A Study in Tornado-Like Vortex Dynamics

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ABSTRACT

Fine-resolution calculations using an axisymmetric numerical model of the flow within a Ward-type vortex chamber are discussed. Particular attention is paid to the vortex-ground interaction. Variations in the swirl ratio $S$ from zero to unity lead to radically different vortex structure in the “corner” region (i.e., near $r = z = 0$). For $S < 1$, a concentrated vortex forms in the upper chamber but not in the corner. At moderate $S$, we observe vortex breakdown, large-amplitude inertial waves, and very intense swirling motion in the corner. When $S = 1$, the central downdraft penetrates to the lower surface and the vortex breakdown occurs within the boundary layer. These results are consistent with experimental observations and suggest the explanation of a number of observed facets of tornadoes.

1. Introduction

A numerical study of the flow within the vortex chamber conceived by Ward\(^1\) (1972) is described herein. The author has previously reported on such calculations (Rotunno, 1977)\(^2\); however, in that study, a zero-stress boundary condition at the lower level surface is emphasized so that the effect of the chamber geometry could be isolated. Some preliminary calculations employing frictional lower boundary conditions were also reported, but the grid resolution was too poor for any definitive statements to be made on the resulting flow.

New numerical simulation experiments using a fourfold (as compared to that used in I) reduction in grid spacing and the application of the zero-slip lower boundary condition have been performed. The numerical model is capable of simulating not only the gross features of the vortex (e.g., the core size and basic radial structure), but also the finer details of the boundary layer, the “vortex breakdown” phenomenon, and the formation and breaking of inertial waves on the vortex core. I believe the present numerical calculations are unique in that no previous numerical vortex study has reported on this combination of phenomena within the context of a single physical model.

The laboratory model is described and a résumé of some past and current findings is given in the following section. Also, the basic flow parameters are introduced.

The particulars of the numerical model are discussed in Section 3. Both laboratory and numerical models indicate that the swirl ratio\(^3\) $S$ is the single most important parameter governing vortex dynamics; for this reason, Section 4, which contains the model results, is divided into sections describing cases of zero, low, moderate and large swirl ratio, respectively. Probably the most dramatic event which occurs in the chamber is vortex breakdown (see Fig. 2.2). The present model simulates this phenomenon and permits the evaluation of a number of analytical explanatory theories. Some discussion of the relationship these calculations have with other laboratory and field observations is interspersed in this section. However, the bulk of these arguments is given in Section 5. Included in this discussion are the following topics: the influence of $S$ on the transition to turbulent flow, an explanation of why a concentrated vortex is observed to occur only at moderate values of $S$, and the role of the far upstream boundary layer in producing large velocities at the vortex base, and an explanation for the field observation that rotation, nearly on the funnel scale (1 km), occurs first at mid-levels within the parent storm and henceforth grows downward.

The flow in the vortex chamber involves a number of complex fluid dynamical phenomena; vortex breakdown, large-amplitude inertial waves, local instability associated with these waves, and a boundary layer with a complex external flow. I believe that the present study will, among other things, serve to emphasize the importance of these phenomena with respect to tornado-vortex theory.

2. Résumé of some laboratory findings

The relevance of the TVC to actual tornadoes is dis-
discussed at length by Davies-Jones (1976, 1978) and Church et al., (1977). The TVC may be described as follows (see Fig. 2.1). A cylindrical chamber is bounded above by a honeycomb grid over which suction is created by an exhaust fan. This simulates the cloud updraft which is driven by the buoyancy associated with latent heating at mid levels within the storm. The rotating screen at the entrance of the convergence layer imparts angular momentum to the incoming air. This corresponds to low-level vorticity concentration below a stable layer; the updraft hole simulates a "puncture" in the stable layer through which the updraft acts to concentrate vorticity.

Ward (1972) succeeded in measuring the vortex core (see I, p. 1447) size as a function of the screen rotation rate, the updraft strength and radius, and the inflow depth. Ward interpreted his results in terms of two dimensionless parameters, the configuration ratio

\[ c = \frac{2R}{h} \]  

(2.1)

and the inflow angle

\[ \alpha = \tan^{-1} \frac{v_R}{u_R} \]  

(2.2)

where \( R, h, v_R \) and \( u_R \) are the updraft radius, inflow depth, and tangential and radial velocity at \( R \), respectively. Davies-Jones (1973) reexamined these data and found the core size is predominantly a function of a single nondimensional parameter, the swirl ratio

\[ S = \frac{RT_R}{2Q} \]  

(2.3)

where \( 2RT_R \) is the circulation at \( R \) and \( 2\pi Q \) the volume flow rate through the apparatus. Hence \( S \) can be expressed in terms of \( c \) and \( \alpha \) as

\[ S = \frac{1}{4} c \tan \alpha. \]  

(2.4)

The swirl ratio is a fundamental parameter which exercises great control over the nature of the flow. Ward found that the vortex remains axisymmetric for \( S \leq O(1) \). The flow becomes highly asymmetric with the formation of multiple vortices for larger \( S \). The occurrence of this phenomenon in the laboratory adds credibility to Ward's concept because multiple vortices are observed in nature (Fujita, 1971; Agee et al., 1976).

