A Stability Theorem for Energy-Balance Climate Models

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ABSTRACT

This paper treats the stability of steady-state solutions of some simple, latitude-dependent, energy-balance climate models. For north-south symmetric solutions of models with an ice-cap-type albedo feedback, and for the sum of horizontal transport and infrared radiation given by a linear operator, it is possible to prove a "slope-stability" theorem; i.e., if the local slope of the steady-state iceline latitude versus solar constant curve is positive (negative) the steady-state solution is stable (unstable). Certain rather weak restrictions on the albedo function and on the heat transport are required for the proof, and their physical basis is discussed in the text.

1. Introduction

The parallel study of climate models within a hierarchy of models of increasing complexity is now a well-established strategy in climate theory [for a discussion see the review of Schneider and Dickinson (1974)]. One hopes that some features will be common to all models from the simplest zero-dimensional globally averaged models to the most complex three-dimensional general circulation models. The one-dimensional Budyko-Sellers models have proven to be useful in exploring such properties. For example, the catastrophic transition to an ice-covered planet if the solar constant is lowered by a few percent was first discovered independently by Budyko (1968, 1969) and Sellers (1969) and appears to be common to a large variety of more complicated models (e.g., Temkin and Snell, 1976) ranging up to general circulation models (Manabe and Wetherald, private communication).

The purpose of this paper is to sketch the proof of a stability theorem for a large class of energy-balance climate models which include the Budyko-Sellers models. The linear stability analysis of solutions of several individual models in the class has been given previously by a number of investigators (Schneider and Gal-Chen, 1973; Held and Suarez, 1974; North, 1975a,b; Ghil, 1976; Su and Hsieh, 1976; Fredericksen, 1976; Drazin and Griffel, 1977; North, 1977). In addition, approaches to the full nonlinear stability problem have been suggested by Ghil (1976) and North *et al.* (1979). Our theorem clarifies the central result of these studies, and applies to a larger class of mean annual models than those represented above.

We shall not dwell on the well-known assumptions and limitations inherent in the Budyko-Sellers approach, since these have been adequately discussed in the recent literature. We shall start by motivating our study with a simple example in this section. In Section 2 we proceed to obtain formal solutions to the class of models being considered. In Section 3 we derive the so-called slope-stability theorem for that class.

We first consider a very simple model, variants of which have been discussed recently by several authors (Sellers, 1974; Crawfoord and Källen, 1978; Fraedrich, 1978). The model is a zero-dimensional energy-balance model (globally averaged) which may be defined by

$$C \frac{d}{dt} T_0 + I(T_0) = Q\bar{a}(T_0), \qquad (1.1)$$

where C is the heat capacity per unit area, T_0 the globally (and annually) averaged temperature, $I(T_0)$ the infrared radiation rate, Q the solar constant divided by 4, and $\bar{a}(T_0)$ the globally (and annually) averaged co-albedo, which is presumed to be a function of T_0 because of the ice-cap albedo feedback. The steady-state solutions to (1.1) are easily obtained by setting d/dt to zero and solving the resulting algebraic relation for T_0 as a function of Q. For a given Q there are typically several roots, as illustrated by the solution curve in Fig. 1, obtained from a model with a linear infrared law and a cubic co-albedo.

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In order to examine the linear stability, let $T_0 = T_0^0 + \delta(t)$, where T_0^0 is a solution to the steady-state problem. Then to the first order in $\delta(t)$,

$$C\dot{\delta}(t) + (I' - Q\bar{a}')\delta(t) = 0, \qquad (1.2)$$

where a prime denotes the derivative, evaluated at $T_0 = T_0^0$. The stability is determined by the sign of the expression in parentheses. This sign can be expressed in terms of the slope of the solution curve $Q(T_0)$, as follows. Differentiating the steady-state equation $I = Q\bar{a}$ and substituting, Eq. (1.2) becomes

$$C\dot{\delta}(t) + \frac{dQ}{dT_0}\bar{a}\delta(t) = 0. \qquad (1.3)$$

This last equation embodies the "slope-stability" theorem

$$\left.\begin{array}{l} \dfrac{dQ}{dT_0} > 0 \leftrightarrow \text{stability} \\ \dfrac{dQ}{dT_0} < 0 \leftrightarrow \text{instability} \end{array}\right\} . \qquad (1.4)$$

The theorem is easily interpreted since branches with negative slope have an apparent negative heat capacity and are therefore unphysical. Budyko (1972) advanced a heuristic argument along these lines as a stability proof, but it is not obvious that it applies to systems with spatial extension. One such example is a model of a star like the sun which has uniform temperature, is held together by its own gravity, and a heat balance is maintained by nuclear reactions in the interior and blackbody radiation at the surface. Increase of the heating rate leads to an increased radius and a *cooler* star (Nauenberg and Weisskopf, 1978). We presume such a star is stable. Counterexamples like this suggest that when possible we should construct rigorous proofs for stability. Moreover, rigorous mathematical results should help physical insight advance into still uncertain areas of climate theory.

This paper concentrates on one-dimensional energy-balance models which retain the sine of the latitude, x, as the single spatial variable. Hemispheric symmetry is assumed, so that only values of x from 0 to 1 need to be considered. The horizontal transport and infrared radiation laws are assumed to be represented by operators which are linear in the temperature field, and whose sum has an inverse with certain physically reasonable properties. Rather than assuming an explicit temperature dependence for the co-albedo as in the global model (1.1), an implicit temperature dependence is introduced through a parameter x_s, the sine of the latitude of the ice cap edge; this edge is assumed to be attached to a given isotherm. We shall consider only temperature fields having a unique x_s, and further assume that the temperature decreases northward across x_s.

