Space-Time Spectral Analysis of Rotary Vector Series

YOSHIKAZU HAYASHI

Geophysical Fluid Dynamics Laboratory, NOAA, Princeton University, Princeton, NJ 08540

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ABSTRACT

The analogy between space-time spectra and rotary spectra is discussed. The space-time spectra can be interpreted as the rotary spectra of a wave vector. These spectra are combined to resolve a rotary vector into clockwise and anticlockwise components as well as progressive and retrogressive components. The space-time rotary spectrum analysis is useful for a statistical identification of traveling vortices.

1. Introduction

Spectral formulas have been derived (Hayashi, 1971, 1973, 1977b) to compute space-time (wavenumber-frequency) spectra[1] by use of time spectral techniques such as the lag-correlation method, the direct Fourier transform method and the maximum entropy method. A space-time spectral analysis resolves disturbances into progressive and retrogressive components. It is also possible to resolve transient disturbances into standing and traveling waves by use of formulas developed by Hayashi (1977a, 1979). The space-time spectral formulas have been extensively applied to the wave analyses of general circulation models (Hayashi, 1974; Hayashi and Golder, 1977, 1978) and observational analysis (Gruber, 1974; Zangvil, 1975a,b; Hartmann, 1976; Sato, 1977; Fraedrich and Böttger, 1978; Deparadine, 1978; Krishnamurti, 1978).

On the other hand, rotary spectra have been formulated by Fofonoff (1969), Gonella (1972) and Mooers (1973). The rotary spectral analysis resolves a velocity vector into clockwise and anticlockwise components. This technique has been applied to observational analysis by a number of oceanographers such as Fofonoff (1969), Perkins (1972), Gonella (1972), Crew and Plutchak (1974), O'Brien and Pillsbury (1974), Leaman and Sanford (1975), Leaman (1976), Müller *et al.* (1978), Thompson (1978) and Weisberg *et al.* (1979a,b). Recently, rotary bispectra (Yao *et al.* 1975) and rotary spectra of a three-dimensional vector (Calman, 1978) have been formulated.

Both the space-time and rotary spectra are essentially the time spectra of complex vectors, namely, a "wave vector" and a "rotary vector." It will be of interest to compare and combine these two spectra. For this purpose the analogy between these complex vectors will be first discussed in Section 2 and formulas for computing space-time rotary spectra will be derived in Section 3. An example of their application will be given in Section 4. In Appendixes A–D, important properties of rotary spectra are summarized as a review.

[1] Alternative methods of computing two-dimensional spectra are the quadrature spectrum method (Deland, 1964, 1972), the two-dimensional direct Fourier transform method (Kao, 1968; Tsay, 1974), the two-dimensional lag correlation method (Leese and Epstein, 1963; Izawa, 1972), the maximum likelihood method (Capon, 1969), the maximum entropy method (McDonough, 1974), the empirical orthogonal cross-spectrum method (Pratt and Wallace, 1976), the Doppler shift method (Chapman *et al.*, 1974, Hirota, 1976), the geostrophic method (Willebrand, 1978) and the covariance fitting method (Bretherton and McWilliams, 1979).

2. Analogy between rotary and wave vectors

In this section, the analogy between a "rotary vector" and a "wave vector" is discussed, since space-time spectra and rotary spectra are reduced to the time spectra of these vectors as will be shown in the next section.

The rotary vector is a complex vector w represented by

$$w(t)=u(t)+iv(t), \tag{2.1}$$

where u and v can be interpreted, for example, as the zonal and meridional components for a wind vector. The absolute value and argument of w correspond to the speed and direction of winds, respectively (see Fig. 1a). This vector rotates clockwise or anticlockwise with time. A rectilinear oscillation consists of rotations of equal magnitude in each direction.

On the other hand, the wave vector U_k is the space-Fourier transform of a real-valued space-time series u

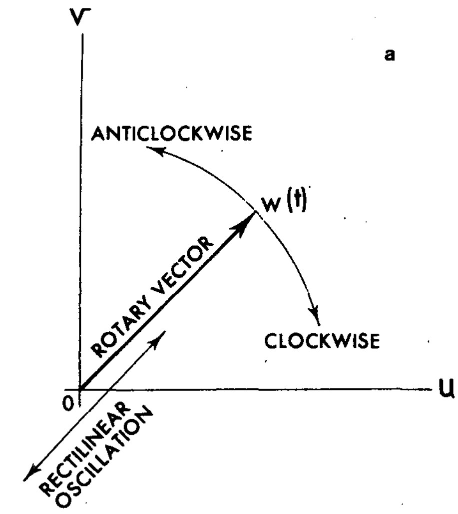

FIG. 1a. Rotary vector $w(t)$ in a complex plane.

