On Normal Mode Linear Initialization on the Sphere

A. Wiin-Nielsen

European Centre for Medium-Range Weather Forecasts, Shinfield Park, Reading, Berkshire, England

(Manuscript received 26 March 1979, in final form 21 June 1979)

ABSTRACT

The normal mode initialization procedure is investigated. It is shown that a balance exists between the wind field and the mass field when the gravity modes have been removed from the initial fields. Adopting a representation in the spectral domain on the sphere it is shown that the vectors consisting of all amplitudes of the streamfunction and the velocity potential, respectively, are related to the vector consisting of all amplitudes of the geopotential by a square matrix which depends entirely on the eigenvalues and eigenvectors of the truncated systems.

The balance which exists after normal mode initialization is compared with the quasi-geostrophic balance when this procedure is applied to the adjusted initial fields which are obtained when the contribution from the gravity waves has been removed. It turns out that the balance from the normal mode procedure is virtually identical to the quasi-geostrophic balance except on the largest scales. The difference on the largest scale between the Rossby or rotational modes obtained by the two procedures is in the linear case entirely due to the sphericity of the earth since the modes would be identical if the Coriolis parameter were constant.

The modifications to the initial state created by the normal mode procedure are investigated in Section 5, and Section 6 contains an analysis of the first baroclinic mode analogous to the basic barotropic mode considered in the main body of the paper.

1. Introduction

The problem of initialization in numerical weather prediction has received much attention. Bengtsson (1975) has given a review of the various procedures which are used or being considered for operational use. One of the most recent developments is the introduction of the so-called normal mode initialization which was first developed by Flattery (1970) although he restricted himself to the nondivergent components of the atmospheric flow. Further developments of the procedure have been carried out by Dickinson and Williamson (1972), Williamson (1976) and Daley (1978).

The present paper is limited to the linear case. Nonlinear aspects of the problem have been considered by Machenhauer (1977) and Baer and Tribbia (1977). The general problem considered in this paper, i.e., a comparison between the normal mode and the quasi-geostrophic initialization procedures, is closely related to the problems treated by Moura (1976) in his comparison of the solutions of the balanced system with the more fundamental linear solutions of the Laplace tidal equations. In the present study we concentrate on the more limited problem of the consequences of normal mode initialization.

The basic tool used in normal mode initialization is to determine the eigenfunctions corresponding to the eigenvalues in a suitably truncated system. The modified initial state is then obtained by subtracting the contributions from the meteorologically unimportant gravity modes from the total fields. Normal mode initialization thus requires a modification of the analyzed fields to assure that no gravity modes exist. It is to be expected that a new type of balance will exist between the wind field and the mass field in the modified fields. The major purpose of this investigation is to determine this balance and to compare it to other more elementary balance conditions. In particular, we shall compare with geostrophic balance or the so-called linear balance equation.

We may in general distinguish between external and internal gravity waves. The speed of propagation of the former is much larger than the wave speed of the latter. In this paper which is restricted to the linear case we can separate the two cases, and they will be treated in different sections.

2. The problem

We shall formulate the normal mode procedure on the spherical earth. In this section we shall consider the shallow water equations. It is most convenient to write these equations in terms of a vorticity, a divergence and a continuity equation. The dependent variable will be the streamfunction $\psi$.  

Unauthenticated | Downloaded 09/18/23 01:40 AM UTC
the velocity potential $\chi$ and the geopotential $\phi$ which we replace by the quantity $\eta$ where $\phi = 2\Omega\eta$ and $\Omega$ is the angular velocity of the earth ($\eta$ has the same dimension as $\psi$ and $\chi$). We introduce also a non-dimensional time $\tau$ defined by $\tau = 2\Omega t$. With these notations and definitions and with $\mu = \sin \phi$ where $\phi$ is latitude the equations are

\[
\begin{align*}
\frac{\partial \nabla^2 \psi}{\partial \tau} + \frac{\partial \psi}{\partial \lambda} + \mu \nabla^2 \chi + (1 - \mu^2) \frac{\partial \chi}{\partial \mu} &= 0 \\
\frac{\partial \nabla^2 \chi}{\partial \tau} + \frac{\partial \chi}{\partial \lambda} - \mu \nabla^2 \psi &= -(1 - \mu^2) \frac{\partial \psi}{\partial \mu} + \nabla^2 \eta = 0 \\
\frac{\partial \eta}{\partial t} + q^2 \nabla^2 \chi &= 0
\end{align*}
\]

where \( q^2 = \frac{g H}{4\Omega^2 a^2} \).

\[ (1) \]

\[ (2) \]

Introducing $s = -mc$ we obtain the infinite set of equations

\[ (3) \]

\[ (4) \]

