The Effect of Nonuniform Wind Shear on the Intensification and Reflection of Mountain Waves

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ABSTRACT

Following the analysis of Klemp and Lilly, a study is made of the intensification and reflection of mountain waves when the shear of the basic wind profile is nonuniform. A two-layer atmospheric model is treated, and the wind profile in the troposphere is assumed to be parabolic. The Scorer parameter includes the wind profile curvature term, $\tilde{u}_z$, which may not be neglected if the Richardson number $R_i$ is finite. When $R_i$ is finite, the optimal phase difference across the troposphere for maximum surface velocity intensifications is found to be slightly greater than $\pi$. As $R_i$ increases, the optimal phase difference decreases with $R_i$ and approaches the limiting value $\pi$. This implies that waves approximately reverse phase between the surface and the tropopause for maximum wave intensifications in most physically realistic atmospheric situations. The concept of Eliassen and Palm concerning the additivity of the vertical wave energy fluxes is expanded (valid at least up to the parabolic wind profile), by which the upward and downward energy transporting modes are identified. The reflection coefficient $r$ decays rapidly with $R_i$ when $R_i_{\text{crit}}$ is smaller than about unity, but the dependence of $r$ on $R_i$ is quite mild when $R_i_{\text{crit}} \gg 3$, attaining the limiting behavior describable by the static stability ratio alone.

1. Introduction

The mechanism underlying the momentum and energy transfer in stationary mountain waves was examined by Eliassen and Palm (1960, hereafter referred to as EP). They employed a multilayer (usually two to four layers) linearized two-dimensional model, in which a constant static stability $N$ as well as a uniform but different mean wind is specified in each layer. Basically, in the model of EP, the Scorer parameter $l$ becomes constant ($l = N/\tilde{u}$) within each layer.

When the conditions in the layer are such that the wave motion is of the internal type, the motion may be divided into two components, one carrying energy upward and the other downward. EP found that the wave component with upward energy flow can be interpreted as the incident wave, set up by the mountain, and that the wave component with downward energy flow is caused by the reflection of the incident wave in higher layers in the atmosphere.

Expanding the analysis of EP, Klemp and Lilly (1975, hereinafter referred to as KL) made a study of the dynamics of wave-induced downslope winds to identify the atmospheric conditions that would generate large-amplitude surface responses. KL used a linear, two-dimensional, steady-state, hydrostatic, multilayer Boussinesq fluid model. In KL's model the static stability $N$ is taken to be constant within each layer. In order to avoid shearing instabilities, continuous wind profiles are treated; however, a constant wind shear is allowed in each layer in the troposphere. Therefore, the Richardson number may be made finite in KL’s model, although KL’s subsequent analysis was concentrated in the cases of small shear, i.e., $R_i \gg 1$. Comparisons of KL's model predictions and observations (e.g., Lilly and Kennedy, 1973) are sufficiently encouraging to support the soundness of the idea of partial reflection of vertically-propagating wave energy in considering the mountain wave intensifications.

However, since the wind profile $\tilde{u}(z)$ is linear in KL's model, the Scorer parameter $l$ still consists of a single term $N/\tilde{u}(z)$, and the effect of the wind profile curvature term, $\tilde{u}_z$, in the full representation of $l \equiv (N/\tilde{u}^2 - \tilde{u}_z)^{1/2}$ has not been adequately taken into account. It is, therefore, of interest to inquire as to the intensification and reflection characteristics of hydrostatic mountain waves when the assumption of a linear wind profile in the troposphere is relaxed.

In this paper, mainly for the sake of analytical tractability and in an attempt to capture the essential of the effect of the wind profile curvature term, we shall deal with a two-layer atmosphere made of a
stable uniform stratosphere and a less stable troposphere having a parabolic wind profile. We shall seek an analytical solution to the linearized hydrostatic gravity wave equation pertinent to this two-year atmosphere. By analyzing the response intensification characteristics, more quantitative information is obtained as to the influence of the wind profile curvature. Another purpose of the present study is to clarify and possibly expand the points regarding the identification of the vertical energy fluxes, first expended by EP with respect to the case of a uniform wind profile. It will be shown that by use of a generalized solution the upward and downward energy fluxes can still be identified when the wind profile is nonuniform.

