psi_R^{(1)}$ reveals that although ψ_0 is stationary in (x_0,y) frame of reference, $\psi_0+|\Delta|^{\frac{1}{2}}\psi_R^{(1)}$ is slowly shifting phase by arctan $[|\Delta|^{\frac{1}{2}}A^{(1)}(T)/A_c]$. Since by (4.28) $A^{(1)}>0$, the phase shift is westward and is generated by the nonlinear interactions [via the Jacobian terms in (4.13)] among the modes constituting the $O(|\Delta|^{\frac{1}{2}})$ perturbation flow. This drift of the basic wave flow pattern will, as we shall see at $O(|\Delta|^{\frac{3}{2}})$, *strongly* affect the time evolution of the perturbation flow.

The complete solution to $O(|\Delta|^{\frac{3}{2}})$ is

$$\psi=\psi_0+|\Delta|^{\frac{1}{2}}(\psi_R^{(1)}+\psi_h^{(1)})+|\Delta|(\psi_R^{(2)}+\psi_h^{(2)}+\psi_f^{(2)}), \tag{4.31}$$

where $\psi_R^{(2)}$, given by (4.11), is added at $O(|\Delta|)$ to handle the inhomogeneities proportional to $e^{\pm iNx_0}$, produced by the Jacobians at $O(|\Delta|^{\frac{3}{2}})$. In (4.31) ψ_0 and $\psi_R^{(1)}$ are given by (2.3) and (4.30), respectively, and $\psi_h^{(1)}$, $\psi_h^{(2)}$ and $\psi_f^{(2)}$ are given in general by (4.9), (4.15) and (4.23), respectively, and to first approximation by (3.10), (4.22) and (4.27), respectively.

From (4.4) and (4.5) the equation which governs the $O(|\Delta|^{\frac{3}{2}})$ problem is

$$\left(\frac{\beta}{N^2}\frac{\partial}{\partial x_0}+NA_c\cos Nx_0\frac{\partial}{\partial y}\right)(\nabla^2\psi^{(3)}+N^2\psi^{(3)})$$
$$=-\frac{\partial}{\partial T}\nabla^2\psi^{(2)}-\frac{\Delta}{|\Delta|}N\cos Nx_0\frac{\partial}{\partial y}(\nabla^2\psi^{(1)}+N^2\psi^{(1)})-J(\psi^{(1)},\nabla^2\psi^{(2)})-J(\psi^{(2)},\nabla^2\psi^{(1)}). \tag{4.32}$$

The right-hand side of (4.32) may be evaluated with the help of (4.31), (4.9), (4.11), (4.15) and (4.23). The result is

$$\left(\frac{\beta}{N^2}\frac{\partial}{\partial x_0}+NA_c\cos Nx_0\frac{\partial}{\partial y}\right)(\nabla^2\psi^{(3)}+N^2\psi^{(3)})=\eta^{(3)}$$
$$=N^2\frac{dA^{(2)}}{dT}e^{iNx_0}+e^{ily}\sum_n(l^2+n^2N^2)\frac{dX_n^{(2)}}{dT}e^{inNx_0}+e^{2ily}\sum_n(4l^2+n^2N^2)\frac{dF_n^{(2)}}{dT}e^{inNx_0}$$
$$-lNe^{ily}\sum_m(l^2+m^2N^2-N^2)[(A^{(1)}X_m^{(2)}+A^{(2)}X_0C_m)e^{i(m+1)Nx_0}-(A^{(1)*}X_m^{(2)}+A^{(2)*}X_0C_m)e^{i(m-1)Nx_0}]$$
$$+lN^3X_0\sum_m\{\sum_p(m-p)(m^2-p^2)[C_mX_p^{(2)}e^{i[2ly+(m+p)Nx_0]}-C_mX_p^{(2)*}e^{i(m-p)Nx_0}]\}$$
$$-lNX_0\sum_m\{\sum_p(2m-p)[3l^2+(p^2-m^2)N^2][C_mF_p^{(2)}e^{i[3ly+(m+p)Nx_0]}-C_mF_p^{(2)*}e^{i[(m-p)Nx_0-ly]}]\}$$
$$+\tfrac{1}{2}ilNX_0e^{ily}\sum_m(l^2+m^2N^2-N^2)C_m[e^{i(m+1)Nx_0}+e^{i(m-1)Nx_0}]+*. \tag{4.33}$$