It is seen that an even (odd) value of $n$ for the streamfunction corresponds to an odd (even) value for the velocity potential, while an even (odd) value for the geopotential corresponds to an even (odd) value for the velocity potential. The system can therefore be divided in two separate systems. We shall restrict ourselves to the system in which $\eta_m$ and $\chi_m$ are symmetric around the equator, but the antisymmetric case, to be considered separately in a later publication, is also of interest because Moura (1976) finds that for the mixed Rossby-gravity modes, i.e., the gravest antisymmetric Rossby mode, there

\[ (5) \]

\[ (6) \]

\begin{align*}
H \text{ is the equivalent depth of the fluid, } g \text{ the acceleration of gravity and } a \text{ the radius of the earth. The system (1) has been used by Eliassen and Machenhauer (1969) and in a more general context by Wiin-Nielsen (1971).}

To obtain the eigenvalues we introduce the perturbations in the form

\[
\begin{pmatrix}
\psi \\
\chi \\
\eta
\end{pmatrix} =
\begin{pmatrix}
\psi_m \\
\chi_m \\
\eta_m
\end{pmatrix} e^{i(m\lambda - ct)},
\]

\[ (3) \]

where the functions with subscript $m$ are functions of $\mu$. For these we write

\[
\begin{align*}
\psi_m &= \sum_{n=m}^{\infty} \Psi(n) P_m^m(\mu) \\
\chi_m &= \sum_{n=m}^{\infty} X(n) P_m^m(\mu) \\
\eta_m &= \sum_{n=m}^{\infty} Z(n) P_m^m(\mu)
\end{align*}
\]

\[ (4) \]

Introducing (3) and (4) in (1) and setting $s = -mc$, we obtain the infinite set of equations

\[ (5) \]

\[ (6) \]

is good agreement between tidal and quasi-geostrophic theory.

The infinite system (5) becomes finite when we put an upper limit to $n$. The first problem is then to determine all the eigenvalues $s$ of the system. For the even components of $X$ and $Z$ we write $n = m + 2r$, while the odd components of $\Psi$ has $n = m + 2r + 1$. The parameter $r$ has the values 1, 2, . . . , $R$. We shall thus have $3(R + 1)$ equations and the same number of eigenvalues. With these notations we may reformulate (5) as a standard eigenvalue problem where the eigenvalues and eigenvectors are determined numerically. The system is
In writing the first equation in (6) for $r = 0$ we must omit the first term, while the last term in the third equation of (6) must be omitted for $r = R$. The last omission closes the system.

3. Linear normal mode initialization

In this section we shall consider the general case as expressed in the Eqs. (5). The $3(R + 1)$ eigenvalues are determined by numerical methods, and they will be denoted $s_i$, $i = 1, 2, \ldots, 3(R + 1)$. To each eigenvalue corresponds an eigenvector with the components

$$X_i(m), \quad \tilde{Z}_i(m), \quad \tilde{\Psi}_i(m + 1), \quad \ldots \quad \tilde{X}_i(m + 2R), \quad \tilde{Z}_i(m + 2R), \quad \tilde{\Psi}_i(m + 2R + 1).$$

The first practical question is the selection of $R$. For each selection of $R$ there will be some eigenvalues which are inaccurate. In analogy with Eliassen and Machenhauer (1969) and Win-Nielsen (1971) some test calculations have been carried out. It turns out that $R = 6$ for each value of $m$ gives excellent accuracy for the modes of interest. The eigenvalues fall in two groups: the numerically small and the numerically large values. The numerically small values correspond to the Rossby modes, while those in the second group correspond to the gravity modes. We shall in the following assume that the eigenvalues and the eigenvectors are ordered in such a way that the first $2(R + 1)$ values are those for the gravity modes while the last $(R + 1)$ values are those for the Rossby modes.

We shall next consider the initial conditions. They are given by

$$X_0(m), \quad Z_0(m), \quad \Psi_0(m + 1), \quad \ldots \quad X_0(m + 2R), \quad Z_0(m + 2R), \quad \Psi_0(m + 2R + 1).$$

For each element among the initial values we know that the sum of contributions from all the modes, i.e., all gravity modes and all Rossby modes, must be equal to the initial value. This condition leads to the following set of equations:

$$\begin{align*}
X_1(m) + X_2(m) + \ldots + X_{3(R+1)}(m) &= X_0(m), \\
Z_1(m) + Z_2(m) + \ldots + Z_{3(R+1)}(m) &= Z_0(m), \\
\Psi_1(m + 1) + \Psi_2(m + 1) + \ldots + \Psi_{3(R+1)}(m + 1) &= \Psi_0(m + 1), \\
X_1(m + 2R) + X_2(m + 2R) + \ldots + X_{3(R+1)}(m + 2R) &= X_0(m + 2R), \\
Z_1(m + 2R) + Z_2(m + 2R) + \ldots + Z_{3(R+1)}(m + 2R) &= Z_0(m + 2R), \\
\Psi_1(m + 2R + 1) + \Psi_2(m + 2R + 1) + \ldots + \Psi_{3(R+1)}(m + 2R + 1) &= \Psi_0(m + 2R + 1).
\end{align*}$$