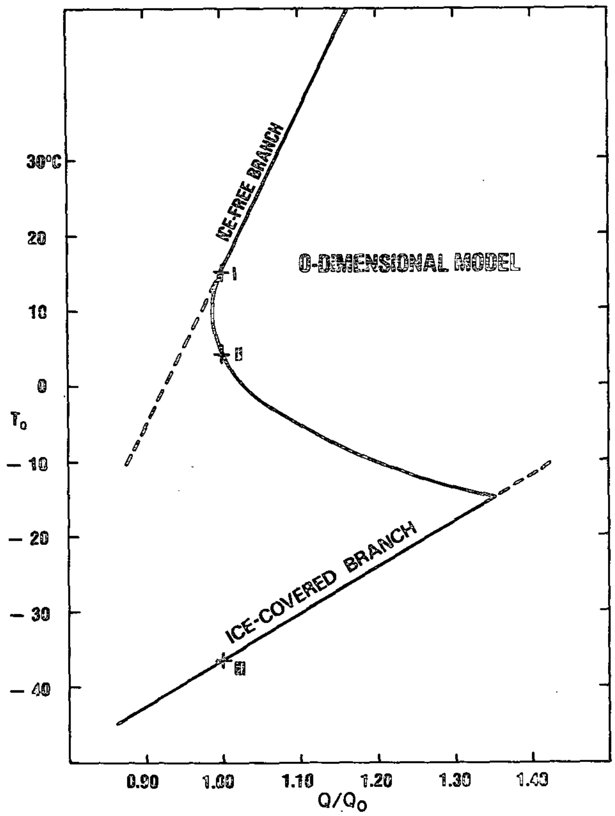

FIG. 1. Global average temperature T_0 as a function of the solar constant Q (in units of the present value Q_0), obtained from a global model having a globally and annually averaged co-albedo depending on T_0 due to ice cap-albedo feedback, as discussed in the text. The crosses indicate three solutions corresponding to the present value of the solar constant.

Just as the multiple solutions of the global model for a given solar constant are specified by the value of T_0, the different solution branches of the one-dimensional models are completely specified by the value of x_s. Fig. 2 shows an example of an exact solution to a specific model of this type (North, 1975a). A given point on the curve corresponds to a unique temperature field. We shall prove a stability theorem related to the sign of the local slope of this graph. Drazin and Griffel (1977) have found that under certain circumstances there can exist north-south unsymmetrical solutions. In such cases more than x_s and Q would be required to specify a given solution and our theorem does not hold.

The outline of the paper is as follows: Section 2 employs the formalism of linear operators and their corresponding Green's functions to construct steady-state solutions to the model equations, and to derive a transcendental equation whose roots determine the stability of these solutions. Section 3 establishes the slope-stability theorem by showing that the sign of the minimum root, which determines whether perturbations will grow in time or decay

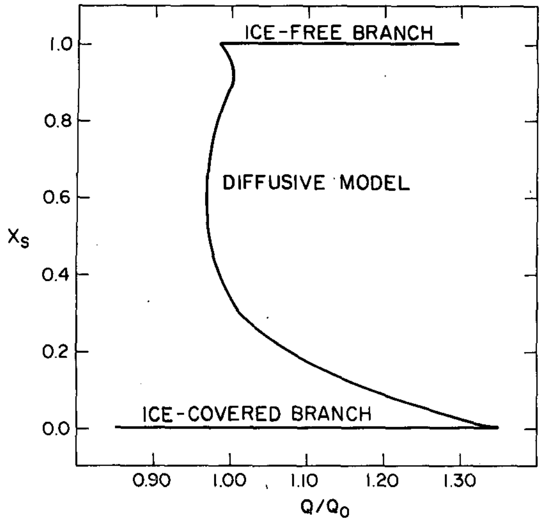

FIG. 2. A graph of the sine of the latitude of the iceline, x_s, as a function of the solar constant Q (in units of the present value Q_0) obtained from a zonal model having diffusive heat transport, as discussed by North (1975a). The co-albedo depends on the temperature field implicitly via x_s, which is attached to an isotherm. Multiple solutions correspond to a given value of the solar constant but different values of x_s.

back to equilibrium, is identical to the sign of the slope of the solution curve $x_s(Q)$ at the point which corresponds to the particular temperature field being perturbed. Also discussed in Section 3 are the assumptions for this theorem, its relation to the infinite instability of small ice caps found in some cases (indicated by the cusp near $x_s = 1$ in Fig. 2), and its relation to the formulation in terms of T_0 rather than x_s. Section 4 gives a brief discussion of results and concludes the paper. Two appendices are devoted to special cases: Appendix A treats the Green's functions of the Budyko and diffusive models, Appendix B the stability for a step-function albedo.

2. Steady-state solutions and stability eigenvalues

The class of models considered may be defined in terms of their corresponding energy balance equation

$$L[T_E](x) + f(x) = QS(x)a(x,x_s). \qquad (2.1)$$

Here $T_E(x)$ is the equilibrium (sea level) temperature field, $f(x)$ a given positive function associated with the outgoing radiation rule, Q the solar constant divided by 4, $S(x)$ the mean annual normalized solar distribution reaching the top of the atmosphere, $a(x,x_s)$ the co-albedo which is a function of the sine of the latitude, x, and the sine of the latitude of the ice cap edge, x_s. L is a linear operator representing horizontal transport and the part of the infrared rule which is linear and homogeneous in $T(x)$.

We may illustrate the various terms in (2.1) with some specific examples. In the diffusive transport model of North (1975a,b)

$$L_D[T](x) = -\frac{d}{dx}D(1-x^2)\frac{d}{dx}T(x) + BT(x) \qquad (2.2)$$

and $f(x) = A$ = constant. The first term here takes account of the heat transported by transient eddies, and $A + BT$ is Budyko's infrared law. A mean circulation term $\mathbf{v}(x)\cdot\nabla T$ as in Sellers (1969) might also be included, as well as possible latitude-dependence in D, A and B. In the Budyko model (Budyko, 1969; Chylek and Coakley, 1975)

$$L[T](x) = \gamma\int_0^1 dy[\delta(x-y) - 1]T(y) + BT(x)$$
$$= \gamma(T(x) - T_0) + BT(x), \qquad (2.3)$$

and again $f(x) = A$.