Suppose we consider a perturbation solution at $O(|\Delta|^{\frac{3}{2}})$,

$$\psi_h^{(3)} = e^{ily} \sum_n X_n^{(3)} e^{inNx_0} + *, \quad (4.34)$$

whose (x_0,y) structure is *identical* to that given by the perturbation solution of the homogeneous problem on the marginal curve, i.e., $\psi_h^{(1)}$ given by (4.9). Substitution of (4.34) into (4.33) gives

$$\sum_n \left(Y_{n-1}^{(3)} + \frac{2\beta}{N^2 A_c l} n Y_n^{(3)} + Y_{n+1}^{(3)} \right) e^{i(ly+nNx_0)} + * = -(2i/NA_c l)\eta^{(3)}, \quad (4.35)$$

where

$$Y_n^{(3)} = (N^2 - n^2N^2 - l^2) X_n^{(3)}. \quad (4.36)$$

Upon extracting from $\eta^{(3)}$ in (4.33) the terms of $e^{\pm i(ly+nNx_0)}$ structure, we have, from (4.35), to first approximation, three equations

$$\begin{aligned} -(2\beta/N^2A_c l) Y_1^{(3)} + Y_0^{(3)} &= -(2i/NA_c l)[(l^2+N^2) dX_1^{(2)}/dT \\ &\quad - \tfrac{1}{2} ilN(N^2-l^2) X_0 (1 - 2iA^{(2)*}) \\ &\quad - lN(3l^2+N^2) X_0^* F_1^{(2)}], \quad (4.37a) \end{aligned}$$

$$Y_{-1}^{(3)} + Y_1^{(3)} = -(2il^2/A_c) C_1 X_0 (A^{(2)} + A^{(2)*}), \quad (4.37b)$$

$$\begin{aligned} (2\beta/N^2A_c l) Y_1^{(3)} + Y_0^{(3)} &= -(2i/NA_c l)[(l^2+N^2) dX_1^{(2)}/dT \\ &\quad - \tfrac{1}{2} ilN(N^2-l^2) X_0 (1 + 2iA^{(2)}) \\ &\quad - lN(3l^2+N^2) X_0^* F_1^{(2)}]. \quad (4.37c) \end{aligned}$$

For $\psi_h^{(3)}$ to have identical structure to $\psi_h^{(1)}$, the coefficients $Y_n^{(3)}$ in (4.35) must satisfy the same recursion relations as the Y_n's, i.e., Eq. (3.3) with $\sigma=0$ [to first approximation $Y_{-1}^{(3)}$, $Y_0^{(3)}$ and $Y_1^{(3)}$ in (4.37) must satisfy the same recursion relations as Y_{-1}, Y_0 and Y_1, i.e., Eq. (3.5) with $\sigma=0$]. On comparing these expressions, we conclude that the left-hand side in (4.35), and to first approximation, the left-hand side of each equation in (4.37) vanish. Later discussion [see Eq. (4.40)] will show that $A^{(2)}$ in (4.37) is pure imaginary. Therefore, right-hand side in (4.37b) is identically zero, whereas the right-hand sides in (4.37a) and (4.37c), which are identical, must be set equal to zero. We have then

$$\frac{dX_1^{(2)}}{dT} - \frac{ilN(N^2-l^2)}{2(l^2+N^2)} X_0 (1+2iA^{(2)}) - \frac{lN(3l^2+N^2)}{(l^2+N^2)} X_0^* F_1^{(2)} = 0,$$

or, with the help of (4.21) and (4.26),

$$\frac{d^2X_0}{dT^2} + \frac{N^3A_c}{\beta}(N^2-l^2)\frac{d}{dT}(X_0A^{(1)}) - \frac{N^2l^2A_c(N^2-l^2)}{2(N^2+l^2)} X_0(1+2iA^{(2)}) - \frac{l^2N^4(N^2+3l^2)}{4(N^2+l^2)^2} X_0|X_0|^2 = 0. \quad (4.38)$$