(7)

We note first that a vertical column in (7) is proportional to an already computed vector, i.e., the eigenvector corresponding to the value $i$. Consequently, we may write

$$\begin{align*}
X_i(m) &= K_i \tilde{X}_i(m), \\
Z_i(m) &= K_i \tilde{Z}_i(m), \\
\Psi_i(m + 1) &= K_i \tilde{\Psi}_i(m + 1), \\
&\vdots \\
\Psi_i(m + 2R + 1) &= K_i \tilde{\Psi}_i(m + 2R + 1).
\end{align*}$$

(8)

Introducing (8) in (7), we may write the system (7) as a single equation

$$\begin{align*}
\{E\} \mathbf{K} &= \mathbf{V}_0 \\
\text{in which } \{E\} \text{ is a matrix with the eigenvectors as the column vectors, } \mathbf{K} \text{ is a column vector with the components } K_i, \ i = 1, 2, \ldots, 3(R + 1) \text{ and } \mathbf{V}_0 \text{ is the vector of all the initial values. Eq. (9) can be solved for } \mathbf{K} \text{ using standard methods, and we may then use (8) to calculate the required result. We note that the procedure outlined above may be used for any set of initial conditions } \mathbf{V}_0 \text{ regardless of how these values have been obtained.}
\end{align*}$$

The normal mode procedure consists in subtracting from the given initial conditions contributions from the gravity modes. We may express the corrected initial conditions in the form

$$\begin{align*}
X_0(m) &= X_0(m) - \sum_{j=1}^{2(R+1)} X_j(m), \\
Z_0(m) &= Z_0(m) - \sum_{j=1}^{2(R+1)} Z_j(m), \\
\Psi_0(m + 1) &= \Psi_0(m + 1) - \sum_{j=1}^{2(R+1)} \Psi_j(m + 1), \\
&\vdots \\
\Psi_0(m + 2R + 1) &= \Psi_0(m + 2R + 1) - \sum_{j=1}^{2(R+1)} \Psi_j(m + 2R + 1).
\end{align*}$$

(10)
The vector $\mathbf{V}'_0$ is the initialized fields under the normal mode procedure. Due to the conditions (10) which remove $2(R + 1)$ modes from the initial conditions we find that the remaining $(R + 1)$ Rossby modes must satisfy $3(R + 1)$ equations. We divide these equations in three groups collecting all the equations pertaining to the velocity potential in the first group, those pertaining to the geopotential in the second, and the remaining equations pertaining to the streamfunction in the third. Each of these groups has $(R + 1)$ equations and $(R + 1)$ unknowns, i.e., $K_{2R+3}, \ldots, K_{3R+1}$. In formal terms we may write

$$\begin{cases}
\{\mathbf{X}_0\}_{K_{R0}} = \mathbf{X}'_0 \\
\{\mathbf{Z}_0\}_{K_{R0}} = \mathbf{Z}'_0 \\
\{\mathbf{\Psi}_0\}_{K_{R0}} = \mathbf{\Psi}'_0
\end{cases}, \quad (11)$$

in which $K_{R0}$ is a column vector with the elements

$$\{\mathbf{X}_0\} = \begin{bmatrix} 
\dot{X}_{2R+3}(m) & \ldots & \ldots & \ldots & \dot{X}_{3R+1}(m) \\
\vdots & & & & \\
\dot{X}_{2R+3}(m + 2R) & \ldots & \ldots & \ldots & \dot{X}_{3R+1}(m + 2R)
\end{bmatrix}$$

with corresponding definitions of $\{\mathbf{Z}_0\}$ and $\{\mathbf{\Psi}_0\}$. It follows from (11) that each equation can be solved for $\mathbf{X}_0$, and we find

$$K_{R0} = \{\mathbf{X}_0\}^{-1}\mathbf{X}'_0 = \{\mathbf{Z}_0\}^{-1}\mathbf{Z}'_0 = \{\mathbf{\Psi}_0\}^{-1}\mathbf{\Psi}'_0. \quad (13)$$

We have therefore

$$\dot{X}_0 = \{\mathbf{X}_0\}\{\mathbf{Z}_0\}^{-1}\mathbf{Z}'_0 = \{\mathbf{X}_0\}\{\mathbf{\Psi}_0\}^{-1}\mathbf{\Psi}'_0.$$ \quad (14)

Eq. (14) shows that the normal mode initialization procedure results in a balance between the wind field and the mass field. Since each of the matrices in (14) is a square matrix of dimension $(R + 1)$ it is seen that the balance involves all the amplitudes in the adjusted initial conditions. The matrices connecting the velocity potential and the streamfunction to the geopotential depend entirely on the eigenvectors of the system and only on those parts included in the definition (12).