A TVC has been constructed (after the Ward concept) at Purdue University (Church et al., 1977) with modifications which have considerably reduced vortex wander. Some preliminary results of internal flow measurements have been reported by Snow et al. (1977). Thus far, only the total velocity magnitude and yaw angle (\( \tan^{-1} v/u \)) have been measured internally (see Fig. 4.9). These measurements show the basic vortex structure is one with vorticity concentrated within an annulus surrounding a central axis. This fact was inferred by Ward (1972) with his flow visualization technique and appears in the numerical results of I. Ward (1972) showed that the radial surface pressure profile has a pronounced maximum thus giving a high pressure ring which surrounds the central low. (Ward also offered some field observational evidence for this effect.) The pressure maximum occurs at the radius where, in the radial momentum equation, the outward centrifugal acceleration exactly cancels the inward inertial acceleration (see I and/or Sections 4a and 4b for a more complete discussion). Recent measurements (J. T. Snow,
Thus, the governing equations are (in cylindrical coordinates)

\[ u_r + u_
u + u_w z - r^{-1} v^2 = -\rho^{-1} \dot{p} + \nu \left[ r^{-1} (ru)_z \right] + u_z, \]  

(3.1)

\[ v_r + v_
u + v_w z + r^{-1} \dot{w} = \nu \left[ r^{-1} (rv)_z \right] + v_z, \]  

(3.2)

\[ w_r + w_
u + w_z z = -\rho^{-1} \dot{p} + \nu \left[ r^{-1} (rw)_z \right] + w_z, \]  

(3.3)

\[ r^{-1} (ru)_z + w = 0. \]  

(3.4)

The quantities \( u, v, w \) and \( p \) are the radial, azimuthal and vertical velocities, and pressure, respectively. The independent variables are the radius \( r \), height \( z \) and time \( t \). Subscripts denote partial differentiation. The radial, azimuthal and vertical components of the vorticity vector are

\[ \xi = -v_z, \quad \eta = u_z - w_r, \quad \zeta = r^{-1} (rv)_z, \]  

(3.5a, b, c)

respectively. The streamfunction \( \psi \) is defined such that

\[ u = r^{-1} \psi_z, \quad w = -r^{-1} \psi_r. \]  

(3.6a, b)

The “streamfunction-vorticity” method is used to solve Eqs. (3.1)–(3.4), i.e., the vertical derivative of Eq. (3.1) minus the radial derivative of Eq. (3.3) is

\[ \eta = -J(\psi, r^{-1} \eta) = r^{-1} (\Gamma^2)_z + \nu \left[ r^{-1} (r\eta)_z \right] + \eta_z, \]  

(3.7)

where \( J(x, y) = x y_2 - x y \) is the Jacobian operator and \( \Gamma = r \eta \). An equation for \( \Gamma \) is obtained by multiplying Eq. (3.2) by \( r \), i.e.,

\[ \Gamma_r - r^{-1} J(\psi, \Gamma) = \nu \left[ r^{-1} (r\Gamma)_z \right] + \Gamma_z. \]  

(3.8)

The result of substituting Eqs. (3.6a) and (3.6b) into Eq. (3.5b) is

\[ \eta = r^{-1} \psi_z + (r^{-1} \psi_r). \]  

(3.9)

Eqs. (3.7)–(3.9) constitute the basic system of three equations in the three unknowns \( \psi, \Gamma \) and \( \eta \).

The boundary conditions are given in Fig. (3.1). The conditions on the center axis satisfy the axisymmetry condition. The requirements on the rim (a fictitious wall placed here for computational expediency) represent zero-stress boundary conditions. The upper conditions require the flow to exit straight out the top. These conditions are discussed at length in I. The inflow boundary conditions are described below. The implementation of the zero-slip lower boundary condition was not described in I, and hence is described herein.

Air flows into the chamber through the lower right-hand portion of the domain \( r = R, 0 \leq z \leq h \) and flow measurements can be made to determine \( u_R \) and \( \Gamma_R \) over the depth \( h \). However, in a thin boundary layer adjacent to the ground the velocity must go to zero. Hence we let \( u(R, z) = u_{Rf}(z) \) and \( \Gamma(R, z) = \Gamma_{Rf}(z) \)

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3. Formulation and method of solution

For a thorough review of confined vortex flows and numerical vortex models, the reader is referred to Lewellen (1971 and 1976, respectively).