The co-albedo in the solar input term on the right-hand side (RHS) of (2.1) has often been assumed to be discontinuous at $x = x_s$, or as a function of temperature it is taken as discontinuous at a given iceline value T_s. These are equivalent formulations. However, there is no observational evidence for such a sharp transition. Even neglecting the effect of clouds and seasonal snow cover, we may expect that the zonal average of an iceline having large longitudinal variations such as Earth's would introduce considerable smoothing. Previous proofs of the stability theorem have been restricted to the step function co-albedo discussed in Appendix B. Unless otherwise stated we shall regard the source terms in (2.1) as smooth functions of x. As examples of smooth co-albedo functions one might generalize the step function of Appendix B to include a linear transition of width Δx_s around x_s, or simply replace it with a smooth function such as $\tanh[(x - x_s)/\Delta x_s]$. The presence of other feedback mechanisms, such as variable cloudiness, might also require some explicit temperature-dependence in the co-albedo which cannot be directly formulated in terms of latitude, and we shall exclude such processes here.

In addition to (2.1) it is necessary to specify the iceline condition

$$T_E(x_s) = T_s, \qquad (2.4)$$

where T_s has a fixed value (Budyko, 1969) usually taken to be −10°C. This condition introduces nonlinearity into the model. Finally, we specify the boundary conditions such that no net heat flows across the equator (north–south symmetry), $x = 0$, or into the pole, $x = 1$. In what follows it is necessary to assume that a Green's function G_0 for L exists satisfying the boundary conditions and

$$L[G_0](x,y) = \delta(x-y), \qquad (2.5)$$

where $\delta(x-y)$ is the Dirac function. The existence of G_0 limits the class of operators L, but this is not a severe restriction since any physical, linear system such as the ones we consider should have a unique response to a localized heat source.

As an example for the case of diffusive transport, $L = L_D$ given by Eq. (2.2), we may construct $G_0(x,y)$ explicitly

$$G_0^D(x,y) = \sum_{n=0}^{\infty} \frac{f_n(x)f_n(y)}{L_n}, \tag{2.6}$$

where $f_n(x)$, $n = 0, 1, 2, \ldots$, are the orthonormal eigenfunctions of L_D, and $L_n > 0$ are the corresponding discrete eigenvalues. [Ghil (1976) has shown that standard Sturm-Liouville results apply here.] For D constant, the eigenfunctions are just the even-indexed, normalized Legendre polynomials, $(2l+1)^{1/2}P_l(x)$, and $L_l = Dl(l+1) + B$ with $l = 2n$. In this latter case a closed form can be found for G_0^D in terms of hypergeometric functions. The spectral form (2.6) is discussed in Appendix A along with its counterpart for the Budyko model.

Returning to the general case, the equilibrium temperature field is given by the nonlinear integral equation

$$T_E(x) = \int_0^1 dyG_0(x,y)[QS(y)a(y,x_s) - f(y)], \tag{2.7}$$

where $T_E(x)$ enters the integrand through x_s.

Evaluating (2.7) at $x = x_s$ yields

$$T_s = \int_0^1 dyG_0(x_s,y)[QS(y)a(y,x_s) - f(y)]. \tag{2.8}$$

For a given x_s, Eq. (2.8) states that Q is determined; in fact, we may solve for it and obtain

$$Q = Q(x_s) = \frac{T_s + \int_0^1 dyG_0(x_s,y)f(y)}{\int_0^1 dyG_0(x_s,y)S(y)a(y,x_s)}. \tag{2.9}$$

From Eq. (2.9) we have the desired relationship between x_s and Q, so that we may plot a graph like that in Fig. 2. Given values x_s and Q, we may compute the unique temperature field $T_E(x)$ corresponding to them from (2.7). Hence, the steady-state problem is formally solved.

For a fixed value of the solar constant, we consider now a small perturbation of the temperature field about the steady-state solution, and apply the standard linear stability technique. The variation in temperature will also produce a variation in ice cap size through the ice-albedo feedback mechanism (provided $0 < x_s < 1$; special cases $x_s = 0, 1$ will be treated separately). Thus in addition to

$$T(x,t) = T_E(x) + \delta T(x,t), \tag{2.10}$$

we must also have

$$x_s(t) = x_s + \delta x_s(t), \tag{2.11}$$

where the ice cap variation δx_s may be determined in terms of δT by expanding the iceline condition to first order in the small quantities. To first order in δx_s and δT we have the temperature at the perturbed iceline

$$\begin{aligned} T(x_s+\delta x_s,t) &= T(x_s,t) + \left(\frac{\partial T}{\partial x}\right)_{x_s} \delta x_s \\ &= T_E(x_s) + \delta T(x_s,t) + T_E'\delta x_s, \end{aligned}$$

where T_E' is $(dT_E/dx)_{x=x_s}$. Due to the iceline conditions $T(x_s+\delta x_s, t) = T_s = T_E(x_s)$, the lowest order terms cancel, and we have

$$\delta x_s = \delta T(x_s,t)/(-T_E'). \tag{2.12}$$

Ice cap models for which x_s may be defined must have $T_E(x)$ decrease as we cross to the north of the iceline so that $T_E' < 0$. Thus Eq. (2.12) says that a positive variation in temperature away from equilibrium causes the ice cap to shrink, and the amount of shrinkage is inversely proportional to the temperature drop across the equilibrium iceline. The dependence on the temperature drop becomes clear if we picture the neighborhood of the iceline having a linear falloff in T_E and constant δT. Thus in order for x_s to follow an isotherm, we need $\delta T/\delta x_s = |\text{slope}|$. Note that in writing Eq. (2.12) we have implicitly assumed that T_E' exists, which may not hold when the source terms have a discontinuity and L contains integral operators as in the Budyko case. No discontinuities in temperature arise in Sturm-Liouville type problems, but in other cases it may be necessary to smooth the source terms.