2. Analysis

a. Formulation

We consider a two-layer atmosphere. In the lower layer, the troposphere, the static stability \( N_0 \) is constant, and the basic wind profile \( \bar{u}(z) \) is described by

\[
\bar{u}(z) = \bar{u}_0 (z/h_0)^2, \quad h_0 \leq z \leq h_s,
\]

where \( \bar{u}_0 \) is the wind speed at the surface level \( z = h_0 \). In the upper layer, which extends from the tropopause \( (z = h_s) \) to infinity, both the static stability \( N_s > N_t \) and the basic wind speed \( \bar{u}_s \) are assumed to be constant.

The linear, two-dimensional, steady-state hydrostatic equation formulated for the vertical velocity \( w \) is

\[
\frac{\partial^2 w}{\partial z^2} + l \frac{w}{\bar{u}} = 0,
\]

where the Scorer parameter \( l \) is defined as

\[
l = \frac{N^2}{\bar{u}^2} - \frac{u_s}{\bar{u}}.
\]

To obtain solutions for flow over a single Fourier component of a mountain contour, we let the velocity perturbations

\[
[u(x,z), w(x,z)] = [\hat{u}(z), \hat{w}(z)]e^{ikx}.
\]

Upon substitution of the prescribed wind profile, i.e., \( \hat{u} = \bar{u}_0 (z/h_0)^2 \) in the troposphere and \( \hat{u} = \bar{u}_s \) in the stratosphere, into Eqs. (2) and (3), we have the solution

\[
\hat{w}(z) = A(z_t + iz_t^2)e^{i(z_t - iz_t^2)} + B(z_t - iz_t^2)e^{-i(z_t - iz_t^2)}
\]

(4)
in the troposphere, and

\[
\hat{w}_s(z) = Ce^{i(2z - h_s)}, \quad l_s = N_s/\bar{u}_s
\]

(5)
in the stratosphere.

Notice that in arriving at Eq. (5), we have adopted the radiation condition (e.g., Yanowitch, 1967) as the proper upper boundary condition of the system, viz., only the upward propagating mode is selected in the stratosphere. On the other hand, the linearized lower boundary condition is straightforward:

\[
\hat{w}_t(h_0) = i\bar{u}_0 H,
\]

in which \( H \) is the mountain amplitude in the Fourier representation of the surface contour \( h(x) = He^{i(kx)} \).

Supplementing the system will be the matching conditions (Vergeiner, 1971) at the layer interface \( z = h_s \)

\[
\hat{w}_t(h_s) = \hat{w}_s(h_s),
\]

\[
\frac{\partial \hat{w}_t}{\partial z}(h_s) = \frac{\partial \hat{w}_s}{\partial z}(h_s) + \frac{2}{h_s} \hat{w}_s(h_s).
\]

(7)

(8)

After laborious algebraic effort, the constants \( A \) and \( B \) are determined as

\[
A = (ik\bar{u}_0 H e^{-i\phi_0}) \frac{D_1}{D},
\]

\[
B = (ik\bar{u}_0 H e^{i\phi_0}) \frac{D_2}{D},
\]

(9)

(10)

where

\[
D = e^{i(k\bar{u}_0 - \phi_0)} \left[ \left( \frac{\phi_s}{\phi_0} - \frac{N_s}{\phi_0} \frac{N_t}{N_t} \right) + i \left( -\phi_s + \frac{N_s}{\phi_0} + \frac{1}{\phi_0} \right) \right],
\]

(11)

\[
D_1 = \frac{N_s}{N_t} - i\phi_0 \left( 1 - \frac{N_s}{N_t} \right),
\]

(12)

\[
D_2 = \frac{N_s}{N_t} - i\phi_s \left( 1 + \frac{N_s}{N_t} \right),
\]