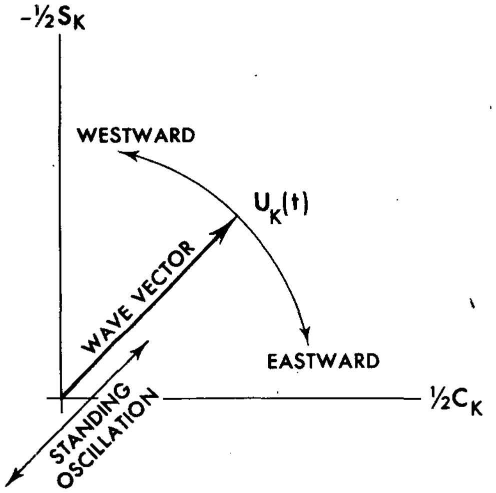

FIG. 1b. Wave vector $U_k(t)$ in a complex plane.

given by

$$u(x,t)=2\,\mathrm{Re}\sum_{k=1}^{\infty}U_k(t)e^{ikx}+U_0(t), \tag{2.2}$$

$$U_k(t)=\tfrac{1}{2}[C_k(t)-iS_k(t)] \tag{2.3a}$$

$$=\tfrac{1}{2}A_k(t)e^{i\phi_k(t)}, \tag{2.3b}$$

where C_k and S_k are the cosine and sine coefficients, while A_k and ϕ_k are amplitude and phase angle, respectively.

The wave vector is analogous to the rotary vector (Fig. 1b). The absolute value and argument of the wave vector correspond to the amplitude and phase angle, respectively. The clockwise (anticlockwise) rotation of the wave vector corresponds to eastward (westward) phase propagations with time. The rectilinear oscillation corresponds to a standing wave oscillation.

More generally, a rotary wave vector W_k is defined as the space-Fourier transform of a complex valued space-time series w given by

$$w(x,t)=u(x,t)+iv(x,t), \tag{2.4}$$

$$w(x,t)=\sum_{k=-\infty}^{\infty}W_k(t)e^{ikx}, \tag{2.5}$$

where the positive and negative values of k correspond to the anticlockwise and clockwise rotations of the rotary vector w with space, respectively.

The rotation of $W_k(t)$ with time is interpreted as that of the wavenumber k component of the rotary vector $w(x,t)$. As illustrated by Fig. 2, if vectors rotating clockwise (anticlockwise) with longitude propagate westward, they also rotate clockwise (anticlockwise) with time.

The rotary wave vector W_k is related to the wave vectors U_k and V_k as

$$W_{\pm k}(t)=U_{\pm k}(t)+iV_{\pm k}(t), \tag{2.6}$$

where

$$U_{\pm k}(t)=\tfrac{1}{2}[C_k^u(t)\mp iS_k^u(t)], \tag{2.7}$$

$$V_{\pm k}(t)=\tfrac{1}{2}[C_k^v(t)\mp iS_k^v(t)]. \tag{2.8}$$

3. Space-time rotary spectrum

In this section, formulas (3.5), (3.6) and (3.7) will be derived to compute space-time rotary spectra.

The rotary wave vector W_k in (2.5) can be represented by a Fourier-Stieltjes integral (see Yaglom, 1962; Lumley and Panofsky, 1964) for a stationary stochastic process as

$$W_k(t)=\int_{-\infty}^{\infty}e^{i2\pi ft}d\hat{W}_k(f), \tag{3.1}$$

where $\hat{W}_k(f)$ denotes the Fourier-Stieltjes transform of $W_k(t)$ and the increment $d\hat{W}_k(f)$ may be interpreted as the complex time-amplitude associated with infinitesimal frequency increment.

Inserting (3.1) into (2.5) gives

$$w(x,t)=\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(kx+2\pi ft)}d\hat{W}_k(f), \tag{3.2}$$

where the positive (negative) value of f corresponds to anticlockwise (clockwise) rotation of the rotary vector w with time. One period corresponds to one rotation of the vector. Also, the positive (negative) value of f/k corresponds to westward (eastward) phase velocities, respectively.

The space-time rotary power spectrum $P_{\pm k,\pm f}$ is defined, based on (3.2), as

$$P_{\pm k,\pm f}(w)df=\langle|d\hat{W}_{\pm k}(\pm f)|^2\rangle, \tag{3.3}$$

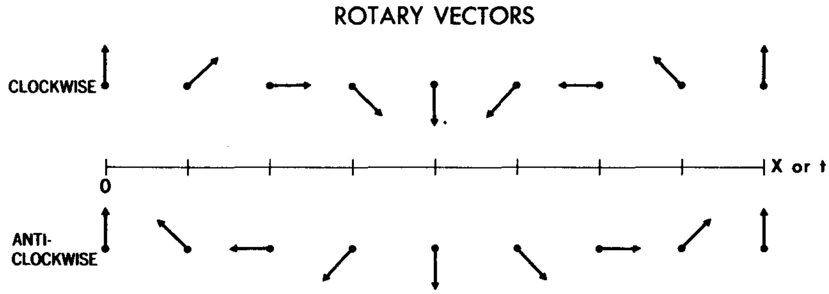

FIG. 2. Rotation of vectors with longitude x or time t.

where the angle braces denote an ensemble average which can be replaced by an average over an infinitesimal frequency band for an ergodic time series of infinite length (see Beran and Parrent, 1964, p. 23).

The coherence between clockwise $(-f)$ and anticlockwise $(+f)$ components for either westward or eastward moving components is defined as

$$\mathrm{coh}_{\pm k,f}(w)=\frac{|\langle d\hat{W}^*_{\pm k}(f)d\hat{W}^*_{\mp k}(-f)\rangle|}{\langle|d\hat{W}_{\pm k}(f)|^2\rangle^{\frac{1}{2}}\langle|d\hat{W}_{\mp k}(-f)|^2\rangle^{\frac{1}{2}}}, \quad (3.4)$$

where the asterisk denotes the complex conjugate of the increment.