The allowed temperature variations are determined by the requirement that the time-dependent energy balance equation must be satisfied to first order in the small quantities. Expanding

$$C\frac{\partial T}{\partial t}(x,t) + L[T](x,t) = QS(x)a(x, x_s+\delta x_s) - f(x)$$

to first order in δx_s and δT and eliminating T_E by employing Eq. (2.1) leads to

$$C\frac{\partial}{\partial t}\delta T(x,t) + L[\delta T](x,t) = QS(x)a_2(x,x_s)\delta x_s, \tag{2.13}$$

where a_2 indicates $\partial a(x,x_s)/\partial x_s$. If we substitute Eq. (2.12) for the iceline shift, we have a linear equation in $\delta T(x,t)$ which has solutions of the form

$$\delta T(x,t) = \delta T(x)e^{-\lambda t/C}. \tag{2.14}$$

Substituting (2.14) into Eq. (2.13) and cancelling exponentials, we obtain for $\delta T(x)$ the equation

$$(L-\lambda)[\delta T](x) = QS(x)a_2(x,x_s)\delta T(x_s)/(-T_E'). \quad (2.15)$$

The possible values of the parameter λ, the "stability eigenvalues" λ_n of this equation, determine the stability of the equilibrium solution $T_E(x)$ under a perturbation given by the associated eigenfunction $\delta T_n(x)$. If any of these perturbations (modes) have an eigenvalue with a negative real part, the equilibrium state is unstable. Equilibrium states having all $\mathrm{Re}\lambda_n > 0$ return to equilibrium with a decay time $\sim C/\mathrm{Re}\lambda_{\min}$.

Eq. (2.15) is an important tool in the study of linear stability and has been previously analyzed for a number of simple climate models. [See for example the appendix of Drazin and Griffel (1977) where the stability eigenvalue equation for the diffusion model with step-function albedo is given by Eq. (A6).] Our interest here, however, is not in determining the stability of a given solution to a given model. Instead we wish to relate the stability of such a solution to the sign of the slope of the steady-state iceline function at the point corresponding to that solution.

We proceed by first eliminating the eigenfunctions from (2.14) in order to obtain a scalar transcendental equation whose roots give the stability eigenvalues. In the next section we then relate these roots to properties of the steady-state solution whose stability is being tested.

We may define a Green's function G_λ which is a generalization of G_0 in Eq. (2.5) and satisfies

$$(L-\lambda)[G_\lambda](x,y) = \delta(x-y), \quad (2.16)$$

along with the boundary conditions at $x = 0, 1$. We may think of G_λ as a function of the continuous parameter λ except at the eigenvalues of L where it is not defined. For the diffusive case G_λ may be constructed explicitly:

$$G_\lambda^D(x,y) = \sum_{n=0}^{\infty} \frac{f_n(x)f_n(y)}{L_n-\lambda}, \quad (2.17)$$

where we have used the notation of Eq. (2.6) (cf. also Appendix A). From Eq. (2.17) one sees directly that G_λ^D is singular at $\lambda = L_n$, $n = 0, 1, 2, \ldots$. In fact, in these examples it is a meromorphic function of λ with simple poles at L_n.

Now the stability equation (2.15) may be rewritten in the form

$$\delta T(x) = \int_0^1 dy G_\lambda(x,y) [QS(y)a_2(y,x_s)\delta T(x_s)/(-T_E')], \quad (2.18)$$

where the boundary conditions are incorporated into G_λ. By evaluating this expression at $x = x_s$ we may cancel $\delta T(x_s)$ and obtain a formula for T_E', i.e.,

$$-T_E' = \int_0^1 dy G_\lambda(x_s,y)[QS(y)a_2(y,x_s)]. \quad (2.19)$$

On the other hand, if we differentiate (2.7) and evaluate the result at $x = x_s$, we obtain an equivalent expression for the slope of T_E at the iceline:

$$\left(\frac{-dT_E}{dx}\right)_{x_s} = -T_E' = \int_0^1 dy \partial G_0(x_s,y)/\partial x_s \times [QS(y)a(y,x_s) - f(y)]. \quad (2.20)$$

Similarly we may differentiate (2.8) with respect to x_s, noting that the first term on the RHS after differentiation is just given by (2.20). Since T_s is constant we obtain, after substitution,

$$-T_E' = \int_0^1 dy G_0(x_s,y) \frac{\partial}{\partial x_s}[QS(y)a(y,x_s)]. \quad (2.21)$$

In the differentiation on the RHS of (2.21) we must allow Q to depend on x_s. Substituting the result into Eq. (2.19) and combining the terms involving $a_2(y,x_s)$ yields

$$\frac{K_s}{Q}\frac{dQ}{dx_s} = \int_0^1 dy[G_\lambda(x_s,y) - G_0(x_s,y)]S(y)a_2(y,x_s), \quad (2.22)$$

where

$$K_s \equiv \int_0^1 dy G_0(x_s,y)S(y)a(y,x_s). \quad (2.23)$$

If we subtract Eq. (2.5) from (2.16) and use the linearity property, we obtain

$$L[G_\lambda - G_0](x,y) = \lambda G_\lambda(x,y), \quad (2.24)$$

so that the term appearing in square brackets may be rewritten as

$$G_\lambda(x,y) - G_0(x,y) = \lambda \int_0^1 dz G_0(x,z)G_\lambda(z,y). \quad (2.25)$$

Substituting this expression into Eq. (2.22) gives the final form of the eigenvalue equation as

$$\frac{K_s}{Q}\frac{dQ}{dx_s} = \lambda F(\lambda), \quad (2.26)$$

where the roots λ_j are determined by the properties of the function $F(\lambda)$, defined by

$$F(\lambda) \equiv \int_0^1 dy \int_0^1 dz G_0(x_s,z) G_\lambda(z,y)S(y)a_2(y,x_s). \quad (2.27)$$

$F(\lambda)$ depends on λ through Green's function G_λ defined by Eq. (2.16). For a given climatic state $T_E(x; Q)$, represented by a single point (x_s,Q) on an iceline curve such as in Fig. 2, the left-hand side of Eq. (2.26) will be fixed. Thus (2.26) is a transcendental equation for λ; the climate $T_E(x; Q)$ will be stable only when there are no roots λ of (2.26) having a negative real part. In the next section we shall argue that for physically reasonable models, the sign of the real part of the smallest value of λ is identical to the sign of the iceline slope, dQ/dx_s.