The above procedure assures that all the inhomogeneous terms in (4.33), which possess the same (x_0,y) structure as the perturbation solution of the homogeneous problem, and which therefore introduce resonance with the linear operator on the left-hand side of (4.33), are removed in order to maintain the validity of the perturbation expansion (4.5). Their removal, to first approximation, yields Eq. (4.38) which describes the long time evolution of $X_0(T)$, the amplitude of the perturbation streamfunction.

The only remaining secular inhomogeneity in (4.33) is that proportional to $e^{\pm iNx_0}$. Again, to ensure the validity of the asymptotic expansion (4.5) it must be removed, i.e., its coefficient must be set to zero. Retaining only terms arising at the first approximation, we find

$$dA^{(2)}/dT - lN(X_0^*X_1^{(2)} - X_0X_1^{(2)*}) = 0$$

or, with the help of (4.21) and (4.29),

$$dA^{(2)}/dT = (i/A_c)[d|X_0|^2/dT + (2N^3A_c/\beta)(N^2-l^2)A^{(1)}|X_0|^2]. \quad (4.39)$$

We note that (4.39) implies

$$A^{(2)}(T) + A^{(2)}(T)^* = A^{(2)}(0) + A^{(2)}(0)^*.$$

But initially all of the basic wave flow structure is confined in ψ_0. Therefore, $A^{(2)}(0) = A^{(2)}(0)^* = 0$ and

$$A^{(2)}(T) = -A^{(2)}(T)^* = -iA_R^{(2)}(T), \quad (4.40)$$

where $A_R^{(2)}$ is real. Therefore, $\psi_R^{(2)}$, the $O(|\Delta|)$ correction to the basic wave flow, becomes

$$\psi_R^{(2)} = 2A_R^{(2)} \sin Nx_0. \quad (4.41)$$

Eqs. (4.28), (4.38) and (4.39) represent three coupled nonlinear ordinary differential equations which describe the long-term evolution of the perturbation and the basic wave state. With $A^{(2)}$ replaced by $-iA_R^{(2)}$, and (4.28) employed in the second term of (4.38), these may be written

$$d^2X_0/dT^2 = -a_1A^{(1)}(dX_0/dT) + a_2X_0(1+2A_R^{(2)}) - a_3X_0|X_0|^2, \quad (4.42a)$$

$$dA^{(1)}/dT = a_1|X_0|^2, \quad (4.42b)$$

$$dA_R^{(2)}/dT = -A_c^{-1}[d|X_0|^2/dT + 2a_1A^{(1)}|X_0|^2], \quad (4.42c)$$

where

$$a_1 = \frac{N^3 A_c}{\beta}(N^2 - l^2),$$

$$a_2 = \frac{\beta l^2}{2N(N^2+l^2)} a_1,$$

$$a_3 = \frac{l^2 N^2}{4(N^2+l^2)^2}(7N^4 - 8l^4 - 3N^2 l^2).$$

Although the derivation of Eq. (4.42) is based on the first approximation it may be shown that the structure of these equations remains *unaltered* irrespective of the degree of approximation. The only quantitative change resulting in (4.42) when higher approximations are considered is a minor modification of coefficients a_1, a_2 and a_3. This can be most easily demonstrated in the case of (4.42b). Removing from (4.14) all secular inhomogeneities proportional to e^{iNx_0}, we obtain the *exact* form of (4.42b):

$$dA^{(1)}/dT = 2lN \sum_{n=1}^{\infty} (2n-1) C_n C_{n-1} |X_0|^2. \quad (4.42\text{b}')$$

