On Solitary Rossby Waves[1]

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ABSTRACT

A variational principle and an associated integral invariant are constructed for two-dimensional (nondivergent) waves of permanent form in a Rossby β-plane. A solitary-wave solution is obtained, and it is shown that the effects of cubic nonlinearity may be comparable with those of quadratic nonlinearity and may limit the amplitude of the wave.

1. Introduction

Solitary Rossby waves in a zonal flow appear to have been discovered (analytically) by Long (1964) and have been studied subsequently by Larsen (1965), Benney (1966), Clarke (1971), Redekopp (1977), Redekopp and Weidman (1978), Hukuda (1978) and Rizzoli (1978). All invoke Rossby's β-plane model, in which the northerly gradient of the vertical component of the earth's rotation (Rossby's β) is constant, and all but Redekopp assume that the wave speed relative to the local flow nowhere vanishes (there are no critical layers). Long, Larsen and Benney neglect vertical variation of the flow (the nondivergent approximation) and assume that the shear, which is essential for the existence of solitary Rossby waves, is weak (the basic flow is almost uniform) but not so weak as to preclude a Boussinesq-like balance between dispersion and quadratic nonlinearity.

I consider here a generalization of Long's basic model, retaining the β-plane and nondivergent approximations and the assumption that there are no critical layers, but allowing for an arbitrary velocity profile (as also do Clarke and Redekopp) and the possibility that cubic nonlinearity may be significant if the shear is sufficiently weak.[2] I also show that the basic partial differential equation for any wave of permanent form in the present model may be derived from a variational integral, which, in turn, implies the existence of an integral invariant that leads rather more directly to the solitary-wave solution than do the perturbation expansions invoked by Long and his successors. [The variational procedure also may be invoked for stratified shear flows (Miles, 1979).]

2. Variational principle

Consider the steady shear flow of a homogeneous, incompressible fluid in a Rossby β-plane, wherein the Cartesian coordinates x and y are directed east and north and have the respective domains $(-\infty, \infty)$ and (y_1, y_2); $f = f_0 + \beta y$ is (the β-plane approximation to) the Coriolis parameter, and f_0 and $\beta \equiv (df/dy)_0$ are constants evaluated at a reference latitude; ψ is the streamfunction, defined such that $[-\psi_y, \psi_x]$ is the local velocity,

$$U(\mathfrak{y}) = -d\psi/d\mathfrak{y} \tag{2.1}$$

is the prescribed velocity, and $\mathfrak{y}$ is the y coordinate of any particular streamline in the upstream flow. Conservation of the total vorticity $\nabla^2\psi + f$ along streamlines then implies (Long, 1964)

$$\nabla^2\psi + \beta y = -U'(\mathfrak{y}) + \beta\mathfrak{y}. \tag{2.2}$$

The flow for a solitary wave (or any wave of permanent form) moving upstream with the wave speed c relative to a basic flow U_* in a fixed reference frame (x_*, y) is steady in a reference frame moving with the wave, in which

$$U = U_* + c, \quad x = x_* + ct. \tag{2.3a,b}$$

The following analysis is carried out in the moving reference frame, but it should be emphasized that

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[2] The possibility that the effects of cubic nonlinearity may be comparable with those of quadratic nonlinearity for an internal solitary wave appears to have been appreciated originally by Long (1956) [see also Miles (1979) and references cited there]. Redekopp (1977) considers solitary Rossby waves for which the dominant nonlinearity is *either* quadratic *or* cubic. Hukuda (1978) gives a formulation in which both quadratic and cubic nonlinearity are significant for solitary Rossby waves, but he does not give explicit solutions.

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c (which enters implicitly through U) is to be determined for prescribed $U_*(\mathscr{y})$.