Attention is confined to situations where the flow is axisymmetric. The working medium, air, has a constant density \( \rho \) and, I will assume, a constant eddy viscosity

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4 My usage of the word “turbulence” will refer to turbulent motion on the resolvable scale. The eddy viscosity refers to subgrid-scale turbulence.
when \( z \leq \delta \), where \( \delta \) is the boundary layer thickness at \( r = R \). The functions \( f_z \) and \( g \) must satisfy the conditions \( f_z(0) = g(0) = 0 \), \( f_z(\delta) = g(\delta) = 1 \) and \( f_z(\delta) = g(\delta) = 0 \) which represent the zero-slip requirement and the matching of the external velocity and stress to the boundary layer values at the top of the boundary layer. By comparison of Eq. (3.8) with the boundary condition in Fig. 3.1, it is evident that \( \Gamma_{z z}(r,0,t) = 0 \), hence one further restriction on \( g \) is \( g_{zz}(0) = 0 \). Functions which satisfy these requirements are

\[
\begin{align*}
    f_z(z) &= (\delta^{-1} z) \{ 2 - (\delta^{-1} z) \}, \\
    g(z) &= 0.5 (\delta^{-1} z) \{ 3.0 - (\delta^{-1} z)^2 \}.
\end{align*}
\]

The inflow value of \( \eta \) is specified as

\[
\eta(R,z) = u_z(R,z),
\]

i.e., we demand \( \theta_z(R,z) = 0 \). This simply states that the boundary layer vorticity is predominantly due to the vertical shear (a standard result of boundary layer theory).

It was found in I that the following numerical technique could simulate the basic features of the vortex chamber flow (i.e., core size vs \( S \), the non-dependence core size on \( N \) for large \( N \), the high-pressure ring, etc.). Eqs. (3.7)–(3.9) are solved by a finite-difference technique. Spatial derivatives are centered and are of second-order accuracy. The Jacobian terms are put into discrete form according to the rules of Arakawa (1966). The leapfrog method is used for the time integration; averaging among time levels is performed periodically to suppress mode-splitting. Viscous terms are lagged by one time step. The time step is chosen to satisfy both the CFL (Courant-Friedricks-Lewy) and diffusive stability criterion. The elliptic equation (3.9) is solved by a system4 routine which employs a cyclic reduction technique. The domain size and shape were chosen to conform closely to that used in the laboratory (see Table 3.1). The computational domain contains \((M+2) \times (N+2)\) equally spaced grid points of spacing \( \Delta \).

We let \( i, j \) and \( \tau \) denote radial, vertical and time indices, respectively, where the origin of each index is unity. The integration proceeds as follows. Given \( \psi_{i-1,j}, \psi_{i,j}, \Gamma_{i,j} \) and \( \Gamma_{i+1,j} \), we predict \( \Gamma_{i+1,j} \) and \( \eta_{i+1,j} \) on the interior \( M \times N \) points. Eq. (3.9) is solved subject to the prescribed boundary conditions to obtain \( \psi_{i+1,j} \). It is natural at this point to describe the implementation of the zero-slip boundary condition. The azimuthal velocity must vanish at the surface, hence \( \Gamma_{i,1} = 0 \) for all \( r \). Since the radial velocity \( u \) does not appear explicitly in Eq. (3.7)–(3.9), we use a procedure introduced by Pearson (1965) and used by Williams (1967). After we obtain \( \psi_{i,j} \), we can use Eq. (3.9) to solve for \( \eta_{i+1,j} \) such that the zero-slip condition is satisfied. At \( z = 0 \), \( \eta = \tau^{-1} \psi_z \); a formula which allows the same order of accuracy for computing the boundary value of \( \eta \) as for the interior values is

\[
\eta_{i,1} = \frac{1}{\Delta (i-1)} \left[ -3.5 \psi_{i,1} - 0.5 \psi_{i,0} + 4 \psi_{i-1,0} - 3 \psi_{i-1,1} \right], \quad \text{O} \left( \Delta^2 \right). \quad (3.13)
\]

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**Table 3.1. Constant and variable parameters**

<table>
<thead>
<tr>
<th>Variable parameters</th>
</tr>
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<tbody>
<tr>
<td>( \Gamma_R ) inflow circulation ( [0, 0.195, 0.743, 1.80] \times 10^{-3} \text{ m}^2 \text{ s}^{-1} )</td>
</tr>
<tr>
<td>( \nu ) eddy viscosity ( [1.86, 0.93] \times 10^{-3} \text{ m}^2 \text{ s}^{-1} )</td>
</tr>
<tr>
<td>( \delta ) inlet boundary layer depth ( [(5, 11, 21) \times \Delta] )</td>
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4 These computations were performed on the CRAY T computer at the National Center for Atmospheric Research. A description of this algorithm may be found in Sweet (1977).
Fig. 4.1. Steady-state streamfunction, radial and vertical velocities for $S=0$, $N=10^4$, $\delta=5\Delta$. The first contour line of a typical field $\phi$, is at a specified $\phi_{min}$ and the remaining contour lines are drawn at a specified $\phi_{int}$. Dashed lines indicate a negative value. Thus, $\phi_{min} = -0.057$ m$^2$ s$^{-1}$ ($-2.0$ (ft$^2$ s$^{-1}$)), $\phi_{int} = 0.0028$ m$^2$ s$^{-1}$ (0.1 ft$^2$ s$^{-1}$), $u_{min} = -0.3048$ m s$^{-1}$ (~1 ft s$^{-1}$) $u_{int} = 0.0305$ m s$^{-1}$ (0.1 ft s$^{-1}$), $w_{min} = 0.0305$ m s$^{-1}$ (0.1 ft s$^{-1}$).