If this coherence is zero, the clockwise and anticlockwise components do not interfere with each other to form a rectilinear oscillation. If the orientation of the axis of elliptical oscillation fluctuates with time, this coherence is not 1.0. This coherence is also called the "stability of the ellipse orientation" by Gonella (1972).

The formulas for computing the power spectrum and coherence are given by use of (2.6) as

$$2P_{\pm k,\pm f}(w)=P_{\pm f}(U_{\pm k}+iV_{\pm k}), \quad (3.5a)$$

$$=P_{\pm k,\pm f}(u)+P_{\pm k,\pm f}(v) -2Q_{\pm k,\pm f}(u,v), \quad (3.5b)$$

$$[1-\mathrm{coh}^2_{\pm k,f}(w)]P_{\pm k,f}(w)P_{\mp k,-f}(w) =[1-\mathrm{coh}^2_{\pm k,f}(u,v)]P_{\pm k,f}(u)P_{\pm k,f}(v). \quad (3.6)$$

The degree of polarization $(D_{\pm k,f})$ is given by

$$(1-D^2_{\pm k,f})=\frac{4P_{\pm k,f}(u)P_{\mp k,-f}(v)}{[P_{\pm k,f}(u)+P_{\mp k,-f}(v)]^2} \times[1-\mathrm{coh}^2_{\pm k,f}(u,v)]. \quad (3.7)$$

The degree of polarization is a measure of circular, elliptical or rectilinear oscillation and coincides with the maximum value of the (u,v) coherence when the coordinates are rotated by a certain angle (see Appendixes B and C).

The right-hand sides of the above formulas can be computed by use of space-time spectral formulas in real representation (Hayashi, 1971) given by

$$4P_{\pm k,\pm f}(u)=P_{\pm f}(C^u_k)+P_{\pm f}(S^u_k)+2Q_{\pm f}(C^u_k,S^u_{\pm k}), \quad (3.8)$$

$$4P_{\pm k,\pm f}(v)=P_{\pm f}(C^v_k)+P_{\pm f}(S^v_k)+2Q_{\pm f}(C^v_k,S^v_{\pm k}), \quad (3.9)$$

$$4K_{\pm k,\pm f}(u,v)=K_f(C^u_k,C^v_k)+K_f(S^u_k,S^v_k) +Q_{\pm f}(C^u_k,S^v_{\pm k})+Q_{\mp f}(S^u_{\pm k},C^v_k), \quad (3.10)$$

$$4Q_{\pm k,\pm f}(u,v)=Q_{\pm f}(C^u_k,C^v_k)+Q_{\pm f}(S^u_k,S^v_k) -K_f(C^u_k,S^v_{\pm k})+K_f(S^u_{\pm k},C^v_k), \quad (3.11)$$

$$\mathrm{coh}^2_{\pm k,\pm f}(u,v)=\frac{K^2_{\pm k,\pm f}(u,v)+Q^2_{\pm k,\pm f}(u,v)}{P_{\pm k,\pm f}(u)P_{\pm k,\pm f}(v)}, \quad (3.12)$$

where P_f, K_f and Q_f are time power spectrum, co-spectrum and quadrature spectrum, respectively, defined by

$$P_f(C_k)df=2\langle|d\hat{C}_k(f)|^2\rangle, \quad (3.13)$$

$$K_f(C_k,S_k)df=2\,\mathrm{Re}\langle d\hat{C}^*_k(f)d\hat{S}_k(f)\rangle, \quad (3.14)$$

$$Q_f(C_k,S_k)df=2\,\mathrm{Im}\langle d\hat{C}^*_k(f)d\hat{S}_k(f)\rangle. \quad (3.15)$$

The plus and minus sign in $S_{\pm k}$ and $Q_{\pm f}$ on the right-hand side terms of (3.8)–(3.11) can be placed in front of each term. In some papers the sign of quadrature spectrum (3.15) is reversed.

In a special case where w does not depend on x, we have by putting $k=0$ in (3.5a):

$$2P_{0,\pm f}(w)=P_{\pm f}(U_0+iV_0). \quad (3.16)$$

This coincides with the time-rotary power spectrum (see Appendix A for its real representation).

On the other hand, if w is real, we have by putting $v=0$ in (3.5a)

$$4P_{k,\pm f}(u)=P_{\pm f}(C^u_k-iS^u_k), \quad (3.17)$$

where the clockwise and anticlockwise components are summed as

$$P_{k,\pm f}(u)\equiv P_{+k,\pm f}(u)+P_{-k,\mp f}(u). \quad (3.18)$$

This formula (3.17) gives the space-time power spectrum in complex representation (Hayashi, 1977b). It should be noted that space-time spectrum can be interpreted as the rotary spectrum of a wave vector.