The transformation

$$x = \mathscr{x}, \quad y = \mathscr{y} + \delta(\mathscr{x},\mathscr{y}) \qquad (2.4a,b)$$

carries (2.1) and (2.2) over to (after some manipulation)

$$U^2\left(\frac{\delta_x}{1+\delta_y}\right)_x + \left[U^2\,\frac{(\delta_y + \tfrac{1}{2}\delta_y^2 - \tfrac{1}{2}\delta_x^2)}{(1+\delta_y)^2}\right]_y + \beta U\delta = 0, \qquad (2.5)$$

which is the Euler condition for the variational integral

$$L = \tfrac{1}{2}\int_{-\infty}^{\infty}\int_{y_1}^{y_2}[U^2(1+\delta_y)^{-1}(\delta_x^2+\delta_y^2) - \beta U\delta^2]\,d\mathscr{x}\,d\mathscr{y} \qquad (2.6)$$

in conjunction with the natural boundary conditions

$$\delta\delta_x = 0 \quad (\mathscr{x} = \pm\infty),$$

$$\delta(\delta_y + \tfrac{1}{2}\delta_y^2 - \tfrac{1}{2}\delta_x^2) = 0 \quad (\mathscr{y} = \mathscr{y}_{1,2}) \qquad (2.7a,b)$$

($\mathscr{y}_{1,2} = y_{1,2}$ for rigid boundaries). Moreover, it follows from Noether's theorem and the fact that U is independent of $\mathscr{x}$ that

$$I = \tfrac{1}{2}\int_{y_1}^{y_2}[U^2(1+\delta_y)^{-1}(\delta_x^2 - \delta_y^2) + \beta U\delta^2]\,d\mathscr{y} \qquad (2.8)$$

is an integral invariant of the flow. Alternatively, (2.8) may be derived by multiplying (2.5) through by δ_x and integrating over $(\mathscr{y}_1,\mathscr{y}_2)$ and indefinitely with respect to $\mathscr{x}$. Note that $\delta = 0$ at $\mathscr{x} = -\infty$ implies $I = 0$.

3. Solitary wave

I now assume that the lateral boundaries are rigid and that $U > 0$ in $(\mathscr{y}_1,\mathscr{y}_2)$ and seek an approximate solution of (2.5) in the form

$$\delta = aA(\xi)\phi_n(\mathscr{y}), \quad \xi = k\mathscr{x}, \qquad (3.1a,b)$$

where a and $1/k$ are scales of amplitude and wavelength; A, ϕ_n and ξ are dimensionless; A satisfies the normalizing and boundary conditions

$$A(0) = 1, \quad A(-\infty) = 0; \qquad (3.2a,b)$$

ϕ_n is determined by the eigenvalue problem

$$(U^2\phi_n')' + \beta U\phi_n = 0 \quad (U > 0,\ \mathscr{y}_1 < \mathscr{y} < \mathscr{y}_2), \qquad (3.3)$$

$$\phi_n = 0 \quad (\mathscr{y} = \mathscr{y}_{1,2}). \qquad (3.4)$$

The implicit eigenvalue is c_n ($n = 1, 2, \ldots$), $c_1 > c_2 > \ldots$, and (here and subsequently) the primes imply differentiation with respect to $\mathscr{y}$.

The differential equation for $A(\xi)$ is obtained by substituting (3.1) into (2.8), expanding I [which, being constant, must vanish identically by virtue of (3.2b)] in powers of $\alpha = a/l$, where $l = \mathscr{y}_2 - \mathscr{y}_1$, and retaining only the dominant terms in the limit $\alpha \to 0$. The resulting approximation, on the hypothesis that $k^2l^2 = O(\alpha)$, is

$$k^2\langle U^2\phi^2\rangle(dA/d\xi)^2 + \langle\beta U\phi^2 - U^2\phi'^2\rangle A^2 + a\langle U^2\phi'^3\rangle A^3 = 0, \qquad (3.5)$$

wherein $\phi \equiv \phi_n$ and the angle brackets represent an average over $(\mathscr{y}_1,\mathscr{y}_2)$. It follows from (3.3) and (3.4), after integration by parts, that [note that $U = U_* + c_n/U_* + c$ in (3.3)/(3.5)]

$$\langle\beta U\phi^2 - U^2\phi'^2\rangle = -\langle(\beta - U'')\phi^2\rangle(c - c_n) - (c - c_n)^2\langle\phi'^2\rangle, \qquad (3.6)$$

which, together with the additional hypothesis that $c = c_n[1 + O(\alpha)]$, permits the reduction of (3.5) to the equivalent approximation

$$k^2\langle U^2\phi^2\rangle(dA/d\xi)^2 = (c - c_n)\langle(\beta - U'')\phi^2\rangle A^2 - a\langle U^2\phi'^3\rangle A^3, \qquad (3.7)$$

wherein $U = U_* + c_n$. This last result agrees with that implied by Clarke's (1971) equations (53)–(56).