The initial data is specified as follows: $\eta(r,0,0)=0$ and $\Gamma(r,0,0)=0$ except at $r=R$, $z<\delta$; Eq. (3.9) is solved to obtain $\psi(r,0)$. $\Gamma(R,z,t)$ for $z<\delta$ is gradually increased to a value $\Gamma_R$. We impose $u_{int}=[0.304, 1$ ft s$^{-1}]$ through the boundary conditions on Eq. (3.9). Thus, given $R$, $h$ and $u_{int}$, the value of $S$ is varied simply by varying $\Gamma_R$. The specification of $\nu$ then gives the radial Reynolds number

$$N = \frac{u_{int} R}{\nu}$$  \hspace{1cm} (3.14)

for the particular experiment.

It was shown in I, using the free-slip condition, that the core-size is independent of $N$. In the course of the present investigation this result was found to be true above the vortex breakdown. The reason for this is given in Section 5b. Certain features such as boundary-layer thickness and the character of the waves produced downstream of the breakdown were found to depend on $N$. However, the basic vortex configurations are found to be mainly a function of $S$. Hence, the results are presented in sections devoted to the flow at a particular $S$; variations of the flow with $N$ or $\delta$ are merely noted within these sections. An explanation of the significance of variations in $\delta$ is offered in Section 4c.

4. Results of the numerical simulations

4.1. Experiments with zero swirl

Snow et al. (1977) display (see their Fig. 3.3) the velocity magnitude $(u^2+w^2)^{1/2}$ as a function of $r$ and $z$ within the chamber when no swirl is imposed. There exists a rather high velocity near the edge of the updraft hole and a separated boundary layer above the lower level surface.

Fig. 4.1 shows the steady-state fields of $\psi$, $u$ and $w$ for $S=0$, $N=10^4$, $\delta=5\Delta$. The model successfully simulates the flow separation although the details of the separated region are probably not represented well due to the likelihood of turbulence (Batchelor, 1967, p. 379) in this region. Large vertical velocities occur near $(R,h)$. The azimuthal vorticity $\eta(R,z)=0$ for $z>h$ to satisfy the free-slip condition and $\eta(R,z)$ is $0$ for $\delta<z<h$ because $\eta_{tu}(R,z)=0$ and, we assume, $w_{\tau}(R,z)=0$. Now since $u(R,h^+)\neq 0$ and $u(R,h^-)=u_R(<0)$, $u_t(R,h)$ is $\infty$. Because $\eta(R,z)=0$ for $\delta<z<\delta$ and in particular $\eta(R,h)=0$, it follows that $w_{\tau}(R,h)=\infty$. Thus $\omega$ in-
creases very rapidly at the edge of the updraft hole. In the laboratory experiment \( \eta(R,h) = \eta_R \), where \( \eta_R \) is boundary layer vorticity associated with the boundary layer on the underside of the upper plate of the convergence zone; \( \eta_R \) is positive. However, extremely large values of \( u_h(R,h) \) occur due to the sharp edge and large values of \( w_h(R,h) \) are to be expected. In any event, due to the fact that the flow is essentially a potential flow, the effect is confined to the vicinity of the sharp edge.

The interior flow is essentially axisymmetric irrotational flow into a corner (see, e.g., Batchelor, 1967, p. 105). In this case

\[
\begin{align*}
u &= -ar, \quad w = +2az. \\
(4.1a, b)
\end{align*}
\]

The inviscid form of Eq. (3.1) shows

\[
\rho_r = -uu_r = -\beta r < 0.
\]

Thus, the pressure gradient is adverse over the entire radius and separation occurs. The result of this numerical integration is (with some minor differences) the axisymmetric analogue to the two-dimensional form of this problem investigated by Leal (1973).

Although this flow is of no direct relevance to tornadoes, it allows a further test of the numerical model and aids in the understanding of the flow with weak imposed swirl.

Variations of \( N \) and \( \delta \) made no qualitative and little quantitative difference in these results.

b. Experiments with weak swirl

Figs. 4.2a and 4.2b are a sequence in the time development of the flow with \( S = 0.105, N = 10^5, \delta = 5\Delta \). The flow through the chamber advects \( \Gamma \) inward and upward, while the streamline pattern changes little over the no swirl case. Eq. (3.1) is approximately

\[
\rho_r = -uu_r = -\frac{\Gamma^2}{r^3}
\]

for the interior flow; thus the effect of the rotation is to counteract the adverse pressure gradient associated with the \(-uu_r\) term. However, when the swirl is weak, as it is in this case, the boundary-layer separation cannot be prevented. Consider the ratio of the terms on the right-hand side of Eq. (4.3) which I designate as \( A \). Since \( \Gamma \) is approximately constant along a streamline, \( \Gamma = \Gamma_0 \), and since Eq. (4.1a) appears to be an adequate approximation of \( u_u = u_0r/R \),

\[
A = \frac{uu_r}{r^{-3} \Gamma^2} = \frac{u_0^2}{\Gamma R^2} (6.55 \times 10^{-5} m^{-4}) r^4
\]

for the parameters used in this case; the adverse for the parameters used in this case; the adverse pressure gradient dominates when \( A \gg 1 \). At \( r = R(0.6096 m), A = 90 \). The quantity \( A = O(1) \) at \( r \approx 40\Delta \); however, the boundary has already separated at approximately \( r \approx 105\Delta \). Some \( \Gamma \) diffuses across the boundary of the recirculating cell and the result is that the contours of \( \Gamma \) are contorted. It is well to recall here that an axisymmetric rotating flow is stable if

\[
\frac{\partial r^2}{\partial r} > 0
\]

(Rayleigh, 1916). Thus, in the contour diagrams as long as \( \Gamma \) increases from left to right, the flow is stable. Note, however, the flow in the separated region does not satisfy Rayleigh's criterion. This structure breaks and the more diffuse structure of Fig. 4.2b is ultimately obtained. Larger values of \( \Gamma \) never penetrate very close to the axis near the corner, but are deflected thus preventing the formation of a strong vortex near the lower surface.