The formula (3.5a) in complex representation is suitable for applying the maximum entropy method to obtain fine frequency resolutions even for a short record (see Hayashi, 1977b), while the formula (3.5b) in real representation is convenient for applying

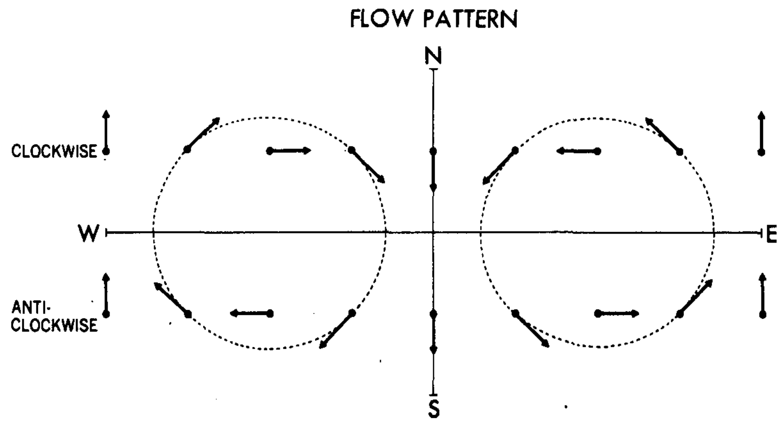

FIG. 3. A schematic flow pattern of mixed Rossby-gravity waves. The circles represent streamlines. The wind vector at the northern (southern) latitude rotates clockwise (anticlockwise) with longitude.

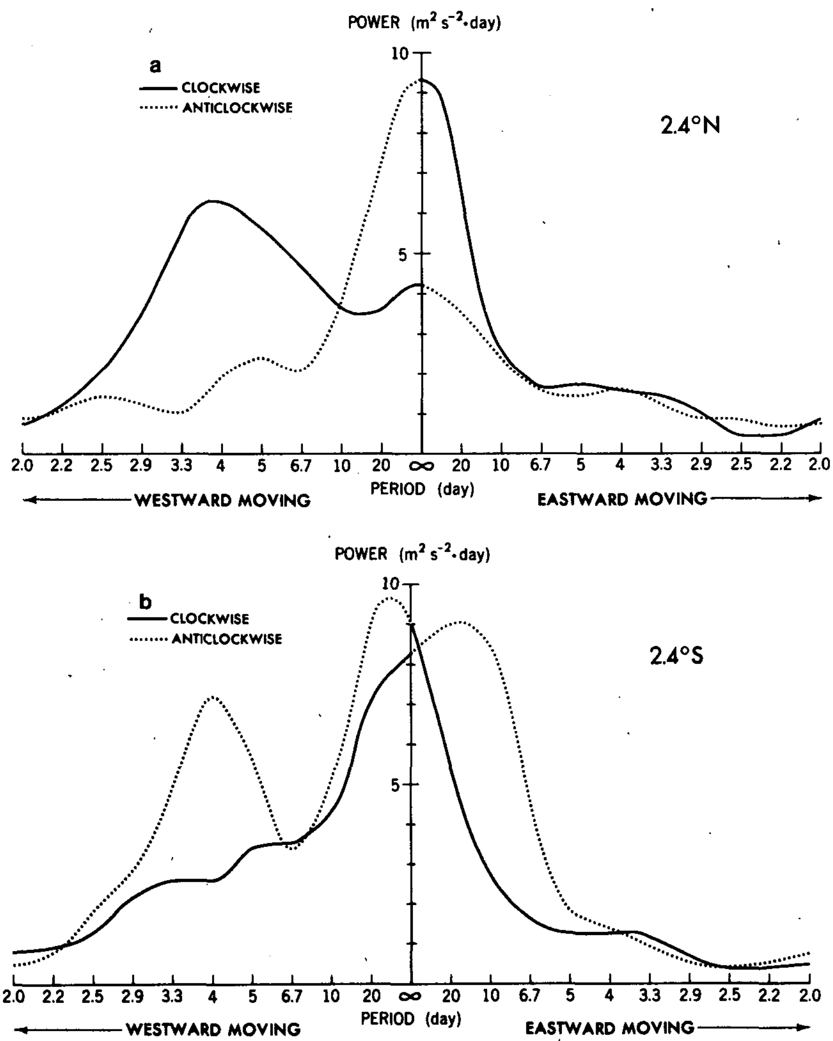

FIG. 4. Space-time rotary power spectrum of horizontal wind (wavenumber 4) at 110 mb at 2.4°N(a) and 2.4°S(b) during the period June–September, from a GFDL general circulation model. The solid and dotted curves denote clockwise and anticlockwise rotations, respectively. Frequency bandwidth 0.05 day^{-1}.

conventional methods of computing cross spectra (see Bendat and Piersol, 1971).

4. Example of application

As an example of the application of the space-time rotary spectrum, an analysis is made of mixed Rossby-gravity waves simulated by a GFDL general circulation model (Manabe *et al.*, 1974). An application of time-rotary spectrum to an analysis of oceanic mixed Rossby-gravity has been made by Weisberg *et al.* (1979a,b).

Atmospheric mixed Rossby-gravity waves are characterized by wavenumber 4, a period of 4 days and a westward phase velocity [see Yanai *et al.* (1968) for observation and Hayashi (1974) for simulation]. Theoretically, these waves take the form of vortices centered over the equator (Matsuno, 1966). As illustrated schematically by Fig. 3 (see also Fig. 2), the flow pattern of mixed Rossby-gravity waves is associated with clockwise (anticlockwise) elliptical rotation of the wind vector with longitude on the northern (southern) side of the equator. When this pattern propagates westward, the wind vectors also rotate clockwise (anticlockwise) with time on the northern (southern) side of the equator. These characteristics of mixed Rossby-gravity waves can be detected by the following spectral analysis.