4. Normal mode and quasi-geostrophic initialization

It is of considerable interest to explore the nature of the normal mode initialization. One way in which this can be done is to compare the procedure with the well-known quasi-geostrophic procedure which consists of determining the streamfunction and the velocity potential from the quasi-geostrophic equations. The quasi-geostrophic relation between the streamfunction and the geopotential is obtained

$$K_{2R+3}, \ldots, K_{3R+1},$$

while $\mathbf{X}'_0$ is a column vector containing all the adjusted amplitudes of the velocity potential in the initial conditions, i.e.,

$$X_0(m), X_0(m + 2), X_0(m + 4), \ldots, X_0(m + 2R).$$

The elements in the column vector $Z'_0$ are

$$Z_0(m), Z_0(m + 2), Z_0(m + 4), \ldots, Z_0(m + 2R)$$

and those in $\Psi'_0$ are

$$\Psi_0(m + 1), \Psi_0(m + 3), \Psi_0(m + 5), \ldots,$$

$$\Psi_0(m + 2R + 1).$$

We mention finally that the matrices $\{\mathbf{X}_0\}, \{\mathbf{Z}_0\}, \{\mathbf{\Psi}_0\}$ depend entirely on the part of the eigenvectors which pertain to the Rossby modes and to the variable in question. Specifically, we write that

$$Z(m + 2r) = \frac{(m + 2r - 1)(2r)}{(m + 2r)(2m + 4r - 1)} \Psi(m + 2r - 1)$$

$$+ \frac{(m + 2r + 2)(2m + 2r + 1)}{(m + 2r + 1)(2m + 4r + 3)} \times \Psi(m + 2r + 1). \quad (15)$$

The relation (15) can be used to calculate the streamfunction from the geopotential. Starting from $r = 0$ we find

$$Z(m) = \frac{(m + 2)(2m + 1)}{(m + 1)(2m + 3)} \Psi(m + 1), \quad (16)$$

which can be used to calculate $\Psi(m + 1)$ from $Z(m)$. Knowing $\Psi(m + 1)$ and $Z(m + 2)$ we may then calculate $\Psi(m + 3)$ from (15) for $r = 1$. It is seen that $\Psi(m + 3)$ will be a linear combination of $Z(m)$ and $Z(m + 2)$. We may continue this process by increasing $r$ by one unit each time and continue to use (15). Since $Z(m + 2R)$ is the last value of the geopotential, it is seen that the final value of the streamfunction will be $\Psi(m + 2R + 1)$. Each value of the amplitude of the streamfunction will be a linear combination of all values of the amplitudes of the geopotential with indices smaller than the index for the streamfunction. With respect to the
matrix which in the quasi-geostrophic case will
connect the streamfunction vector to the geopotential
vector it means that the matrix has zeroes above
the diagonal. It is obvious that the procedure
described above can be used to calculate all the ele-
ments in the matrix in the quasi-geostrophic case.
In the following we shall compare the normal mode
and the quasi-geostrophic cases by calculating the
matrices for \( m = 1 \) and \( m = 5 \). Table 1 gives all
the elements in the matrix \( \{ \Psi_{ij} \} \{ Z_{ij} \}^{-1} \) for the
normal mode case for \( m = 1 \). It is seen that the largest
values appear along the diagonal, and that the values
above the diagonal are small compared to the diag-
onal values. Table 1 must be compared to the values
listed in Table 2 which gives the corresponding
matrix in the quasi-geostrophic case. The two
procedures are very close to each other except on the
very largest scale where considerable differ-
ces appear as seen from the first column in each ma-
trix. The difference on the largest scale is about 16%.
There is much better agreement between the two
matrices for larger values of \( m \). To illustrate this
point we have repeated the calculations for \( m = 5 \),
and the results are given in Tables 3 and 4. It is ob-
served that the two matrices are almost identical
below the diagonal, and that the numbers appearing
above the diagonal in Table 3 are extremely small.

The next problem is to investigate the velocity
potential in a similar manner. For this purpose we
use the last part of the first equation in (14) relating
\( X_0 \) to \( \Psi_0 \) through the matrix \( \{ X_{ij} \} \{ \Psi_{ij} \}^{-1} \). It turns
out that this relation is more convenient than the
relation between \( X_0 \) and \( Z_0 \) because the former
relationship is easier to compare with the quasi-
geostrophic case. To derive the quasi-geostrophic
equation for the velocity potential we return to (6)
into which we shall introduce the relation (15). This
is done in the third equation of (6). The next step is
to write the first equation in (6) for \( (m + 2r - 1) \)
and \( (m + 2r + 1) \). From the resulting equations
we eliminate those terms which contain \( s \), i.e.,
the elimination of the time derivatives. The resulting
equation can be written in the form