Fig. 4.3 shows the steady-state azimuthal velocity. The maximum swirling occurs rather high up in the chamber. As will be discussed in Section 5, dual-Doppler radar observes rotation at mid-cloud levels first. The following section will show how this maximum swirl descends and intensifies.

Variations of \( N \) and \( \delta \) again have little influence on the basic flow features.

c. Experiments with moderate swirl

Figs. 4.4a and 4.4b represent the time development of the flow with \( S = 0.4, N = 10^5, \delta = 11\Delta \). The quantity \( A = O(1) \) at \( r = 0.396 m \); this is the radius at which \( \rho_r \) changes sign and hence the radial pressure profile will have a relative maximum. This is observed both in the experiment and in the free-slip numerical simulations of \( I \). Thus, the pressure gradient is adverse only over a short interval, and the boundary thickens in the flow direction but does not separate. The radial pressure gradient is favorable for \( r \leq 0.396 m \) and hence the boundary layer thins and the flow accelerates. This is a most important development because now the favorable pressure gradient can drive nonzero \( \Gamma \) very close to the axis near \( z = 0 \), thus producing extremely large swirling velocities at these levels.

The numerical model simulates the "vortex breakdown" phenomenon (compare Fig. 4.4b with Fig. 2.1). The term "vortex breakdown" refers to "an abrupt change in the (vortex) structure with very pronounced retardation of the flow along the axis and a corresponding divergence of the stream surfaces near the axis."
Fig. 4.2a. Streamfunction and circulation for $S=0.105$, $N=10^5$, $\delta = 5\Delta$ at $t=5.0$ s. $\psi_{min} = -0.0567$ m$^2$ s$^{-1} (-2.0$ ft$^2$ s$^{-1})$, $\psi_{max} = 0.0028$ m$^2$ s$^{-1} (0.1$ ft$^2$ s$^{-1})$, $\Gamma_{min} = 0.0$ m$^2$ s$^{-1}$, $\Gamma_{max} = 0.0056$ m$^2$ s$^{-1} (0.06$ ft$^2$ s$^{-1})$. The $\psi$ field is very similar to that of the no swirl case.

Fig. 4.2b. Steady-state streamfunction for $S=0.105$, $N=10^5$, $\delta = 5\Delta$ with the same contouring as in Fig. 4.2a.
(Hall, 1972). The breakdown of a vortex in contact with a lower boundary was first observed by Maxworthy (1972) and Ward (1972).

There are three prominent theories to explain vortex breakdown, one of which relies heavily on flow asymmetries (Ludwig, 1961). Since the present model is axisymmetric and vortex breakdown occurs, one may eliminate that theory from consideration. The two remaining theories are as follows.

Gartshore (1962) and Hall (1967) propose that vortex breakdown is a type of flow separation phenomenon. Benjamin (1962, 1967) believes that vortex breakdown is the rotating fluid analogue to the hydraulic jump phenomenon of open-channel flows. As explained by Hall (1972), neither of these theories can completely account for the observed flow behavior; however, both theories have relevant components.

Benjamin (1962) argues that vortex breakdown is a spontaneous transition between a supercritical (no waves allowed) and a subcritical flow, both of which are solutions of the same nonlinear governing equation. It is argued that the subcritical flow has a greater flow force \(2\pi \int_0^a (aw^2 + p)rdr\), where \(a\) is the containing tube radius than its supercritical conjugate. Benjamin finds that for weak jumps waves which form downstream of the jump exert enough drag on the subcritical flow to reduce the flow force to its upstream (supercritical) value. For strong jumps the excess flow force is lessened by wave breaking.

Hall (1972) describes the flow separation analogy as follows. Based on the steady quasi-cylindrical approximation, it can be demonstrated that a slight divergence of the flow stream tubes, in the flow direction, induces a much larger adverse pressure gradient on the axis than at the edge of the core; the adverse pressure gradient can cause strong axial flow retardation and perhaps backflow which causes greater stream tube divergence. The resulting flow can no longer be regarded as being quasi-cylindrical.

Hall (1972) synthesizes these two theories by noting that the supercritical and subcritical conjugate flows are two distinct solutions of the governing equation at the transition point; hence, axial gradients must necessarily be large in this region, thus destroying the quasi-cylindrical approximation.