The horizontal wind at 110 mb in the tropics is analyzed during the period June through September. The time spectrum was computed by the use of a lag-correlation method with the hanning lag window. The maximum lag is 10 days and the equivalent degree of freedom is 24 (see Blackman and Tukey, 1958). Figs. 4a and 4b show a space-time rotary power spectrum for wavenumber 4 at 2.4°N and 2.4°S, respectively. It is seen that the 4-day spectral peak for westward phase velocity is dominated at 2.4°N(S) by clockwise (anticlockwise) rotation. This result is consistent with the westward propagation of the flow pattern of mixed Rossby-gravity waves. Even for the ideal flow pattern (Fig. 3), the clockwise or anticlockwise component is not expected to vanish completely on either side of the equator.

Fig. 5 shows the latitudinal distribution of the space-time rotary power spectrum (left) and the degree of polarization and the coherence between clockwise and anticlockwise components (right) for wavenumber 4, period of 4 days (westward moving). It is seen that the clockwise and anticlockwise components attain their maximum amplitudes at 4.8°N and 2.4°S, respectively, while they have the same amplitude at the equator. For the idealized flow pattern (Fig. 3), these maxima should occur symmetrically with respect to the equator.

The degree of polarization is a measure of elliptical or rectilinear rotation. For the ideal flow (Fig. 3) the degree of polarization is expected to be 1.0 for all

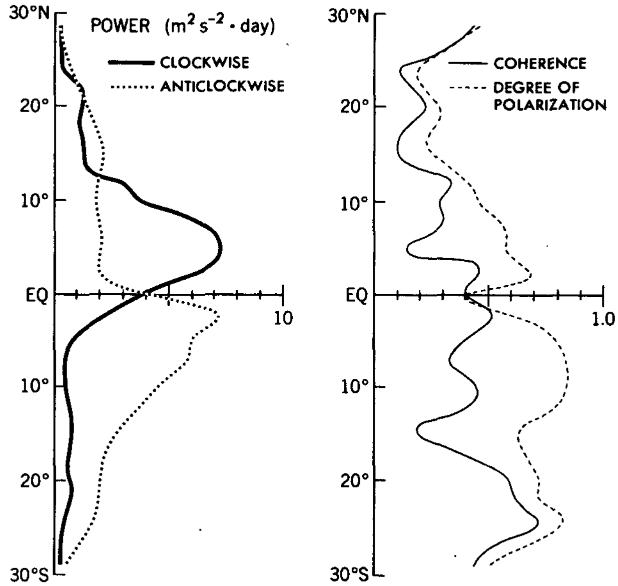

FIG. 5. The latitudinal distribution (110 mb level) of the space-time rotary power spectrum (left) and the coherence between clockwise and anticlockwise components and the degree of polarization (right) of the horizontal wind of wavenumber 4, period of 4 days (westward moving frequency bandwidth 0.05 day^{-1}).

latitudes in the absence of random noise. In the present example (Fig. 5), the degree of polarization exceeds 0.5 a few degrees away from the equator, being consistent with the elliptical rotation of the wind vectors. However, it does not exceed 0.5 at the equator where a rectilinear oscillation is expected for the ideal flow pattern. This is probably due to the fact that the centers of the vortices often drift from the equator, resulting in alternating clockwise and anticlockwise rotation.

The coherence between clockwise and anticlockwise components is a measure of interference between these components. For the ideal pattern (Fig. 3) it is expected to be 1.0 for all latitudes in the absence of noise. If either clockwise or anticlockwise component is completely absent (circular rotation), the coherence is zero, while the degree of polarization is 1.0. If these components have the same amplitude, the coherence coincides with the degree of polarization as is the case at the equator in the present example (Fig. 5). The coherence never exceeds the degree of polarization (see Appendix C) as is the case with the present example (Fig. 5).

5. Remarks

The analogy between space-time and rotary spectrum analyses is helpful in interpreting the results of space-time spectrum analysis. By this analogy (see Appendix B) we also find that the coherence and phase difference between the zonal cosine and sine coefficients depend on a choice of the origin of the zonal coordinate, while the coherence between progressive and retrogressive components is invariant. The degree of polarization (see Appendix C) is also invariant and is analogous to a measure of regular traveling or standing waves (see Schäfer, 1979; Hayashi, 1979). The degree of rectilinear oscillation (see Appendix D) is invariant and is analogous to a measure of standing waves (see Pratt, 1976; Hayashi, 1977a).

The space-time spectral analysis of a rotary vector series is useful for a statistical identification of traveling vortices. Another approach is the empirical orthogonal (principal component) analysis of a vector series as proposed by Hardy and Walton (1978). By combining these two techniques, traveling vortices can be further decomposed into meridional or vertical principal components.

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APPENDIX A

Formulas for Rotary Spectra

This appendix is essentially based on Gonella (1973), Mooers (1973) and Calman (1978) and is also analogous to Hayashi (1971, 1977a).