(13)

in which

\[
\phi_0 = (N_t h_0 / \bar{u}_0),
\]

\[
\phi_s = (N_t h_s / \bar{u}_s)(h_0/h_s).
\]

By means of the continuity equation, the horizontal component of the velocity perturbation \([\hat{u}(z)e^{ikx}]\) in the troposphere is shown to be

\[
\hat{u}(z) = \left( \frac{i}{k\phi_0 h_0} \right) \left[ A \left( 2 + 2iz_t - i \frac{z_t}{z_t} \right) e^{i(z_t - iz_t)} + B \left( 2 - 2iz_t - i \frac{z_t}{z_t} \right) e^{-i(z_t - iz_t)} \right].
\]

(14)
Thus, by evaluating $|\hat{u}(h_0)|$ from Eq. (14) with the aid of Eqs. (9)–(13), we seek optimal conditions for maximum perturbation of the surface wind.

b. Maximum surface velocity intensification

With the wind profile $\bar{u}(z) = \bar{u}_0(z/h_0)^3$, the Richardson number can be expressed as

$$ Ri = N_z^2/\bar{u}_z^2 = (\bar{u}_0)^2/4(h_0/z)^2, $$

(15)

which indicates that $Ri$ is a monotonically decreasing function of $z$. Consequently, $Ri_{\text{max}}$ and $Ri_{\text{min}}$ occur at the surface and the tropopause, respectively. Furthermore, $\bar{u}_0$ and $\Phi_0$ can be recognized to be related to $Ri_{\text{max}}$ and $Ri_{\text{min}}$ in the following fashion:

$$ Ri_{\text{max}} = \Phi_0^2/4, $$

$$ Ri_{\text{min}} = \Phi_0^2/4. $$

At first, we shall confine our attention to situations when the wind shear in the troposphere is small so that $Ri_{\text{min}} \gg 1$ is satisfied. Taking advantage of the fact that $\Phi_0 \gg 1$ in this case, we have

$$ |\hat{u}(h_0)|/(N_z H) $$

$$ = \left[ \frac{(N_x^2 + N_y^2)}{(N_x^2 + N_z^2) - (N_x^2 - N_y^2) \cos(2\Phi_0 - 2\Phi_0)} \right]^{1/2} $$

$$ + O\left(\frac{1}{\Phi_0}\right). $$

(16)

As was pointed out by KL, since $N_z H$ is the maximum perturbation of the horizontal surface velocity in a uniform-wind one-layer atmosphere, the quantity on the right-hand side of Eq. (16) is the amplification factor pertinent to the present model. It can readily be seen from Eq. (4) that, subject to $\Phi_s \gg 1$, $\hat{u}_s(z)$ reduces to

$$ \hat{u}_s(z) = A_s z e^{iz} + B_s e^{-iz}. $$

(17)

Eq. (17) indicates that $1/z_t$ represents the phase of the wave motion, and, consequently, $(\Phi_0 - \Phi_s)$ denotes the phase difference of the wave across the troposphere when $Ri \gg 1$. We note here that the Scorer parameter $l$ in the troposphere can be expressed as

$$ l^2 = \frac{N_x^2}{\bar{u}_0^2} - \frac{\bar{u}_x}{\bar{u}(z)} $$

$$ = \frac{N_x^2}{\bar{u}_0^2} \left(1 - \frac{1}{2 \cdot Ri}\right). $$

(18)

Thus, by making the assumption $Ri \gg 1$ we are neglecting the $\bar{u}_x/\bar{u}(z)$ term in the expression for $l^2$. Eq. (17) can, therefore, be derived as the solution to Eq. (2) when $l$ is represented by $N_z/\bar{u}(z)$ only.