We see that the exact and approximate forms of (4.42b) are identical in structure, but the exact form of a_1 is an infinite series

$$a_1 = 2lN \sum_{n=1}^{\infty} (2n-1) C_n C_{n-1}.$$

The convergence of this series may be established with the help of (3.3), (3.4) and (4.8). We find that, for large n, $(2n-1)C_nC_{n-1} \rightarrow 0$ about as rapidly as

$$\left(\frac{N^2-l^2}{N^2}\right)^2 \left(\frac{N^2 A_c l}{2\beta}\right)^{2n-1} \frac{(2n-1)}{n^2(n-1)^2 n!(n-1)!}.$$

Therefore, larger values of n contribute very little to a_1. In practice we can compare the values of a_1 for the first three approximations ($M=n=1$, 2, or 3) for $N=6$ and $\beta=1$ with the help of Table 1. For $l=1$, to first approximation $a_1=8.244$ whereas to third approximation $a_1=8.246$; for $l=5$, to first approximation $a_1=18$ whereas to third approximation $a_1=18.5$. Clearly, even the first approximation yields an excellent estimate of a_1 (especially at smaller values of l). The remaining equations (4.42a and 4.42c) and coefficients a_2 and a_3 cannot be found exactly. Consideration of higher approximations (algebraically a very tedious process which was carried out as far as the third approximation) yields identical structure of the equations and a small modification of a_2 and a_3 which, like a_1, take the form of a rapidly converging series. We conclude that Eqs. (4.42) correctly describe the evolution of the perturbation and the basic wave, irrespective of the level of truncation.

It is instructive to compare Eqs. (4.42) with the amplitude equation derived by Pedlosky (1970) describing the finite-amplitude evolution of baroclinic waves [his Eq. (4.29)]. The difference in structure arises from the presence of $A^{(1)}(T)$, which, we recall, measures the phase shift in the basic Rossby wave flow; otherwise, with $A^{(1)}(T)=0$, Eq. (4.42c) would yield $A_R^{(2)}(T)=(-1/A_c)[|X_0|^2-|X_0(0)|^2]$, which substituted into (4.42a) would produce an equation for $X_0(T)$ identical to Pedlosky's. To determine how the phase shift in the basic wave flow, absent when the basic flow is zonally uniform, alters the finite-amplitude evolution of the perturbation and the basic wave from the vacillatory character found by Pedlosky, we must solve the coupled set (4.42). Since analytical solutions could not be found, a numerical approach was used.

Numerical solutions of (4.42) have been obtained for N in the integer range 2–10, l in the integer range 1–$(N-1)$ and β in the range 0.1–10. In all calculations it was assumed that $X_0(T)$ is real and that initially $X_0(0)=\epsilon A_c$, $0.1 \leqslant \epsilon \leqslant 5$. As is physically most meaningful, in the bulk of the calculations the initial growth of the disturbance was taken as predicted by the linear theory, i.e., $(dX_0/dT)_{T=0}=\sigma_i X_0(0)$, and the initial structure of the basic wave was confined to ψ_0, i.e., $A^{(1)}(0)=A_R^{(2)}(0)=0$. For completeness, however, the initial values of dX_0/dT, $A^{(1)}$ and $A_R^{(2)}$ were also relaxed from above to a range $(dX_0/dT)_{T=0}=\epsilon\sigma_i X_0(0)$, $A^{(1)}(0)=\pm\epsilon A_c$ and $A_R^{(2)}(0)=\pm\epsilon A_c$, $0.1 \leqslant \epsilon \leqslant 5$.