The time-cross spectra between the clockwise $(-f)$ and anticlockwise $(+f)$ components of a rotary vector can be computed by use of the following formulas:

$$w(t)=u(t)+iv(t), \tag{A1}$$

$$P_{\pm f}(w)=\tfrac{1}{2}[P_f(u)+P_f(v)]\mp Q_f(u,v), \tag{A2}$$

$$K_f(w,w^*)=\tfrac{1}{2}[P_f(u)-P_f(v)], \tag{A3}$$

$$Q_f(w,w^*)=-K_f(u,v), \tag{A4}$$

$$\mathrm{coh}_f^2(w,w^*)=\frac{K_f^2(w,w^*)+Q_f^2(w,w^*)}{P_f(w)P_{-f}(w)}, \tag{A5}$$

$$\mathrm{Ph}_f(w,w^*)=\tan^{-1}[Q_f(w,w^*)/K_f(w,w^*)], \tag{A6}$$

where P_f, K_f, Q_f, coh_f, Ph_f are the power spectrum, cospectrum quadrature spectrum, coherence and phase difference, respectively. The asterisk denotes the complex conjugate. In some papers, the sign of the quadrature spectrum is reversed. For the cross spectra between two rotary vectors, see Mooers (1973).

The rotary spectra are interpreted as follows:

$P_f(w)=0$	and	$\mathrm{coh}_f(w,w^*)=0$	(circular)
$P_f(w)\neq P_{-f}(w)$	and	$\mathrm{coh}_f(w,w^*)=1$	(elliptical)
$P_f(w)=P_{-f}(w)$	and	$\mathrm{coh}_f(w,w^*)=1$	(rectilinear)
$P_{\pm f}(w)\neq 0$	and	$\mathrm{coh}_f(w,w^*)=0$	(irregular).

The above formulas can be rewritten by use of cross-spectrum matrices $\mathbf{C}_f(w,w^*)$ and $\mathbf{C}_f(u,v)$ as

$$\begin{pmatrix} w \\ w^* \end{pmatrix}=\sqrt{2}\,\mathbf{I}^*\begin{pmatrix} u \\ v \end{pmatrix}, \tag{A7}$$

$$\mathbf{C}_f(w,w^*)=\mathbf{I}\mathbf{C}_f(u,v)\mathbf{I}^{-1}, \tag{A8}$$

where

$$\mathbf{C}_f=\begin{bmatrix} P_f & K_f+iQ_f \\ K_f-iQ_f & P_f \end{bmatrix}, \tag{A9}$$

$$\mathbf{I}=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, \tag{A10}$$

$$P_f(w^*)=P_{-f}(w). \tag{A11}$$

Since the above two cross-spectrum matrices in (A8) are related through a unitary matrix $\mathbf{I}$, their traces (Tr), determinants (Det) and eigenvalues $(E_\pm)$ coincide with each other, i.e.,

$$\mathrm{Tr}=P_f(w)+P_{-f}(w) \tag{A12a}$$

$$=P_f(u)+P_f(v), \tag{A12b}$$

$$\mathrm{Det}=[1-\mathrm{coh}_f^2(w,w^*)]P_f(w)P_{-f}(w) \tag{A13a}$$

$$=[1-\mathrm{coh}_f^2(u,v)]P_f(u)P_f(v), \tag{A13b}$$

$$2E_\pm=P_f(w)+P_{-f}(w)\pm\{[P_f(w)-P_{-f}(w)]^2+4P_f(w)P_{-f}(w)\,\mathrm{coh}_f^2\,(w,w^*)\}^{\frac{1}{2}}, \tag{A14a}$$

$$=P_f(u)+P_f(v)\pm\{[P_f(u)-P_f(v)]^2+4P_f(u)P_f(v)\,\mathrm{coh}_f^2\,(u,v)\}^{\frac{1}{2}}. \tag{A14b}$$

These values are non-negative and invariant with coordinate rotation (see Appendix B). The trace is a measure of the magnitude of oscillation, while the determinant vanishes for a polarized oscillation (see Appendix C). The eigenvalues give the magnitude of empirical orthogonal components in the frequency domain (Wallace and Dickinson, 1972).

APPENDIX B

Coordinate Transform of u-v Spectra

This appendix is essentially based on Fofonoff (1969) and Mooers (1973).

If the (u,v) coordinate is rotated by an angle θ, the rotary vector and the cross-spectrum matrix $\mathbf{C}_f(u,v)$

are transformed as

$$w' = we^{-i\theta}, \quad \text{(B1)}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \Theta \begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{(B2)}$$

$$\mathbf{C}_f(u',v') = \Theta \mathbf{C}_f(u,v)\Theta^{-1}, \quad \text{(B3)}$$

where

$$\Theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \quad \text{(B4)}$$

The maximum (P_+) and minimum (P_-) values of $P_f(u)$ occurring for $\theta = \theta_N$ and $\theta = \theta_N \pm \pi/2$, respectively, are given in rotary representation (see Appendix A) by

$$2P_\pm = P_f(w) + P_{-f}(w) \pm 2P_f^{\frac{1}{2}}(w)P_{-f}^{\frac{1}{2}}(w) \operatorname{coh}_f(w,w^*), \quad \text{(B5)}$$

$$2\theta_N = -\operatorname{Ph}_f(w,w^*). \quad \text{(B6)}$$

These values are interpreted as follows

$P_+ = P_-$	(circular or irregular)
$P_+ > P_-$	(elliptical or irregular)
$P_- = 0$	(rectilinear).