3. Slope-stability theorem

In the previous section we derived a transcendental equation whose roots λ_j are the stability eigenvalues. If the lowest eigenvalue has a positive (negative) real part, the system is stable (unstable). In this section we shall argue that for a certain class of models the slope-stability theorem

$$\left.\begin{aligned} \frac{dQ}{dx_s} > 0 &\leftrightarrow \text{stability} \\ \frac{dQ}{dx_s} < 0 &\leftrightarrow \text{instability} \end{aligned}\right\} \tag{3.1}$$

follows from the properties of the eigenvalue equation (2.26). First, we state the main assumptions which limit the class of models considered. Aside from the assumption of north-south symmetric solutions where the single index x_s may be used [implicit is also the assumption that $T_E(x)$ is decreasing at $x = x_s$], we impose the following conditions:

(i) The feedback is of the ice-cap type. Specifically $a_2(x,x_s) \geq 0$ for all x, x_s between 0 and 1.
(ii) The Green's function $G_0(x,y)$ is positive.
(iii) The generalized $G_\lambda(x,y)$ is positive for negative real values of λ with the asymptotic behavior $\delta(x - y)/(-\lambda)$ as $\lambda \rightarrow -\infty$.

The first assumption is obvious but does eliminate certain cloud band feedback mechanisms from consideration.

To clarify condition (ii), we first note that when an arbitrary heat source $\rho(y)$ is introduced it produces a temperature distribution $T(x) = \int dy \times G_0(x,y)\rho(y)$. If in addition $\rho(y)$ is increased by adding heat at a rate q at latitude x_0, so that $\Delta\rho(y) = q\delta(y - x_0)$, then the temperature distribution changes by $\Delta T(x) = qG_0(x,x_0)$. Thus saying that G_0 is positive is equivalent to the statement that heat added at one latitude will not lead to a decrease in temperature at any other latitude. Fig. 3 shows an example of the behavior of the Green's function corresponding to L_D, Eqs. (2.2) and (2.6) with D constant. This condition holds for L of Sturm-Liouville type, and a proof of this is sketched in Appendix A. We suspect that it holds for any physically reasonable L including some integral operators such as Budyko's (cf. Appendix A). Similar conditions have been applied to other stability problems, e.g., Joseph (1976).

The increasing localization of G_0 in Fig. 3 for decreasing values of D (or increasing B) is related to condition (iii). For the Sturm-Liouville type systems

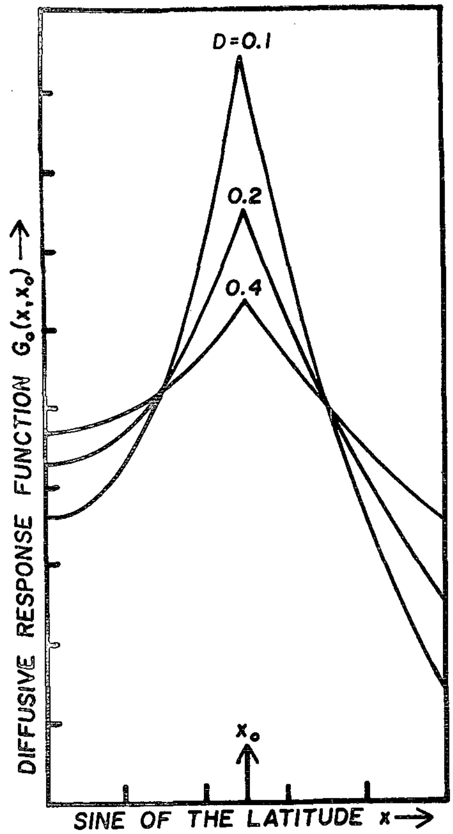

FIG. 3. The diffusive response function $G_0(x,x_0)$ given in Eq. (2.6) of the text, computed with $B = 1$ and various (constant) values of D. The area under the curve is independent of D (units = 0.2).

the expression $\delta(x - y)/(-\lambda)$ corresponds to keeping the first term in an asymptotic expansion for large negative λ and using the completeness relation $\sum_n f_n(x)f_n(y) = \delta(x - y)$. Adding this relation divided by $-\lambda$ to (2.17) leads to

$$G_\lambda^D(x,y) = -\delta(x - y)/\lambda + \sum_n f_n(x)f_n(y)\frac{L_n}{(L_n - \lambda)\lambda}; \tag{3.2}$$

the second term on the RHS is equivalent to $L[G_\lambda^D](x,y)/\lambda$, which can also be obtained formally from rearranging (2.16). Although more singular at $x = y$ than the first term, since it contains derivatives of delta functions, it is smaller by a factor of order $1/\lambda$. This suggests that if $G_\lambda^D(x,y)$ only appears as a factor in an integrand multiplied by sufficiently well-behaved functions, (iii) will hold. It is possible to exhibit this asymptotic behavior of L explicitly in certain specific cases.

This ends the discussion of the conditions (i), (ii), and (iii) which are sufficient for our proof to be valid. The theorem may hold also for a weakened form of these conditions in certain cases. In what follows the reader may wish to refer to Appendix B where a special case is discussed.

We now return to the discussion of the stability eigenvalues. From (ii) we clearly have

$$K_s > 0, \tag{3.3}$$

since each factor in the integrand of (2.23) is positive.

The behavior of $F(\lambda)$ may be determined from that of G_λ. According to Eq. (2.25) $G_\lambda \to G_0$ as $\lambda \to 0$, and for small positive values of λ we have

$$G_\lambda(x,y) \approx G_0(x,y) + \lambda G_0^{(2)}(x,y) > 0; \tag{3.4}$$

the "iterated kernel," given by

$$G_0^{(2)}(z,y) \equiv \int dz' G_0(z,z')G_0(z',y), \tag{3.5}$$

represents the first term in a series expansion of G_λ in powers of λ. Eq. (2.25) is in fact an *inhomogeneous Fredholm equation of the second kind*, for G_λ. Its solution can be represented by such a power series in λ, a *Neumann series*, which converges as long as $|\lambda| < L_0$, where L_0 is the first eigenvalue of L. The coefficient of λ^n in the Neumann series is the nth iterate of the kernel G_0. Hence each term is positive for positive λ. Thus, G_λ is positive for $-\infty < \lambda < 0$ because of assumption (iii) and for $0 \leqslant \lambda < L_0$ by the argument above; as $\lambda \to L_0$, $G_\lambda \to +\infty$. Furthermore, if we compute the derivative with respect to λ of Eq. (2.16) the solution of the resulting equation may be written in the form

$$dG_\lambda(x,y)/d\lambda = \int dz G_\lambda(x,z)G_\lambda(z,y), \tag{3.6}$$

which again is positive in this interval so that G_λ is strictly monotonic for $\lambda < L_0$. This is sufficient information to conclude that as λ decreases through zero, the stability function $\lambda F(\lambda)$ will also decrease through zero, and will monotonically approach its asymptotic value, determined by the behavior of G_λ to be

$$\lambda F(\lambda) \xrightarrow[\lambda\to-\infty]{} - \int_0^1 dy G_0(x_s,y)S(y)a_2(y,x_s). \tag{3.7}$$