Over the considered range of N, l and β *all* of the solutions of (4.42) were found to exhibit a *qualitatively similar* behavior. Fig. (2) illustrates this behavior when initially $X_0(0)=A_c$, $(dX_0/dT)_{T=0}=\sigma_i X_0(0)$ and $A^{(1)}(0)=A_R^{(2)}(0)=0$ for $N=6$, $l=1$ and $\beta=1$. Initially, while the perturbation amplitude $X_0(T)$ is small, it undergoes an exponential growth in agreement with the linear theory. When $X_0(T)$ increases substantially above $X_0(0)$ the perturbation begins to alter the structure of the basic wave by altering both its phase and amplitude. This is manifested by the growth of $A^{(1)}(T)$ and $A_R^{(2)}(T)$ and the reduction in growth of $X_0(T)$. A time T_M is reached when the basic wave is so altered as to make further extraction of energy from the basic wave impossible (In Fig. 2 T_M corresponds to the time when perturbation is at a maximum amplitude while the basic wave is undergoing a maximum rate of change). When $T>T_M$, the structure of the basic wave and the perturbation is such that the direction of energy flow is reversed. Eventually, the perturbation decays out (i.e., loses all its energy), while the basic wave equilibrates with a phase and amplitude which are *different* from their initial values.

The direction of energy flow depends on the phase relationship between the basic wave and the perturbation. With the help of (2.3), (4.1), (4.30) and (4.41) the streamfunction for the basic wave flow may, to $O(|\Delta|)$, be written as

$$\psi_R = -Uy + A_R \sin(Nx_0 + \theta_R), \quad (4.43)$$

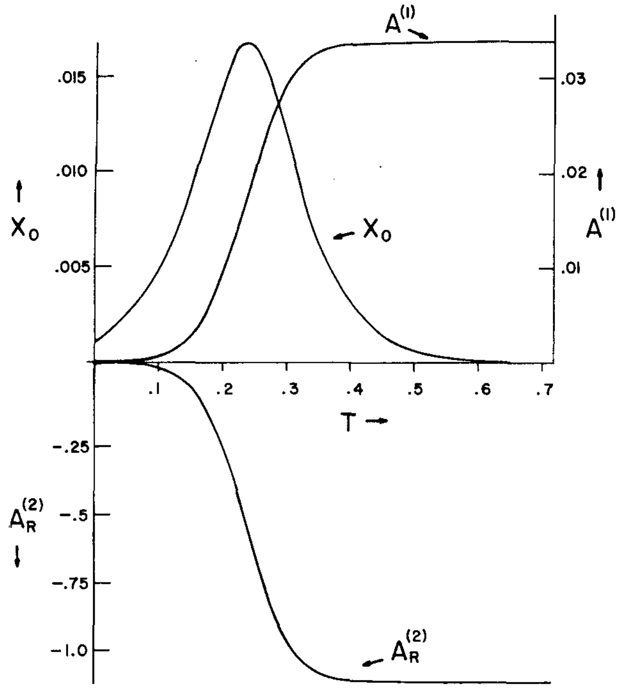

FIG. 2. Long time evolution of the perturbation amplitude $X_0(T)$ and the basic Rossby wave amplitude corrections $A^{(1)}(T)$ and $A_R^{(2)}(T)$ for $\beta=1$, $N=6$ and $l=1$; initial conditions are $X_0(0)=A_c$, $(dX_0/dT)_{T=0}=\sigma_i A_c$ and $A^{(1)}(0)=A_R^{(2)}(0)=0$.

where

$$A_R=[(A_c+\Delta+2|\Delta|A_R^{(2)})^2+4|\Delta|A^{(1)2}]^{\frac{1}{2}}$$

$$\approx A_c\left[1+\frac{\Delta}{A_c}+\frac{2|\Delta|}{A_c}\left(\frac{A^{(1)2}}{A_c}+A_R^{(2)}\right)\right], \quad (4.44)$$

$$\theta_R=\tan^{-1}\left(\frac{2|\Delta|^{\frac{1}{2}}A^{(1)}}{A_c+\Delta+2|\Delta|A_R^{(2)}}\right)\approx|\Delta|^{\frac{1}{2}}\frac{2A^{(1)}}{A_c}. \quad (4.45)$$