These explanations can be tested by considering the radial profiles of \(v\) and \(w\) upstream of the breakdown and asking whether or not they represent a supercritical flow. Benjamin (1962) suggests a method by which to make this determination. A supercritical flow is one which does not allow wave motion. For a steady, inviscid flow with axial and azimuthal velocities which depend solely on \(r\), the equation governing infinitesimal perturbations to this flow is

\[
\frac{r}{r} \left( \frac{\psi_r}{r} \right) + \frac{\psi_{rr}}{r^2} + \frac{\left( \frac{r^2}{\omega} \frac{\psi_r}{r} - \frac{\psi_r}{r} \right)}{r^2} = 0, \quad (4.6)
\]

where \(\psi\) is the perturbation streamfunction; \(\psi(r)\) and \(\Gamma(r)\) are the mean axial velocity and circulation, respectively. If we assume \(\psi \approx \exp(\gamma z)\), Eq. (4.6) becomes

\[
\frac{r}{r} \left( \frac{\psi_r}{r} \right) + \frac{\gamma^2}{r^2} + \frac{\psi_r}{r^2} = 0. \quad (4.7)
\]

Eq. (4.7) together with the boundary conditions

\[
\psi(0) = 0, \quad \psi(a) = 0 \quad (4.8a, b)
\]
define an eigenvalue problem for the eigenvalues $\gamma_r$. If any of the $\gamma_r < 0$, the flow is subcritical, i.e., axial standing waves exist. The supercriticality test is as follows. A function $\psi_c(r)$ is defined such that

$$ r \left( \frac{\psi_{cr}}{r} \right) + \left[ \frac{r^{-3}(\Gamma)^2_r}{\psi} \frac{\psi_r}{\psi} \frac{\psi_{rr}}{\psi} \right] \psi_c = 0 \quad (4.9) $$

and $\psi_c$ satisfies one of conditions (4.8a, b). Suppose the complete system Eq. (4.7) is subcritical (at least one $\gamma_r < 0$). It follows from Sturm's fundamental comparison theorem (Ince, 1956, Section 10.3) that $\psi_c$ oscillates more rapidly within $(0,a)$. This fact leads to the conclusion that if $\psi_c$ has a zero within $(0,a)$ the flow is subcritical; if the complete system is supercritical (all $\gamma_r > 0$), $\psi_c$ oscillates less rapidly within $(0,a)$. Thus, if $\psi_c$ does not have a zero within $(0,a)$ the flow is supercritical. If $\Gamma$ is constant, Eq. (4.9) may be written as

$$ \left( \frac{\psi_{cr}}{r} \right) - \frac{1}{\psi} \frac{\psi_r}{\psi} \psi_c = 0, $$

thus $\psi_c \approx \psi$. Hence $\psi_c$ will not have a zero unless $\psi = 0$. Thus, in the following calculations we compute $\psi_c$ from $r = 0$ to $r = 0.5R$ (where $\Gamma$ is constant); if no zeroes occur, the flow is supercritical.

Fig. 4.5 show radial profiles of $w, v$ and $\psi_c$ at selected axial stations approaching the impending vortex breakdown. Upstream of (below) the breakdown the flow is quasi-cylindrical, supercritical, and approaches critical as the breakdown is approached. The breakdown is characterized by a separation of the flow off the axis with attendant downflow and the establishment of standing waves. The breakdown is, essentially, the breaking of the lead wave; this was predicted by Benjamin (1962). Fig. 2.1 shows that after the lead wave breaks, the downstream flow becomes turbulent. For this reason, I have not employed a radiation condition which would allow these waves to pass unreflected at $z = H$ since the flow in the core will not be well represented after wavebreaking which occurs much before waves have formed all the way to $z = H$. These calculations support almost all of the main theoretical assertions of Benjamin (1962) and Hall (1972) and one may thus be more assured of the correctness of those assertions.

In the present experiment $\delta = 11\Delta$ because for larger $\delta$ the vortex breakdown initially occurs higher up on the axis, thus giving a longer region of quasi-cylindrical flow before the breakdown. The smaller values of $\delta$ lead

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$\delta r^2/\delta r < 0$ locally.
to more \( \Gamma \) “getting into” the boundary layer and thus forced to converge close to axis to produce a very large swirling velocity. This induces a larger adverse pressure gradient which precipitates a vortex breakdown much closer to the lower surface. This is a very important result because it implies that a very thin upstream surface boundary layer can produce very large swirling velocities at the base of the vortex. Fig. 4.6 is a comparison of the swirling flows for \( N = 10^3 \), \( S = 0.4 \), \( \delta = 5, 11, 21 \) at times near where each case reaches its maximum swirl. The larger swirl velocity is indicative of a larger vertical vorticity close to the axis. The equation for the vertical vorticity is (neglecting friction)

\[
\frac{d\zeta}{dt} = \xi w_x + \xi w_z. \tag{4.10}
\]

Near the axis the azimuthal vorticity \( \eta = -w_r \), so that

\[
\frac{d\xi}{dt} = -\xi \eta + \xi w_z. \tag{4.11}
\]

Since \( \xi \) and \( w_z \) are positive in the region below the vortex breakdown, the effect of the second term on the right-hand side of Eq. (4.11) is to increase the vertical vorticity. The azimuthal vorticity \( \eta \) is seen (Fig. 4.7) to be large and positive near the axis; this is due to the term \( r^{-3}(\Gamma^2) \), in Eq. (3.7) which is large and positive in the boundary layer. The radial vorticity \( \xi = -r^{-1}\Gamma \), which is negative in the boundary layer. Thus the first term on the right-hand side of Eq. (4.11) is positive. Furthermore, both \( \xi \) and \( \eta \) behave roughly as \( r^{-1} \), thus a partial explanation for the behavior in Fig. (4.6) has been established. Physically speaking, the amount of radial vorticity \( \xi \) available for upward tilting and the ability to perform the tilting (\( w_r \) or \( -\eta \)) are both enhanced given a smaller boundary layer thickness.