We may now establish the slope-stability theorem by considering the possible values of λ determined by the relation

$$\frac{K_s}{Q}\frac{dQ}{dx_s} = \lambda F(\lambda), \tag{2.26}$$

for a fixed point on the equilibrium iceline curve $x_s = x_s(Q)$. The graphical solution of this transcendental equation is illustrated schematically in Fig. 4. The constants L_1, L_2, etc., are the higher eigenvalues of L, but for now we concentrate on the region $\lambda < L_0$. When the iceline slope is positive, indicated by the solid horizontal line in Fig. 4, the smallest possible stability eigenvalue λ_0 is also positive, and determined by the intersection of the horizontal line with $\lambda F(\lambda)$, computed from the Neumann series for G_λ. Perturbations in this region decay to zero in a characteristic time C/λ_0. This is the situation in the positive slope branches in Fig. 2. As we approach one of the critical points indicated in Fig. 2, dQ/dx_s as well as λ_0 approach zero and we have a situation of neutral stability. Finally, in the unstable regions

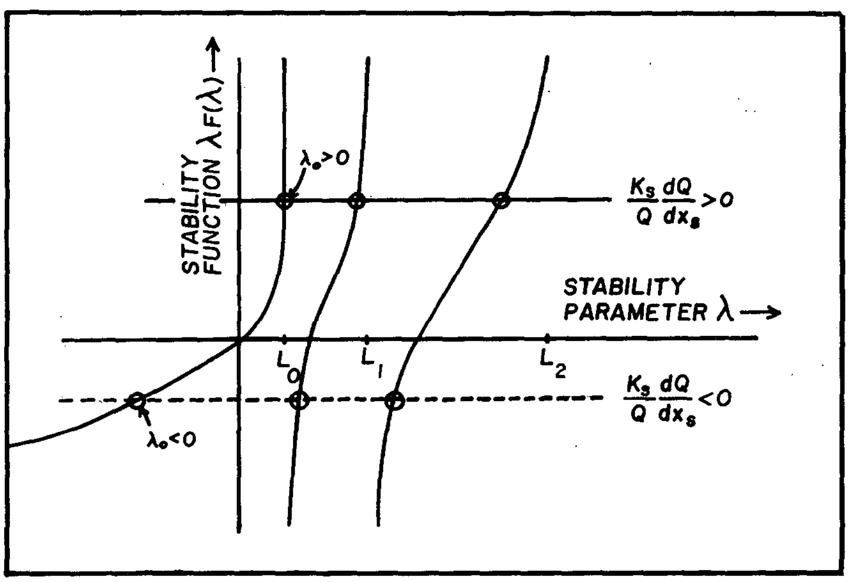

FIG. 4. A schematic graph of the right-hand-side of Eq. (2.26) of the text versus the stability parameter λ. The solid and dashed horizontal lines represent positive and negative values of the left-hand-side of Eq. (2.26), determined by the slope at a given point on an iceline curve such as in Fig. 2. Intersections marked with circles give the stability eigenvalues, and λ_0 is the minimum stability eigenvalue.

(negative slope) shown in Fig. 2, the iceline slope is negative, and we have the situation indicated by the dotted horizontal line in Fig. 4. In this case, the stability eigenvalue λ_0 is negative, so that the corresponding perturbation δT_0 increases in time according to Eq. (2.14). Eventually the linearization of $T[x_s(t),t]$ and $a[x,x_s(t)]$ breaks down, and the exact solution approaches one of the available stable solutions for the given value of Q, for example, that of an ice-covered or ice-free planet.

If, as Drazin and Griffel (1977) have suggested, the iceline curve can have a cusp in some cases, so that $dQ/dx_s \to -\infty$ as $x_s \to 1$, the dotted line in Fig. 4 might conceivably fall below the asymptotic value of $\lambda F(\lambda)$, so that a region of tremendous instability ($\lambda_0 \to -\infty$) would suddenly become stable. It remains then in our proof to show that the dashed line never falls below the asymptotic level of $\lambda F(\lambda)$ as given by (3.7).

The slope of the iceline curve for equilibrium solutions may be computed from Eq. (2.21) and has the form

$$\frac{K_s}{Q}\frac{dQ}{dx_s} = (-T_E')/Q - \int_0^1 dy G_0(x_s,y)S(y)a_2(y,x_s), \quad (3.8)$$

where the positive constant K_s is defined in Eq. (2.23). Since for the class of models (or solutions) considered the temperature decreases as one crosses the ice cap poleward, the first term in (3.8) is positive. The second term, on the other hand, is strictly negative. This latter is in fact just the asymptotic value of $\lambda F(\lambda)$ which is given in (3.7). Hence, $(K_s/Q)(dQ/dx_s)$ is always larger than the asymptotic plateau by the positive amount $(-T_E')/Q$. This concludes the proof of the slope-stability theorem.

We should note that the cusp behavior of the x_s vs Q curve pointed out by Drazin and Griffel (1977) and shown near $x_s = 1$ in Fig. 2 comes about for the case of $a(x,x_s)$ a step function. In this case $a_2(x,x_s)$ is a delta function so that the second term of (3.8) can be evaluated explicitly. The cusp comes about because the resulting expression is proportional to $G_0(x_s,x_s)$, which diverges as $x_s \to 1$. Any smoothing of the albedo at the ice cap edge eliminates this divergence. (In fact, we have found numerically that a smoothing width Δx_s of the order of 0.1 is sufficient to remove the negative slope portion of the curve in Fig. 2 near $x_s = 1$ altogether.) One also notes that the cusp is removed even in the discontinuous albedo case if $S(1) = 0$, in agreement with Drazin and Griffel.