With the help of (3.10) and (4.22) the perturbation streamfunction may, to $O(|\Delta|)$, be written as

$$\psi_h=|\Delta|^{\frac{1}{2}}2X_0[\cos ly-A_h\sin ly\sin(Nx_0+\theta_h)], \quad (4.46)$$

where

$$A_h=-\frac{a_1}{Nl}\left[1+\frac{4|\Delta|}{A_c^2a_1^2X_0^2}\left(\frac{dX_0}{dT}+a_1X_0A^{(1)}\right)^2\right]^{\frac{1}{2}}, \quad (4.47)$$

$$\theta_h=\tan^{-1}\left[\frac{2|\Delta|^{\frac{1}{2}}}{A_ca_1X_0}\left(\frac{dX_0}{dT}+a_1X_0A^{(1)}\right)\right]$$

$$\approx\frac{2|\Delta|^{\frac{1}{2}}}{A_ca_1X_0}\left(\frac{dX_0}{dT}+a_1X_0A^{(1)}\right). \quad (4.48)$$

It follows from (4.45) and (4.48) that the zonal phase difference between the perturbation and the basic wave is approximately

$$\theta_h-\theta_R\approx\frac{2|\Delta|^{\frac{1}{2}}}{A_ca_1X_0}\frac{dX_0}{dT}. \quad (4.49)$$

Therefore, in the region $0<T<T_M$, where the perturbation extracts energy from the basic wave (i.e., where $X_0>0$ and $dX_0/dT>0$), the perturbation lags the basic wave. In the region $T_M<T<\infty$, where the perturbation returns the energy to the basic wave (i.e., where $X_0>0$ but $dX_0/dT<0$) the perturbation leads the basic wave. At $T=T_M$ (where $dX_0/dt=0$) the perturbation and the basic wave are in phase.

Eq. (4.42b) may be combined with (4.42c) and the result integrated between $T=0$ and an arbitrary T to give an integral relation between X_0, $A^{(1)}$ and $A_R^{(2)}$. Recalling that $A^{(1)}(0)=A_R^{(2)}(0)=0$, we obtain

$$A_R^{(2)}+A^{(1)2}/A_c=[X_0^2(0)-X_0^2]/A_c. \quad (4.50)$$

Thus, the amplitude of the basic Rossby wave A_R [given by (4.44)] may be written

$$A_R\approx A_c\left\{1+\frac{\Delta}{A_c}+\frac{2|\Delta|}{A_c^2}[X_0^2(0)-X_0^2]\right\}. \quad (4.51)$$

Eq. (4.51) shows that while $X_0>X_0(0)$ the basic wave is depleted of some of its energy and that eventually, as $X_0\rightarrow 0$, the basic wave amplitude equilibrates at an amplitude

$$A_{R_\infty}\equiv\lim_{T\to\infty}A_R\approx A_c\left[1+\frac{\Delta}{A_c}+\frac{2|\Delta|X_0^2(0)}{A_c^2}\right]. \quad (4.52)$$

[A_{R_∞} exceeds $A_R(0)$ because the model is inviscid and so whatever energy is initially in the perturbation state, finds itself eventually in the basic wave state.] Unfortunately, an analytical expression for the equilibrated value of the phase θ_R of the basic wave could not be found. This value can, however, be easily computed for each β, N and l from (4.45) using the numerical result for the equilibrated value of $A^{(1)}$.