\[
A(m,r)X(m + 2r - 2) + B(m,r)X(m + 2r) \\
+ C(m,r)X(m + 2r + 2) \\
= D(m,r)\Psi(m + 2r - 1) \\
+ E(m,r)\Psi(m + 2r + 1),
\]

where

\[
A(m,r) = \frac{(m + 2r - 2)(2r)(2r - 1)}{(m + 2r)(2m + 4r - 1)(2m + 4r - 3)} \\
B(m,r) = \frac{(m + 2r - 1)(m + 2r + 1)(2r)(2m + 2r)}{(m + 2r)^2(2m + 4r - 1)(2m + 4r + 1)} \\
+ \frac{(m + 2r)(m + 2r + 2)(2r + 1)(2m + 2r + 1)}{(m + 2r + 1)^2(2m + 4r + 1)(2m + 4r + 3)} + q^2(m + 2r)(m + 2r + 1) \\
C(m,r) = \frac{(m + 2r + 3)(2m + 2r + 1)(2m + 2r + 2)}{(m + 2r + 1)(2m + 4r + 3)(2m + 4r + 5)} \\
D(m,r) = \frac{m(2r)}{(m + 2r)^2(2m + 4r - 1)} \\
E(m,r) = \frac{m(2m + 2r + 1)}{(m + 2r + 1)^2(2m + 4r + 3)}.
\]

Eq. (17) corresponds to the \( \omega \)-equation in the
quasi-geostrophic theory, but the equation is in this
case expressed for the velocity potential in the spec-
tral domain. In the linear case treated here we find
that the only forcing comes from the beta effect.
We note here that the linear approximation for the
\( \omega \)-equation, leaving a beta term as the only remain-
ing forcing term, cannot be considered as very
realistic since \( \omega \) is determined essentially by advec-
tion processes. It is thus important to compare
calculations of \( \omega \) from normal mode and quasi-
geostrophic procedures in the future. On the other
hand, the beta-effect may be realistic for large-scale
external Rossby modes, and the following calcula-
tions, especially for \( m = 1 \) and \( q^2 = 0.1 \), may thus
illustrate the difference between the two procedures
in a realistic way. For purposes of comparison
it is most convenient to express (17) as a matrix
equation. We define \( X \) as a column vector with the
elements

\[
X(m), X(m + 2), \ldots, X(m + 2R)
\]

and \( H \) as a column vector with the elements

\[
E(m,0)\Psi(m + 1), D(m,1)\Psi(m + 1) + E(m,1) \\
\times \Psi(m + 3), \ldots, D(m,R)\Psi(m + 2R - 1) + E(m,R) \\
\times \Psi(m + 2R + 1).
\]

Eq. (17) may then be written in the form
\[ \{M\} X = H, \]  
(19)

where the matrix \( \{M\} \) depends entirely on the

\[ \{M\} = \begin{bmatrix} B(m,0) & C(m,0) \\ A(m,1) & B(m,1) \\ 0 & A(m,2) \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \]

The solution to (19) is

\[ X = \{M\}^{-1} H \]  
(20)

and from the form of \( H \) it is then seen that

\[ X = \{a(i,j)E(m,j-1) + a(i,j+1)D(m,j)\} \Psi, \]  
(21)

where \( a(i,j) \) are the elements of the matrix \( \{M\}^{-1} \).

---

**Table 2.** The matrix analogous to Table 1 but for the quasi-geostrophic case for \( m = 1 \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( a_{11} )</th>
<th>( a_{12} )</th>
<th>( a_{13} )</th>
<th>( a_{14} )</th>
<th>( a_{15} )</th>
<th>( a_{16} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1111</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 10^0 )</td>
<td>( 10^{-1} )</td>
<td>( 10^{-2} )</td>
<td>( 10^{-3} )</td>
<td>( 10^{-4} )</td>
<td>( 10^{-5} )</td>
<td>( 10^{-6} )</td>
</tr>
</tbody>
</table>

---

**Table 4.** The matrix analogous to Table 3 but for the quasi-geostrophic case for \( m = 5 \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( a_{11} )</th>
<th>( a_{12} )</th>
<th>( a_{13} )</th>
<th>( a_{14} )</th>
<th>( a_{15} )</th>
<th>( a_{16} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0130</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 10^0 )</td>
<td>( 10^{-1} )</td>
<td>( 10^{-2} )</td>
<td>( 10^{-3} )</td>
<td>( 10^{-4} )</td>
<td>( 10^{-5} )</td>
<td>( 10^{-6} )</td>
</tr>
</tbody>
</table>
Table 5. Percentage difference between the matrices connecting X and \( \Psi \) for the normal mode and the quasi-geostrophic cases for \( m = 1 \) and \( q^2 = 0.1 \).