We may now examine the special cases, $x_s = 0$ and $x_s = 1$, the ice-covered and ice-free situations. In both cases we may neglect the ice feedback since, for example, in the $x_s = 0$ case the equator is well below $-10°C$ so that an enormous perturbation would be required to cause the ice cap to recede from the equator. Hence an infinitesimal perturbation of the steady-state solution causes only a perturbation in the temperature field and $\delta x_s = 0$. One easily deduces that in this case the stability eigenvalues are $L_n > 0$, and the solution is stable. The same argument works for $x_s = 1$.

Finally, we may relate the slope-stability theorem as stated for the $x_s(Q)$ iceline curve to the formulation in terms of the corresponding $T_0(Q)$ curve, as discussed for the globally averaged model in the introduction. We shall see that the "heat capacity" argument does not invariably follow in the present class of spatially extended systems. First note that in the special cases $x_s = 0$, 1 as discussed in the preceding paragraph, the connection is correct since dT_0/dQ is clearly positive and the solution is stable. Elsewhere we may relate the iceline and T_0 slopes by first integrating (2.7) over the hemisphere and then differentiating with respect to Q:

$$dT_0/dQ = \int_0^1 dx \int_0^1 dy G_0(x,y)S(y)a(y,x_s) + Q\frac{dx_s}{dQ}\int_0^1 dx \int_0^1 dy G_0(x,y)S(y)a_2(y,x_s).$$

Since both of the integrals above are bounded and strictly positive, and dx_s/dQ changes sign at an ordinary bifurcation by passing through infinity, we may assert that these derivatives have the same sign at least when $|dx_s/dQ|$ is large enough. However they could have opposite sign near a cusp for which $dx_s/dQ \to 0$. Hence, the theorem must be stated as (3.1) in terms of x_s rather than as (1.4).

4. Discussion

We have proven a theorem that states that the stability of a model climate solution depends on a property of the curve which determines the steady-state solutions, the latitude of the ice cap edge versus the solar constant. If the local slope of this curve is positive (negative) the corresponding steady-state solution will be stable (unstable). The theorem covers a fairly broad class of models of the ice-cap-feedback type, but certain assumptions are necessary. The most important of these relate to the Green's function for the linear part of the problem. Existence, positivity and a certain type of asymptotic behavior are sufficient for our proof to hold. These conditions are satisfied by generalized diffusive (Sturm-Liouville) models and hold for some nonlocal models such as Budyko's.

We have seen that positive values of the global "heat capacity" do not guarantee stability in these models, although negative values definitely imply instability. Instability may be simply shown by an opportune choice of perturbation, but to demonstrate stability of a solution under an arbitrary perturbation is less simple, particularly in systems with spatial extension, which may not follow intuitions appropriate for their global or homogeneous counterparts.

We have also pointed out that the stability of small ice caps, as determined by the slope of the iceline curve near $x_s = 1$, is highly sensitive to the smoothness of the albedo across the iceline. Such dependence unfortunately suggests that the true sensitivity and stability of small ice caps cannot be estimated from such simple models. This question is clearly important, considering the possibility of global warming and the effect of a melting ice cap on the present level of the oceans. The question of whether energy and mass balance requirements are sufficient to model the problem, or whether specific dynamical mechanisms must be included, can only be answered through continued study of ice cap variations on many time scales.

Finally, one may imagine attempting to generalize our theorem to models having more degrees of freedom, with additional dynamics and feedback mechanisms. Rather than resort to pure speculation, we shall remain within the present framework of one-dimensional energy-balance models, and briefly discuss three types of extensions which may affect the stability: different iceline conditions, nonlinearities due to other feedbacks and climatic noise.

Generalizations of our conditions of isothermal icelines having north-south symmetry are of interest. Pollard (1976), for instance, has shown that if the iceline lags slightly behind the $-10°$C isotherm in time rather than following it instantaneously the theorem still holds. We do not yet know if some similar generalization holds for cases when more than one index might be required to specify the branch of a solution. For instance, in unsymmetrical solutions with only one ice cap, one must specify which ice cap.

Several nonlinearities merit further study:

1) As Stone (1973) has suggested, the diffusion coefficient may be proportional to some power of the temperature gradient.

2) Any dependence of atmospheric carbon dioxide, water vapor or cloudiness on the surface temperature would tend to introduce some nonlinearity in the infrared emission. The net effect has been much contested, but since analytical solutions to linear models are known, it would be of much interest to study the effect of a nonlinear perturbation.

3) Although we have generalized the discontinuous coalbedo to allow for smooth variations in latitude, a more realistic treatment of cloudiness would also allow the co-albedo to vary smoothly with temperature. Even if no analytical solution to such a nonlinear problem can be found, it may be possible to solve the linear stability problem analytically.

Climatic noise is typically modeled by the addition of stochastic forcing terms having given statistical properties. These then determine the statistics of the response, or in this case the temperature fluctuations. Such fluctuations will normally be governed by our stability theorem, but new stability questions arise and here we mention two which merit further study:

1) The probability of finite-amplitude fluctuations may be sufficiently large to effect transitions between different branches of the solution curve. One must then consider the relative stability of the states involved. For example, states which have marginal stability in the deterministic sense may be only metastable in the stochastic sense.

2) The statistics of the forcing may itself depend on the climatic state; for example, its variance may depend on the temperature. In that case the stability may not be related in a simple way to the deterministic solution curve.

Many questions remain unanswered concerning climatic stability, and we hope that the present paper will help generate further interest in and study of such questions.

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APPENDIX A

Response Functions in the Budyko and General Diffusion Models

In the Budyko model, the combined infrared and meridional transport terms are given by

$$L[T](x,t) = \int_0^1 dz[(\gamma + B)\delta(x - z) - \gamma]T(z,t), \quad \text{(A1)}$$

and in this case it is easily verified that the generalized response function defined in Eq. (2.16) is given by

$$G_\lambda(x,y) = \frac{\delta(x - y)}{\gamma + B - \lambda} + \frac{\gamma}{(B - \lambda)(\gamma + B - \lambda)}, \quad \text{(A2)}$$

which clearly satisfies our conditions (ii) and (iii) discussed in Section 3. Note that in the absence of ice feedback, any heat added to a given latitude belt does not affect the temperature gradient at any other latitude in this model, since G_0 is constant for $x \neq y$. Also, the delta function implies that any discontinuity in albedo yields a discontinuity in temperature, so that one must be very careful in applying the iceline condition.