We conclude this section by describing how the solutions of (4.42) are affected if we individually vary the parameters β, l and N. Examination of numerical results shows that 1) as β is increased while N and l are held fixed the time T_∞ required for equilibration of the basic wave decreases, the maximum amplitude $X_{0_{MAX}}$ reached by the perturbation increases, the equilibrated value of the $O(|\Delta|^{\frac{1}{2}})$ basic wave correction $A_\infty^{(1)}$ increases, but the equilibrated value of the $O(|\Delta|)$ basic wave correction $A_{R_\infty}^{(2)}$ remains unaltered; 2) as l is increased while β and N are held fixed, T_∞ decreases while the remaining quantities $X_{0_{MAX}}$, $A_\infty^{(1)}$ and $A_{R_\infty}^{(2)}$ all increase; and 3) as N is increased while β and l are held fixed, T_∞ increases, $X_{0_{MAX}}$ and $A_\infty^{(1)}$ decrease while $A_{R_\infty}^{(2)}$ remains unaltered.

5. Summary and conclusions

The preceding analysis shows that a finite-amplitude (linearly unstable) perturbation superimposed on a basic Rossby wave flow undergoes a "life cycle" characterized first by a growth stage up to a certain amplitude $X_{0_{MAX}}$, determined numerically from (4.42), then followed by a decay state. The growth of the perturbation is possible for as long as its phase lags that of the basic Rossby wave. As the perturbation develops, however, it begins to alter the phase and the amplitude of the Rossby wave, the phase correction being to the west. As a result, as the perturbation amplitude X_0 becomes larger, the phase difference becomes smaller, until, at $X_0=X_{0_{MAX}}$, the perturbation and the basic wave find themselves in phase and the growth of the perturbation is halted. At this time, however, the basic wave phase correction is undergoing a maximum rate of change [see Eqs. (4.42b) and (4.45)]; thus, subsequently, the perturbation finds itself leading the Rossby wave and the direction of energy flow is reversed. Over the considered range of N, l and β this phase relation never reverses in time again, and eventually the perturbation totally decays out while the Rossby wave equilibrates with an amplitude greater than initially [because, as we can see from (4.52), the Rossby wave now contains the energy which was initially in the perturbation] and with a phase which is shifted to the west of the initial position. Thus, even though the initially unstable perturbation, having undergone its "life cycle," is no longer present, it leaves the Rossby wave flow in an *altered* form.

Although altered, the basic Rossby wave is the sole survivor of the finite-amplitude stability dynamics. In this sense, the Rossby wave is ultimately stable with respect to the originally superimposed perturbation. After the finite-amplitude process has taken place, however, nothing prevents the equilibrated Rossby wave from encountering a new "properly phased" perturbation and the whole process from repeating itself. After each such consecutive encounter the Rossby wave would undergo further alterations. Because of the uncertainty as to when and where each consecutive encounter could take place and as to how many encounters could occur in any given long time period, it is felt that this study implies a certain degree of inherent unpredictability in large-scale atmospheric flows. It must be kept in mind, however, that in physically more realistic situations the basic Rossby wave would, in addition to the possible unpredictable alterations discussed here, undergo continuing long-time changes as a result of resonant interactions, absent in this study.

The total streamline pattern, corresponding to a time when the perturbation is in the amplifying stage, is given by Lorenz (1972) in his Fig. 2. (The reader is referred to that paper for a detailed description.) The figure clearly illustrates jetlike features and horizontal tilting of the streamlines away from the north–south direction. These flow characteristics, observed in the atmosphere, would not occur here in the absence of the perturbation. During the decaying stage of the perturbation, the streamline pattern would appear as a mirror image of Lorenz' Fig. 2. If we consider a number of encounters between the basic Rossby wave and "properly phased perturbations," the (propagating) flow pattern would in time give an impression of vacillating troughs and ridges. It is possible, therefore, that the explanation of the "tilted trough" vacillation, observed in the anulus, may lie in the barotropic instability of the initially generated baroclinic wave flow patterns. The finite-amplitude barotropic instability of established large-amplitude baroclinic waves is currently being studied.

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