<table>
<thead>
<tr>
<th>( X )</th>
<th>( \Psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-11.6</td>
<td>-16.9</td>
</tr>
<tr>
<td>6.4</td>
<td>-0.9</td>
</tr>
<tr>
<td>6.7</td>
<td>0.5</td>
</tr>
<tr>
<td>6.9</td>
<td>0.0</td>
</tr>
<tr>
<td>7.1</td>
<td>0.0</td>
</tr>
<tr>
<td>7.2</td>
<td>0.0</td>
</tr>
<tr>
<td>7.3</td>
<td>0.0</td>
</tr>
</tbody>
</table>

It is seen that the major differences occur on the components of largest scale. A similar table for \( m = 5, q^2 = 0.1 \) (not reproduced) shows very small differences.

At the conclusion of this section it should be mentioned that all the results apply to the external gravity waves because we have used the shallow-water equations with \( H_0 = 8800 \) m.

5. Modifications of the initial state

The normal mode initialization methods require a modification of the initial fields to ensure that the gravity waves are removed. It is naturally of importance to investigate how large this modification is. Data for October 1957 as given by Eliassen and Machenhauer (1969) have been used to produce some results which will provide an example to illustrate the required modification. The data quoted above are for the 500 mb geopotential field giving the amplitudes of the spherical harmonic functions for selected values of \( m \) and \( n \). From these data we may compute the amplitudes of the quasi-geostrophic streamfunction from (15). Initial estimates of the quasi-geostrophic velocity potential can be obtained from (17), but an even simpler procedure has been applied. Eq. (17) is complicated because it incorporates the full variation of the Coriolis parameter. A first estimate of the velocity potential can be obtained by setting \( \mu = \mu_0 = \sin \phi_0 \), where \( \phi_0 = 45^\circ \). In that case the vorticity and the continuity equations are

\[
\begin{align*}
    s \nabla^2 \psi_m + m \psi_m + \mu_0 \nabla^2 \chi_m &= 0, \\
    s \eta_m + q^2 \nabla^2 \chi_m &= 0; \quad \eta_m = \mu_0 \psi_m
\end{align*}
\]

Eliminating the time derivatives we find from (22) that

\[
X(m + 2r) = \frac{m}{q^2 (m + 2r)(m + 2r + 1)(m + 2r + 1 + q^2)},
\]

where

\[
q^2 = \frac{g H_0}{4 \Omega^2 a^2 \mu_0^2}.
\]

Using (15) and (23), we may in a very straightforward way calculate preliminary values of \( \Psi \) and \( \chi \). Together with \( Z \) we have then all values on the right-hand side of (7), and it is possible to compute by the procedure described in Section 3 the partitioning of the initial amplitudes on the modes contained in the system. It is, in particular, possible to calculate the sum of all the contributions from the gravity waves and the sum of all contributions from the Rossby waves. The first sum, taken as a percentage of the total initial value for the component, is called \( G \), while the second sum, expressed in the same way, is denoted \( R \). The values of \( G \) and \( R \) in percent are given in Table 6 for \( m = 1 \).

Table 6, calculated from \( m = 1 \), shows that the major modifications are found for the largest scales, particularly for the velocity potential which was calculated using a simple approximate procedure.

It is naturally not necessary to provide an initial estimate of \( \Psi \) and \( \chi \) to use the normal mode procedure. To illustrate this point the calculation summarized in Table 6 was repeated with \( X_0(m + 2r) = 0 \) for \( 0 \leq r \leq R \). Table 7 shows the results arranged as in Table 6 for the geopotential and the streamfunction. A comparison between the velocity potential values for the calculations in Tables 6 and 7 is shown in Table 8 indicating that the only minor differences are found on the largest scales.

Calculations were also carried out for other values of \( m \). When \( m \) is large we find only negligible changes in \( Z \) and \( \Psi \), but there are substantial changes in \( X \) on the largest scale for each \( m \). This is undoubtedly due to the approximate nature of (23).

6. Vertical modes

All calculations in Section 4 have been carried out with the shallow water equations using an equivalent depth of \( H_0 = 8800 \) m. This value is characteristic of the basic atmospheric vertical mode. The question is if the results will change radically if we consider other vertical modes. As shown by Wiin-Nielsen (1971) using a constant lapse rate atmosphere and linearized equations it is possible to calculate the equivalent depths for the higher vertical modes. Some of the previous calculations

The table gives the percentage of the total initial value, \( m = 1, R = 6 \), data from October 1957, \( \psi \), from the linear balance equation, \( X_0 \), from the simplified quasi-geostrophic Eq. (23).