Next we consider a class of models given by energy-balance equations of the form

$$C \frac{\partial T}{\partial t} - \frac{\partial}{\partial x} D(x)(1 - x^2) \frac{\partial}{\partial x} T + A(x) + B(x)T - QS(x)a(x,x_s) = 0, \quad \text{(A3)}$$

along with the boundary conditions of vanishing heat flux at the equator and pole, appropriate for a prototype planet with hemispheric symmetry, so that

$$D(x)[(1 - x^2)]^{1/2} \left. \frac{dT}{dx} \right|_{x=0,1} = 0. \quad \text{(A4)}$$

The Green's function for the equilibrium solution of such a model satisfies

$$\left[- \frac{d}{dx} D(x)(1 - x^2) \frac{d}{dx} + B(x) \right] G_0(x,y) = \delta(x - y), \quad \text{(A5)}$$

as well as the boundary conditions (A4). We wish to show that

$$G_0(x,y) \geqslant 0 \quad \text{if} \quad B(x), D(x) > 0. \quad \text{(A6)}$$

The proof is identical for G_λ as long as $B - \lambda > 0$.

We may easily demonstrate the positivity of G_0 at the point $x = y$ from the so-called "spectral representation." The eigenvalue equation

$$\left[- \frac{d}{dx} D(x)(1 - x^2) \frac{d}{dx} + B(x) \right] f_n(x) = L_n f_n(x), \quad \text{(A7)}$$

determines the orthonormal eigenfunctions f_n as well as the corresponding eigenvalues L_n of the linear operator. [See Ghil (1976) for a comparison with standard Sturm-Liouville systems.] Multiplying Eq. (A7) by $f_n(x)$, integrating over all x, and employing the boundary conditions to integrate the first term by parts yields

$$\int_0^1 dx \left[D(x)(1 - x^2) \left(\frac{df_n}{dx} \right)^2 + B(x) f_n^2 \right] = L_n > 0, \quad \text{(A8)}$$

so that the spectrum is positive definite. It can be shown that the eigenfunctions f_n form a complete system; hence we may represent G_0 as

$$G_0(x,y) = \sum_n \frac{f_n(x) f_n(y)}{L_n}, \quad \text{(A9)}$$

which along with Eq. (A8) clearly implies

$$G_0(x,x) > 0. \quad \text{(A10)}$$

For the case in which B and D are constant, we obtain $f_n = (2n + 1)^{1/2} P_n$ and $L_n = Dn(n + 1) + B$, where the P_n are the even-indexed Legendre polynomials. In general, for reasonable $D(x)$, $B(x)$, the f_n are regular everywhere and we shall, therefore, assume in the following that $G_0(x,y)$ is everywhere finite [with the exception of the singular point in Eq. (A4), *viz.*, $x, y = 1$].

In order to show that G_0 is positive for $x \neq y$ we return to Eq. (A5) and choose a new independent variable $z(x)$ such that

$$\frac{dx}{dz} = D(x)(1 - x^2) > 0. \quad \text{(A11)}$$

The integral of (A11) is

$$z(x) = \int_0^x \frac{dx'}{D(x')(1 - x'^2)}, \quad \text{(A12)}$$

so that $z(0) = 0$ and $z(1) = \infty$. Multiplying (A5) by dx/dz leads to

$$\left[- \frac{d^2}{dz^2} + C(z) \right] G_0(z,z_0) = \delta(z - z_0), \quad \text{(A13)}$$

where $C(z) = (dx/dz)B(z) > 0$ and we have employed the well-known rule that $\delta[x(z) - y] = \delta(z - z_0)/|dx/dz|$, where $x(z_0) = y$. According to Eq. (A13), the curvature of G_0 is determined by the sign of G_0. G_0 is concave upward in the upper half plane, concave downward in the lower half plane, and has a point of inflection if it vanishes. Thus, if G_0 is to remain finite, it must be asymptotically flat as $z \to \infty$, and according to the boundary conditions it is also asymptotically flat as $z \to 0$. Since G_0 is positive at $z = z_0$, and thus concave upward on either side of $z = z_0$, it cannot vanish anywhere and still become flat as $z \to 0, \infty$. Thus, it must be positive everywhere.

APPENDIX B

Step-Function Albedo

In order to make contact with previous studies, we may specialize our stability condition in Eq. (2.26) to the case in which the absorption changes discontinuously at the iceline, so that

$$a(x,x_s) = a_W \theta(x_s - x) + a_I \theta(x - x_s), \quad \text{(B1)}$$

where a_W and a_I are given constants for which $a_W > a_I$, and θ is the unit step function. Since the derivative of θ is a delta function, Eq. (2.27) reduces to

$$F(\lambda) = S(x_s)(a_W - a_I) \int_0^1 dz G_0(x_s,z) G_\lambda(z,x_s). \quad \text{(B2)}$$

If we insert the spectral representation

$$G_\lambda(z,x_s) = \sum_n \frac{f_n(z)f_n(x_s)}{L_n - \lambda}, \tag{B3}$$

where the f_n are the orthonormal eigenfunctions of L which we assume to be complete, and the L_n are the eigenvalues which we assume to be positive definite (as proven for general diffusive models in Appendix A), then Eq. (B2) takes the form

$$F(\lambda) = S(x_s)(a_W - a_I) \sum_n \frac{[f_n(x_s)]^2}{L_n(L_n - \lambda)}. \tag{B4}$$

For $\lambda > 0$, $F(\lambda)$ has simple poles at $\lambda = L_n$ and changes sign as λ passes each L_n. However, for $\lambda < 0$ both $F(\lambda)$ and $dF(\lambda)/d\lambda$ are positive definite, and $\lambda F(\lambda)$ decreases monotonically to $-\infty$. Thus, for this case the graphical solution of Eq. (2.26),

$$\frac{K_s}{Q}\frac{dQ}{dx_s} = \lambda F(\lambda),$$

will have the qualitative form shown in Fig. 4, already discussed for the general case in Section 3.

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