The maximum (C_+) and minimum (C_-) values of the (u,v) coherence occurring for $\theta = \theta_N \pm \pi/4$ and $\theta = \theta_N$, respectively, are given in rotary representation by

$$C_+^2 = \frac{[P_f(w) + P_{-f}(w)]^2 - 4 \text{ Det}}{[P_f(w) + P_{-f}(w)]^2}, \quad \text{(B7)}$$

$$C_-^2 = \frac{[P_f(w) - P_{-f}(w)]^2}{[P_f(w) - P_{-f}(w)]^2 + 4 \text{ Det}}, \quad \text{(B8)}$$

where Det is the determinant given by (A13).

These values are interpreted as follows

$C_+ = C_- = 1$	(circular or elliptical)
$C_+ = 1, \quad C_- = 0$	(rectilinear)
$C_+ = C_- = 0$	(irregular).

The following quantities are invariant with the coordinate rotation:

$$\begin{cases} P_{\pm f}(w), \quad \operatorname{coh}_f(w,w^*), \quad P_\pm, \quad C_\pm, \\ Q_f(u,v), \quad P_f(u) + P_f(v), \quad [P_f(u) - P_f(v)]^2 + 4K_f^2(u,v) \\ \text{Tr, Det, } E_\pm. \end{cases}$$

On the other hand, the following quantities are not invariant:

$$\begin{cases} P_f(u), \quad P_f(v), \quad K_f(u,v), \quad \operatorname{coh}_f(u,v), \\ K_f(w,w^*), \quad Q_f(w,w^*), \quad \operatorname{Ph}_f(w,w^*). \end{cases}$$

It should be noted that

$$E_- \leqslant P_-, \quad P_+ \leqslant E_+, \quad \text{(B9)}$$

$$C_-^2 \geqslant 0, \quad \text{(B10)}$$

where $E_\pm$ are the eigenvalues given by (A14a).

The equality in the above relations holds when the clockwise and anticlockwise components are of equal magnitude.

APPENDIX C

Degree of Polarization

This appendix is essentially based on Born and Wolf (1975).

The vector series w can be decomposed into polarized w^p and unpolarized w^q components which are incoherent with each other as

$$w(t) = w^p(t) + w^q(t). \quad \text{(C1)}$$

The polarized component represents a circular, elliptical or rectilinear oscillation, while the unpolarized component represents irregular oscillations.

The cross-spectrum matrix $\mathbf{C}_f(u,v)$ is partitioned into the polarized and unpolarized parts as

$$\mathbf{C}_f(u,v) = \begin{bmatrix} P_1 & P_3 \\ P_3^* & P_2 \end{bmatrix} + \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}, \quad \text{(C2)}$$

where

$$P_1P_2 - |P_3|^2 = 0. \quad \text{(C3)}$$

This means that

$$\operatorname{coh}_f(u^p,v^p) = 1 \quad \text{or} \quad P_f(u^p) = 0 \quad \text{or} \quad P_f(v^p) = 0, \quad \text{(C4)}$$

$$\operatorname{coh}_f(u^q,v^q) = 0 \quad \text{and} \quad P_f(u^q) = P_f(v^q). \quad \text{(C5)}$$

Alternatively, (u,v) in the above partition can be replaced by (w,w^*) by virtue of (A2)–(A4). The magnitudes (traces) of these parts are given respectively by solving (C2) with (C3) as

$$P_1 + P_2 = E_+ - E_-, \quad \text{(C6)}$$

$$2Q = 2E_-, \quad \text{(C7)}$$

where $E_\pm$ are the eigenvalues of the matrix given by (A14).

The degree of polarization D_f is defined by

$$D_f = \frac{P_1 + P_2}{(P_1 + P_2) + 2Q} = \frac{E_+ - E_-}{E_+ + E_-} = C_+, \quad \text{(C8)}$$

where C_+ is given by (B7).

Thus the degree of polarization is interpreted as the maximum value of the (u,v) coherence when the u,v coordinates are rotated by an angle $\theta_N \pm \pi/4$, where θ_N is given by (B6). The orientation of the major axis of the polarized component coincides with θ_N.

The degree of polarization is related to the (u,v) coherence as

$$(1-D_f^2)=\frac{4P_f(u)P_f(v)}{[P_f(u)+P_f(v)]^2}[1-\mathrm{coh}_f^2(u,v)]. \quad \text{(C9)}$$

By use of (A12) and (A13), this relation (C9) is rewritten in terms of rotary coherence as

$$(1-D_f^2)=\frac{4P_f(w)P_{-f}(w)}{[P_f(w)+P_{-f}(w)]^2}[1-\mathrm{coh}_f^2(w,w^*)]. \quad \text{(C10)}$$

It follows from (C10) that

$$D_f \geqslant \mathrm{coh}_f(w,w^*), \quad \text{(C11)}$$

where the equality holds if $P_f(w)=P_{-f}(w)$ or $\mathrm{coh}_f(w,w^*)=1$. Thus the rotary coherence never exceeds the degree of polarization.