To lowest order, Eq. (16) is identical to the expression KL obtained for the two-layer, linear wind profile model in the limit of weak shear, $Ri \gg 1$. Obviously, in this case $|\hat{u}(h_0)|$ is maximized when the phase difference across the troposphere $(\Phi_0 - \Phi_s) = n\pi$, $n = 1, 2, 3, \ldots$. In particular, the phase difference of $\pi$, which is the most physically meaningful case (KL), corresponds to the troposphere having a thickness of one-half its vertical wavelength. Using a three-layer system, KL demonstrated that for a linear profile, if $Ri \gg 1$ the maximum surface velocity amplification is independent of the shear in each layer. The wind profile in each layer is only important in governing the phase shift across that layer. Thus, when $Ri \gg 1$, it is to be expected that changing the shape of the wind profile as we have done here should not alter the optimal conditions for maximum wave intensification, which is indeed verified in Eq. (16).

Now, we turn to the cases when the tropospheric Richardson number is finite. As was stated in Eq. (18), when $Ri$ is finite the $\bar{u}_x/\bar{u}(z)$ term cannot be neglected in the expression for $l^2$, and the exact solution [Eq. (4)] has to be utilized. It should also be noted that since the formulation is based on the hydrostatic approximation ($l^2 \gg k^2$) the solution ceases to be valid as $Ri_{\text{min}}$ approaches $\frac{1}{2}$. If $Ri_{\text{min}} < \frac{1}{2}$, Eq. (18) indicates that $l^2 < 0$ and thus the waves become evanescent in the vertical. Therefore, it should be understood that the finite Richardson number in the ensuing discussions is bounded from below, i.e., $Ri_{\text{min}} > \frac{1}{2}$. It is advantageous to recast Eq. (4) in the form

$$ \hat{w}_s(z) = \hat{w}_0(1 + z^2)(A e^{i\Phi} + B e^{-i\Phi}), $$

where

$$ \Phi = (1/z_t) + \tan^{-1}(z_t). $$

(19)

Clearly, the phase difference across the troposphere is now given as

$$ \Phi_0 - \Phi_s = \Phi(h_0) - \Phi(h_s) $$

$$ = (\Phi_0 - \Phi_s) + [\tan^{-1}(1/\Phi_0) $$

$$ - \tan^{-1}(1/\Phi_s)]. $$

(20)

Thus, if $Ri \gg 1$ then $(\Phi_0 - \Phi_s) = (\Phi_0 - \Phi_s)$, and the problem essentially reduces to the cases discussed earlier.

Since analytical manipulation is rather limited for the case of finite $Ri$, recourse must be made to numerical computations to evaluate $|\hat{u}(h_0)|$. Fig. 1 illustrates the optimal phase difference across the troposphere, $(\Phi_0 - \Phi_s)$, that would generate maximum surface velocity amplification. In general, the optimal phase difference is slightly greater than $\pi$, and the deviation from $\pi$ becomes very small in the range of realistic $Ri$ in actual atmospheres, i.e., say, $Ri_{\text{min}} \gg 10$. As $N_z/N_x$ decreases the optimal phase difference increases somewhat, but the amplification is dropping so this situation is of less interest. We may conclude that for typical values
of atmospheric situations the velocity profile curvature term has only a minor effect on the optimal conditions for mountain wave intensification.

c. Vertical transport of wave energy

Beginning with the perturbation hydrodynamic equations, EP showed that the vertical wave energy flux $p_w$ may be derived in terms of $\tilde{w}$, i.e.,

$$
\overline{p_w} = \left[ \frac{1}{2} \rho_0(h_0) \frac{\tilde{u}}{k} \right] \text{Im}(\tilde{\omega}^* \tilde{w}),
$$

(21)

where the asterisk denotes the complex conjugate.