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Z )</th>
<th>( \Psi )</th>
<th>( \chi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi(1) ) = 48.9</td>
<td>( \psi(1) ) = 51.1</td>
<td>( \psi(2) ) = 8.8</td>
<td>( \psi(2) ) = 91.2</td>
</tr>
<tr>
<td>( \psi(3) ) = 19.3</td>
<td>( \psi(3) ) = 80.7</td>
<td>( \psi(4) ) = 1.5</td>
<td>( \psi(4) ) = 101.5</td>
</tr>
<tr>
<td>( \psi(5) ) = 12.9</td>
<td>( \psi(5) ) = 87.1</td>
<td>( \psi(6) ) = 0.2</td>
<td>( \psi(6) ) = 99.8</td>
</tr>
<tr>
<td>( \psi(7) ) = 9.4</td>
<td>( \psi(7) ) = 90.6</td>
<td>( \psi(8) ) = 0.0</td>
<td>( \psi(8) ) = 100.0</td>
</tr>
<tr>
<td>( \psi(9) ) = 8.0</td>
<td>( \psi(9) ) = 92.0</td>
<td>( \psi(10) ) = 0.0</td>
<td>( \psi(10) ) = 100.0</td>
</tr>
<tr>
<td>( \psi(11) ) = 2.4</td>
<td>( \psi(11) ) = 102.4</td>
<td>( \psi(12) ) = 0.0</td>
<td>( \psi(12) ) = 100.0</td>
</tr>
<tr>
<td>( \psi(13) ) = 2.1</td>
<td>( \psi(13) ) = 102.1</td>
<td>( \psi(14) ) = 0.0</td>
<td>( \psi(14) ) = 100.0</td>
</tr>
</tbody>
</table>
have been repeated for the first baroclinic mode for which a value of $H = 340$ m was selected. The only change in the calculations is that $q^2 = 0.003$ replaces $q^2 = 0.1$.

The value of $H = 340$ m selected above is determined by a particular stratification of the atmosphere. It should be stressed that $H$ varies considerably depending on the stratification, the upper boundary condition and the vertical discretization and may be as large as a few thousand meters. The following calculations represent therefore an example only.

We consider first the matrix connecting the streamfunction vector and the geopotential vector for $m = 1$. The matrix for $q^2 = 0.1$ (basic mode) is given in Table 1. This table was compared with Table 2 which is the corresponding matrix for the quasi-geostrophic case. We note from (15) and (16) that the latter matrix is independent of $q^2$, and it remains therefore unchanged. The matrix for $m = 1$ and $q^2 = 0.003$ is given in Table 9 which should be compared with Table 2. The two tables show good agreement for the smallest scales (lower right corners) although small differences occur. It is, however, apparent that very large differences are found for the large scales (upper left corner and left columns). A comparison between Tables 9 and 1 shows also that the terms above the diagonal are much smaller for $q^2 = 0.1$ than for $q^2 = 0.003$.

The corresponding calculation was made for $m = 5$, $q^2 = 0.003$. The differences are much smaller in this case and are everywhere below the 10% level.

We shall next proceed to compare the velocity potentials computed from normal mode initialization and from the quasi-geostrophic procedure for the first baroclinic mode ($q^2 = 0.003$). This will be done as in Section 4 by calculation of the matrices $(\{X_{ij}\} \times \{\tilde{\psi}_{ij}\})^{-1}$ given in (14) and the quasi-geostrophic matrix (21) for $q^2 = 0.003$ and $m = 1$. The result is given in Table 10 which shows the percentage difference between the matrices for the normal mode and quasi-geostrophic procedures.

Table 10 shows that very appreciable differences occur between the two methods for $m = 1$. We note particularly that the component $X(1,1)$ is systematically smaller in the mode procedure than in the quasi-geostrophic procedure for $m = 1$, $q^2 = 0.003$. A similar calculation was carried out for $m = 5$, $q^2 = 0.003$. In that case the differences are much smaller and everywhere less than 10% in absolute magnitude.

### Table 8. Values of $X(1, 1 + 2r)$ from quasi-geostrophic methods (first column) and $X_r = 0$ (second column). $m = 1, R = 6$, data from October 1957.

<table>
<thead>
<tr>
<th>$X_r = 0$</th>
<th>$X_r = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>X(1)</td>
<td>$4.6115 \times 10^6$</td>
</tr>
<tr>
<td>X(3)</td>
<td>$5.4127 \times 10^4$</td>
</tr>
<tr>
<td>X(5)</td>
<td>$9.5763 \times 10^4$</td>
</tr>
<tr>
<td>X(7)</td>
<td>$2.3555 \times 10^4$</td>
</tr>
<tr>
<td>X(9)</td>
<td>$1.0119 \times 10^4$</td>
</tr>
<tr>
<td>X(11)</td>
<td>$4.1970 \times 10^2$</td>
</tr>
<tr>
<td>X(13)</td>
<td>$1.0073 \times 10^2$</td>
</tr>
</tbody>
</table>

### Table 7. As in Table 6, $m = 1, R = 6$, data from October 1957, $\tilde{\psi}_0$ from the linear balance equation, $\tilde{\psi}_0(m + 2r) = 0$ for all $r$.