The effect of lowering \( \nu \) is to make the breakdown more violent, i.e., higher wavenumbers are excited and even traveling waves exist in the solutions.

In summary, the numerical model simulates the vortex breakdown phenomena; this is characterized by strong boundary layer convergence, axial upflow over the lower portion of the flow, a sudden divergence of the stream surfaces with attendant axial downflow and (probably) turbulence above. The entrant boundary-layer thickness \( \delta \) can be an important upstream indicator of the maximum swirl velocity of the vortex near the ground.
5. Implications for theory, experiments and observations

a. Theory

The experiments at low $S$ emphasize the importance of the radial inflow velocity in connection with boundary layer separation. The boundary layer separation acts to deflect circulation about the corner region, and hence prevents the formation of a strong vortex near $z=0$. The maximum swirl velocity is found well above the convergence layer.

The experiments with intermediate $S$ reveal a complex flow structure radically different from that in the low $S$ regime. The boundary layer remains attached and consequently a large radial acceleration occurs. This convergence induces an extremely large swirling velocity close to the lower surface and near the axis, much larger than what is obtained in the “free-slip” experiments of I. This has also been noted by Lewellen and Teske (1977). The numerical results indicate a sensitivity of $r_{max}$ to the entrant boundary layer depth $\delta$.

Vortex breakdown occurs in the numerical simulations. The complete flow field is simulated (unlike the boundary-layer type calculations of Hall (1967) which fail at or near the point of flow separation). This affords a unique opportunity to test a number of theoretical predictions concerning the nature of this phenomenon.

At larger $S$ the separation point moves down the axis, to the lower surface at some radius. The vortex breakdown takes the character of a “drowned” hydraulic jump as described by Mullen and Maxworthy (1976).

The present model indicates that there are two major sources of turbulent motion (within the context of the axisymmetric restriction). The first source of turbulence is a boundary-layer separation phenomenon which occurs for small values of the swirl ratio $S$; the turbulence is confined to the boundary layer. The second source is related to the axisymmetric stability criterion of Howard and Gupta (1962) (Section 4c). However, the basic flow in the model satisfies this criterion and it appears that this criterion is violated locally only after large-amplitude inertial waves have grown to the point of breaking. I believe the mechanism at work is an instability due to the large-amplitude inertial wave connected to the vortex breakdown region. The flow instability takes the form of a train of breaking inertial waves with (necessarily) axial symmetry. It is unlikely the flow downstream of the lead breaking wave is axisymmetric and, although our model can resolve the wave breaking, one cannot be sure that the resulting turbulent diffusion is correct. Hence, one must adopt the philosophy that the axisymmetric numerical model can give much insight into a number of phenomena and, in particular, can indicate where turbulence is likely to occur, and why; however, one will be left in doubt over the quantitative effect of the turbu-

d. Experiments with large swirl

Fig. 4.8 represents the solution at $t=6.0$ s, for $N=10^4$, $S=1$, $l=5a$. The flow separates off the lower surface; the downdraft penetrates to $z=0$. The separated zero streamline makes a rather small angle with respect to the horizontal. The flow in this case is reminiscent of what Mullen and Maxworthy (1977) term a “drowned vortex jump”. The expansion (with height) of the stream surface is so sudden that it is no longer distinguishable from the boundary layer flow and the resulting flow resembles a fat turbulent column normal to the lower surface. Experiments with larger $N(=2\times10^4)$ support this view.
Fig. 4.6. Azimuthal velocities for $S=0.4$, $N=10^4$; however, $\delta = 5\Delta$, $11\Delta$ and $21\Delta$ in the three figures. The display time is chosen so that $v$ is near the maximum it achieved during an integration. Thus, for $\delta = 5\Delta$, $t = 10$ s, for $\delta = 11\Delta$, $t = 11$ s and for $\delta = 21\Delta$, $t = 15$ s. $\eta_{\text{min}} = 0.0$ and $\eta_{\text{inc}} = 0.1524$ m s$^{-1}$ (0.5 ft s$^{-1}$).

b. Experiments

The author obtained good agreement in I between the numerically predicted and experimentally observed dependence of the core size on $S$, even though a free-slip lower boundary condition was used. Fig. (4.4b) indicates (compare with Fig. 4.4 of I) that the flow above the breakdown is largely (with exception of waves) the same as that obtained in the free-slip solutions. If the hydraulic jump analogy is valid, and I believe it is, the explanation for this agreement is that the upstream supercritical flow undergoes a hydraulic jump to match the downstream subcritical flow which is

Fig. 4.7. Azimuthal vorticity for $S=0.4$, $N=10^4$, $\delta = 5\Delta$ at $t = 10$ s. Large positive $\eta$ exists near $r = 0$ below a certain height. $\eta_{\text{min}} = -25.00$ s$^{-1}$ and $\eta_{\text{inc}} = 4$ s$^{-1}$.
determined by $S$. A similar conclusion was reached by Maxworthy (1972).