In general, since $l^2$ is real-valued the solution to Eq. (2) may be expressed as

$$
\tilde{w}(z) = C_1 e^{i\omega z} + C_2 e^{-i\omega z},
$$

(22)

where $C_1$ and $C_2$ are complex constants and $g(z)$ is a complex-valued function. Placing Eq. (22) into Eq. (21) and rearranging terms, we obtain the general expression

$$
\overline{p_w} = \left[ \frac{1}{2} \rho_0(h_0) \frac{\tilde{u}}{k} \right] \text{Re} \left[ \frac{dg}{dz} \right] \exp[-2 \text{Im}(g)](|C_1|^2 - |C_2|^2),
$$

(23)

where $\text{Re}$ and $\text{Im}$ denote the real and imaginary part, respectively. Note that the hydrostatic assumption is a sufficient but not a necessary condition for Eqs. (22) and (23) to be applicable. [In the case of a nonhydrostatic situation, $l^2 - k^2 > 0$ is needed for Eq. (22) to be the general solution to Eq. (2).] It can be recognized in Eq. (23) that for a superposition of two modes of solutions in Eq. (22), the resulting vertical wave energy flux is additive. The additivity of the energy fluxes enables us to identify the upward and downward energy transporting modes, respectively. The positiveness of $\text{Re}[dg/dz]$ indicates that the $C_1$ wave is upward energy transporting and the $C_2$ wave downward. (If $\text{Re}[dg/dz] < 0$, the reverse is true.)

In order to demonstrate the usefulness of the considerations given in Eqs. (22) and (23), we take examples of simple $\tilde{u}(z)$ profiles. First, if $\tilde{u}(z) = \text{constant}$, then $l^2 = \text{constant}$; thus

$$
\tilde{w} = A_0 e^{-i\varphi z} + B_0 e^{+i\varphi z},
$$

(24)

Comparing Eqs. (22) and (24), we have

$$
g(z) = l z, \quad \text{Re}[dg/dz] = l, \quad \text{Im}[g] = 0.
$$

(25)

Application of Eq. (23) yields

$$
\overline{p_w} = \frac{1}{2} \rho_0(h_0) \tilde{u} \frac{l}{k} (|A_0|^2 - |B_0|^2),
$$

(26)

which is the formula given by EP [Eq. (4.6) of EP]. In view of Eq. (26) it is now possible to identify that the $A_0$ wave transporst energy upward and the $B_0$ wave downward—a well-known result for the case of a problem with constant Scorer parameter.

Next, for the constant-shear, linear wind profile $\tilde{u}(z) = \tilde{u}_0(z/h_0)$, which was considered by KL, the solution to Eq. (2) may be expressed as

$$
\tilde{w}(z) = A_1 z^{1/2} e^{i\varphi z} + B_1 z^{1/2} e^{-i\varphi z},
$$

(27)

where

$$
\varphi = m \log(z/h_0),
$$

$$
m^2 = (N/\tilde{u}_0)^2 - 1/4.
$$

Rewriting Eq. (27) in the form of Eq. (22), we have

$$
\begin{align*}
\text{Re}[dg/dz] &= m/z \\
\text{Im}[g] &= -1/2 \log z
\end{align*}
$$

(28)

Consequently, in this case $\overline{p_w}$ becomes

$$
\overline{p_w} = \frac{1}{2} \rho_0(h_0) \frac{\tilde{u}}{k} m(|A_1|^2 - |B_1|^2),
$$

(29)
which allows us to consider that the $A_1$ wave transports energy upward and the $B_1$ wave downward. (Incidentally, in this connection there appears to be a misprint in KL. It is believed that the $a_1$ and $b_1$ on p. 330 of KL should refer to $A_1$ and $B_1$, respectively, in the present notation.)

Finally, as to the present problem where $u(z) = \tilde{u}_0 (z/h_0)^2$, by rewriting Eq. (4) in the form of Eq. (22), we have

$$g(z) = \frac{1}{z_t} + \tan^{-1}(z_t) - \frac{i}{2} \log(z_t^2 + z_t^4)$$

$$\text{Re}[dg/dz] = -\left(\frac{\tilde{u}_0}{N_t h_0^2}\right) \frac{1}{z_t^2 + z_t^4}$$

$$\text{Im}[g] = -\frac{1}{2} \log(z_t^2 + z_t^4)$$

Application of Eq. (23) in this case gives

$$p_w = \left[\frac{1}{2} \rho_0 h_0 \frac{\tilde{u}_0}{k} \left(\frac{\tilde{u}_0}{N_t h_0}\right)\right] |B|^2 - |A|^2,$$  \hspace{1cm} (31)

indicating that the $B$ wave transports energy upward and the $A$ wave downward. [Notice that $\text{Re}[dg/dz] < 0$ in Eq. (30).]