The above quantities are interpreted as follows:

$D_f=1$ and $\mathrm{coh}_f(w,w^*)=0$ (circular)

$D_f=\mathrm{coh}_f(w,w^*)=1$ (elliptical or rectilinear)

$D_f=\mathrm{coh}_f(w,w^*)=0$ (irregular).

APPENDIX D

Degree of Rectilinear Oscillation

This appendix is analogous to Hayashi (1977a, 1979).

The vector series w can be decomposed into rectilinear (w^L) and nonrectilinear (w^N) components which are assumed to be incoherent with each other as

$$w(t)=w^L(t)+w^N(t). \quad \text{(D1)}$$

The rectilinear component represents a rectilinear oscillation, while the nonrectilinear component represents a circular or irregular oscillation.

The cospectrum matrix can be partitioned into the rectilinear and nonrectilinear parts as

$$\begin{pmatrix} P_f(u) & K_f(u,v) \\ K_f(u,v) & P_f(v) \end{pmatrix}=\begin{pmatrix} L_1 & L_3 \\ L_3 & L_2 \end{pmatrix}+\begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}, \quad \text{(D2)}$$

where

$$L_1L_2-L_3^2=0. \quad \text{(D3)}$$

Comparing (D2) with (C2) in Appendix C, we find that the above parts can be obtained by replacing the u-v cross spectrum (K_f+iQ_f) in the polarized and unpolarized parts by the u-v cospectrum (K_f).

The above partition means that

$$\mathrm{Cor}_f(u^L,v^L)=\pm 1 \text{ or } P_f(u^L)=0 \text{ or } P_f(v^L)=0, \quad \text{(D4)}$$

$$\mathrm{Cor}_f(u^N,v^N)=0 \text{ and } P_f(u^N)=P_f(v^N), \quad \text{(D5)}$$

where Cor_f is the correlation coefficient at frequency f defined by

$$\mathrm{Cor}_f(u,v)=K_f(u,v)/P_f^{\frac{1}{2}}(u)P_f^{\frac{1}{2}}(v). \quad \text{(D6)}$$

By virtue of (A2)–(A4), the partition (D2) is equivalent to the partition of the cross spectrum matrix as

$$\mathbf{C}_f(w,w^*)=\begin{pmatrix} |L| & L \\ L^* & |L| \end{pmatrix}+\begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}. \quad \text{(D7)}$$

This means that the clockwise and anticlockwise components of the rectilinear component are coherent with each other and of equal magnitude, while those of nonrectilinear component are incoherent.

The magnitude of the rectilinear (L_1+L_2) and nonrectilinear $(2N)$ parts are given by

$$L_1+L_2=P_+-P_-, \quad \text{(D8)}$$

$$2N=2P_-, \quad \text{(D9)}$$

where P_+ and P_- are the maximum and minimum values of $P_f(u)$ given by (B5).

The above parts (D8) and (D9) are analogous to the "standing" and "traveling" parts given by Hayashi (1977a). The orientation of the rectilinear oscillation coincides with θ_N given by (B6).

The degree of rectilinear oscillation (L_f) is defined by

$$L_f=\frac{L_1+L_2}{(L_1+L_2)+2N}=\frac{P_+-P_-}{P_++P_-}. \quad \text{(D10)}$$

It can be proven that L_f coincides with the maximum value of $|\mathrm{Cor}_f(u,v)|$ when the u-v coordinate is rotated by an angle $\theta_N\pm\pi/4$.

The degree of rectilinear oscillation is expressed as

$$(1-L_f^2)=\frac{4P_f(u)P_{-f}(v)}{[P_f(u)+P_f(v)]^2}[1-\mathrm{Cor}_f^2(u,v)] \quad \text{(D11)}$$

or

$$L_f=\frac{2P_f^{\frac{1}{2}}(w)P_f^{\frac{1}{2}}(w)}{P_f(w)+P_{-f}(w)}\,\mathrm{coh}_f(w,w^*). \quad \text{(D12)}$$

It follows from (D12) that

$$L_f \leqslant \mathrm{coh}_f(w,w^*), \quad \text{(D13)}$$

where the equality holds if

$$P_f(w)=P_{-f}(w) \quad \text{or} \quad P_{\pm f}(w)=0. \quad \text{(D14)}$$

From (D13) and (C11) we then have

$$L_f \leqslant \mathrm{coh}_f(w,w^*) \leqslant D_f. \quad \text{(D15)}$$

Thus the degree of rectilinear oscillation never exceeds the rotary coherence nor the degree of polarization D_f and is interpreted as

$L_f=1$ (rectilinear)

$L_f=0$ and $D_f=1$ (circular)

$L_f=D_f=0$ (irregular).

It should be noted that if the rectilinear and circular components are of the same origin and hence are coherent with each other a decomposition into rectilinear and nonrectilinear components is not possible by the present method. In this case, L_f should be interpreted as merely the maximum value of the *u-v* correlation or a measure of anisotrophy rather than the degree of rectilinear oscillation. On the other hand, if the elliptical and circular components are not of the same origin and hence are incoherent with each other, a decomposition into polarized and unpolarized components is not possible by the present method. In this case D_f should be interpreted as merely the maximum value of the *u-v* coherence rather than the degree of polarization.

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