It is observed in the laboratory that a single concentrated vortex forms only for intermediate values of $S$ (Davies-Jones, 1978). A large $S$ does not produce a single concentrated vortex because the updraft cannot cause sufficient radial convergence to counteract the large outward centrifugal force (Section 4d). The reason why a concentrated vortex does not form for small $S$ is not quite so obvious. It would appear that a parcel of fluid with a nonzero amount of angular momentum ($mr$) could obtain a very large azimuthal velocity $v$ however weak the updraft, provided only the updraft is maintained for a sufficiently long time. It is demonstrated in Section 4b that the boundary layer over the lower surface separates under the influence of the external decelerating radial flow, thus deflecting the outer flow around the corner region ($r=z=0$). Somewhat larger values of $S$ (Section 4c) set up a favorable external pressure gradient which forces fluid (with nonzero angular momentum) to penetrate very close to the axis, thus producing an extremely large swirling velocity.

The interior flow measurements made by Snow et al. (1977) are not sufficiently accurate to warrant a detailed comparison with the numerical results. However, we may note the salient features of the measurements. Fig. 4.9 is taken from Snow et al. (1977). There appears to be a relatively stagnant region near $r=0$, surrounded by a region of high shear over the mid to upper levels in the chamber. The velocity magnitude becomes quite intense in the corner region and appears to have a secondary maximum at a higher level. I believe there is a good qualitative agreement between the experimentally observed flow and the numerical solutions.

c. Observations

Recent Doppler radar studies (Brown and Lemon, 1976) indicate that rotation, nearly on the funnel scale ($\sim 1$ km) occurs first at mid-levels within the parent storm and henceforth grows downward. This observation has led some investigators to abandon the concept of low-level convergence in favor of a model which emphasizes upper-level convergence (Eskridge and Das,
1976). At low $S$, the maximum swirl is at high levels within the chamber due to the boundary-layer separation mechanism. At moderate to large $S$ the boundary layer remains attached and the maximum swirling velocity is found at very low levels (essentially within the boundary layer). Thus, I advance the hypothesis that the lowering maximum in swirl is due to a steadily intensifying low-level circulation which changes the swirling field from one resembling Fig. (4.3) to one resembling Fig. (4.6a) in a continuous fashion. Support is lent to this hypothesis by the observations of Brandes (1977, 1978).

Golden and Purcell (1977) present a photogrammetric velocity analysis of the tornado which struck Great Bend, Kansas, on 30 August 1974. According to their analysis, the really intense winds occur very close to ground level (<100 m) and decay rapidly aloft. This is the behavior of the no-slip numerical solutions, in fact, the no-slip conditions can\footnote{Depending on $\delta$ (see Section 4c).} increase the maximum swirl by as much as 50\% over that obtained in the free-slip experiments. This increase is not an entirely unexpected result. The magnitude of the effect, however, points to the importance of boundary layer dynamics for the determination of the maximum wind speed.

The axial downdraft (present in I) is intensified by the no-slip condition. Axial downflow has been observed in waterspouts (Golden, 1973) and dust devils (Sinclair, 1973a). There is even some evidence of downflow within tornadoes (Sinclair, 1973b; Hokecker, 1960, 1961).

6. Summary

Fine-resolution calculations of the flow within the vortex chamber conceived by Ward (1972) to study
tornado-like vortices are presented. A short review of the major experimental findings to date is given.

The numerical results indicate the single most important parameter governing vortex structure is the swirl ratio $S$. The solutions for $S=0$, $0.1$, $0.4$, $1.0$ at a radial Reynolds number $N$ comparable ($=10^3$) to that used in the laboratory are discussed. It is found that when

1) $S=0$, the flow separates off the lower surface under the influence of an adverse pressure gradient.

2) $S=0.1$, flow separation still occurs which deflects angular momentum around the corner region, thus preventing the formation of a concentrated vortex at low levels.

3) $S=0.4$, the flow does not separate off the lower surface boundary-layer, convergence induces large swirl and vertical velocity in the corner region. Vortex break-down occurs above this region with the establishment of large-amplitude inertia waves downstream.

4) $S=1.0$, the axial downdraft penetrates to the lower surface, the vortex jump is very close to the surface.

When these experiments were also carried out with $N=2 \times 10^3$, the general effect was to make the breakdown (with its attendant wave instabilities) more violent. The effect of the entrant boundary-layer thickness is found to have a large effect on the maximum swirl attained.

These results are consistent with the laboratory experiments in that boundary-layer separation at low $S$, vortex-breakdown and transition to turbulence, and a high shear core wall surrounding a relatively stagnant inner region are all observed and simulated.

The observational findings of a lowering maximum swirl speed from cloud base and the large intensity of tornadic winds being confined to very low levels within the vortex can be explained within the context of the present model.

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