<table>
<thead>
<tr>
<th>$G \times R$</th>
<th>$\tilde{\psi}_0(2)$</th>
<th>$\tilde{\psi}_0(4)$</th>
<th>$\tilde{\psi}_0(6)$</th>
<th>$\tilde{\psi}_0(8)$</th>
<th>$\tilde{\psi}_0(10)$</th>
<th>$\tilde{\psi}_0(12)$</th>
<th>$\tilde{\psi}_0(14)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z(1)</td>
<td>11.1%</td>
<td>88.9%</td>
<td>-4.1</td>
<td>104.1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z(3)</td>
<td>-0.8</td>
<td>100.8</td>
<td>0.4</td>
<td>99.6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z(5)</td>
<td>0.1</td>
<td>99.9</td>
<td>0.0</td>
<td>100.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z(7)</td>
<td>0.0</td>
<td>100.0</td>
<td>0.0</td>
<td>100.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z(9)</td>
<td>0.0</td>
<td>100.0</td>
<td>0.0</td>
<td>100.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z(11)</td>
<td>0.0</td>
<td>100.0</td>
<td>0.0</td>
<td>100.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z(13)</td>
<td>0.0</td>
<td>100.0</td>
<td>0.0</td>
<td>100.0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. Concluding remarks

The main purpose of this investigation is to explore the normal mode initialization procedure. It has been shown that a balance exists between the wind field and the mass field when the procedure has been completed. The balance involves in general all the spherical harmonic amplitudes within a given truncation when the spectral method is used to express the various fields.

Only the linear case has been treated in this paper. Within this limitation it has been possible to compare the normal mode procedure with the quasi-geostrophic methods for the determination of the streamfunction and the velocity potential from the height field. It is emphasized that the quasi-geostrophic procedure is applied to the adjusted initial fields which are obtained when the contribution from the gravity modes has been removed. It turns out that the normal mode procedure is equal to the quasi-geostrophic method for all practical pur-
Table 10. The percentage difference between the matrices for normal mode and quasi-geostrophic procedures. Matrix: \( \{X_{ij}\}\{\Psi_{ij}\}^{-1} \).

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-9.6</td>
<td>-22.1</td>
<td>-30.3</td>
<td>-38.9</td>
<td>-46.3</td>
<td>-52.8</td>
</tr>
<tr>
<td>47.8</td>
<td>-10.6</td>
<td>-15.3</td>
<td>-19.1</td>
<td>-22.2</td>
<td>-24.8</td>
</tr>
<tr>
<td>47.8</td>
<td>19.4</td>
<td>-6.0</td>
<td>-7.4</td>
<td>-8.4</td>
<td>-9.3</td>
</tr>
<tr>
<td>47.8</td>
<td>19.4</td>
<td>5.7</td>
<td>-1.9</td>
<td>-2.2</td>
<td>-2.4</td>
</tr>
<tr>
<td>47.8</td>
<td>19.4</td>
<td>5.7</td>
<td>1.2</td>
<td>-0.4</td>
<td>-0.4</td>
</tr>
<tr>
<td>47.8</td>
<td>19.5</td>
<td>5.7</td>
<td>1.2</td>
<td>0.2</td>
<td>-0.1</td>
</tr>
<tr>
<td>47.9</td>
<td>19.5</td>
<td>5.7</td>
<td>1.2</td>
<td>0.2</td>
<td>0.0</td>
</tr>
</tbody>
</table>

poses except on the largest scale where significant differences appear. If we assume that the geopotential field is given by the observations we may thus conclude that the streamfunctions and velocity potentials determined by two procedures are quite different on the largest scale. This difference between the Rossby modes obtained from the two methods is in the linear case entirely due to the sphericity of the earth since the Rossby or rotational modes would be identical to the quasi-geostrophic modes if the Coriolis parameter was constant.

There is no doubt that the normal mode procedure, including the nonlinear effects as discussed by Machenhauer (1977) and Daley (1978), has definite numerical advantages because it removes rapid propagation of the gravity modes from the initial fields and greatly reduces the generation of these modes during the forecast. Whether or not the procedure also improves the accuracy of the forecasts must await extended testing of the method in operational predictions.

The investigation has also included a study of the behavior of the first baroclinic mode under the two procedures. The main effect of the much smaller effective height for the baroclinic modes is to create a larger difference between the streamfunctions and velocity potentials obtained from normal mode and quasi-geostrophic procedures. Since the first baroclinic mode is closely related to the thermal field in the atmosphere it is to be expected that the normal mode procedure will result in rather large changes in the baroclinic processes in the atmosphere and, in particular, in the vertical velocities on the largest scales.

References