EP considered only the case of a uniform wind profile, and concluded that the additivity of the energy flux is due to the unique property of the particular fundamental set of solutions, i.e., $\exp(ilz)$ and $\exp(-ilz)$. EP further stated that with any other choice of fundamental set, the energy fluxes would not be additive. By using the general solutions we have shown here, however, the concept of the additivity of energy fluxes may not be limited to the case of a uniform wind profile. It is demonstrated that fundamental set of solutions other than $\exp(ilz)$ and $\exp(-ilz)$, such as the ones shown in Eqs. (4) and (27), can also lead to additivity of the energy fluxes. Although we have not tested the workability of Eqs. (22) and (23) to a more rapidly and arbitrarily varying $\tilde{u}(z)$ profile, the above results suggest that Eqs. (22) and (23) are the generalized method (valid at least up to the parabolic wind profile) by which the upward and downward energy flux can be identified. Whether the $C_1$ wave or the $C_2$ wave of Eq. (22) corresponds to the upward- or downward-energy transporting mode is determined by the sign of $\text{Re}[dg/dz]$.

Based on the energy flux consideration of Eq. (31), we may define the reflection coefficient $r$ as the ratio of the downward to the upward energy flux, i.e.,

$$r = \frac{|A|^2}{|B|^2}.$$  \hspace{1cm} (32)

An approximate analytic formula for $r$ may be found when the Richardson number is large. Using Eqs. (9) and (10), and invoking the largeness of $R_i$, $r$ can be approximated as

$$r = \left(\frac{N_s - N_t}{N_s + N_t}\right)^2 + O\left(\frac{1}{R_{i\text{min}}^2}\right),$$

implying that, to the lowest order in $(1/R_i)$, the reflection is principally caused by the discontinuity in the static stability. As was observed by EP, Eq. (33) describes the reflection coefficient in optics if the stability is replaced by the index of refraction.

When the Richardson number is finite, the behavior of $r$ is shown in Fig. 2. The $r$ curves display a rather strong dependence on $N_s/N_t$. In the range of small $R_i$ the reflection coefficient is seen to decay rapidly with $R_i$, but when $R_{i\text{min}}$ is greater than about 3 the $r$ curves become almost flat, attaining the limiting values of Eq. (33) for $R_i \gg 1$. It is noted that if the stratosphere is very stable so that $N_s/N_t \gg 1$, the reflection of wave energy becomes almost total. In the opposite limit of $N_s/N_t \to 1$, the entire atmosphere is represented by a single static stability, in which case the wind profile is chiefly responsible for the reflection at the tropopause.

3. Conclusions

When the Richardson number is finite so that the $\tilde{u}_z/\tilde{u}$ term in the Scorer parameter cannot be ignored, the present study shows that the optimal phase difference across the troposphere for maximum surface responses is slightly greater than $\pi$. In most realistic atmospheric situations, the deviation of the optimal phase difference from $\pi$ is quite minimal. This reinforces the KL's finding that if $R_i \gg 1$ the shape of the wind profile is unimportant for maximum surface velocity amplification.

The reflection coefficient $r$ decreases rapidly with $R_i$ when $R_i$ is small. However, as $R_{i\text{min}}$ increases beyond about 3, $r$ becomes almost independent of $R_i$, approaching the limiting value given for $R_i \gg 1$. It is worth noting here that the identification of the upward and the downward energy
transporting modes is made possible by means of the generalized solution to the basic equation.

In view of the fact that $R_i$ is usually large in most physically realistic atmospheric situations, the conclusions reached under the assumption of small shear may have wider classes of applicability. On the other hand, for finite $R_i$ ($R_{i,\min} \approx 3$ in the present model) the effect of the wind profile curvature term is not insignificant, especially with respect to the reflection characteristics of mountain waves.

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