The solution of (3.7), subject to (3.2), is given by

$$A = \operatorname{sech}^2\xi, \qquad (3.8a)$$

$$4k^2\langle U^2\phi^2\rangle = (c - c_n)\langle(\beta - U'')\phi^2\rangle = a\langle U^2\phi'^3\rangle. \qquad (3.8b)$$

The parameter $\langle U^2\phi'^3\rangle$ may be either positive or negative and dictates the sign of a. It vanishes for a uniform flow, in consequence of which some shear is necessary for the existence of a solitary wave. This last point is made explicit by the identity

$$\langle U^2\phi'^3\rangle = \tfrac{1}{3}\langle[(\beta - U'')U' + UU''']\phi^3\rangle, \qquad (3.9)$$

which may be established through integration by parts and the invocation of (3.3) and (3.4).

The hypothesis that $k^2l^2 = O(\alpha)$ fails if $\langle U^2\phi'^3\rangle$ is sufficiently small, and it then is necessary to include the additional term $-a^2\langle U^2\phi'^4\rangle A^4$ on the left-hand side of (3.5).[3] The resulting generalization of (3.8) is [cf. Kakutani and Yamasaki (1978) and Miles (1979)]

$$A = (\cosh^2\xi - \mu\sinh^2\xi)^{-1} \quad (0 < \mu < 1), \qquad (3.10a)$$

and

$$4k^2\langle U^2\phi^2\rangle = (c - c_n)\langle(\beta - U'')\phi^2\rangle = a\langle U^2\phi'^3\rangle - a^2\langle U^2\phi'^4\rangle, \qquad (3.10b)$$

where

$$\mu = \frac{a}{2a_n - a}\ (0 < \mu < 1), \qquad (3.11a)$$

[3] The inclusion of cubic nonlinearity in a time-dependent formulation leads to a mixed KdV (Korteweg-deVries)-modified KdV evolution equation (cf. Miles, 1979).

$$a_n = \tfrac{1}{2}\frac{\langle U^2\phi'^3\rangle}{\langle U^2\phi'^4\rangle}. \quad (3.11b)$$

The parameter μ is a measure of the mass associated with the wave, say M, and $\mu \uparrow 1$ implies $M \uparrow \infty$. The parameter a_n, which has the same sign as $\langle U^2\phi'^3\rangle$, is the maximum possible amplitude ($|a| < |a_n|$) within the present approximations. The limit $|a_n| \to 0$ with M fixed implies $|a| \to 0$ and $k \to 0$ (vanishing amplitude and infinite length).

4. Liouville approximation

The eigenvalue problem posed by (3.3) and (3.4) may be solved in terms of known functions for a few special velocity profiles, but the results are complicated. It is more informative to consider the Liouville approximation

$$\phi_n = \left(\frac{U_0}{U}\right)^{3/4}\sin\left[\int_{y_1}^{y}\left(\frac{\beta}{U}\right)^{1/2}dy\right] + O(\epsilon), \quad (4.1a)$$

$$\epsilon = \max\left(\frac{U''}{\beta} + \frac{1}{4}\frac{U'^2}{\beta U}\right), \quad (4.1b)$$

where c_n is determined by

$$\langle U^{-1/2}\rangle = \beta^{-1/2}(n\pi/l) \quad (l \equiv y_2 - y_1), \quad (4.2)$$

and the subscript zero implies $y = y_0 \equiv \tfrac{1}{2}(y_1 + y_2)$.

It is consistent with the Liouville approximation and the further approximation of weak shear to expand $U \equiv U_* + c_n$ about $y = y_0$. The resulting approximations are

$$U_0 \equiv U_{*0} + c_n \sim \beta(l/n\pi)^2, \quad (4.3)$$

$$\langle U^2\phi^2\rangle \sim \tfrac{1}{2}U_0^2, \quad \langle(\beta - U'')\phi^2\rangle \sim \tfrac{1}{2}\beta, \quad (4.4a,b)$$

$$\langle U^2\phi'^3\rangle \sim (2/9n\pi)\beta\begin{cases}2U_0' & (n = 1, 3, \ldots)\\ [7/4(U_0'^2/U_0) - U_0'']l & (n = 2, 4, \ldots),\end{cases} \quad (4.4c)$$

$$\langle U^2\phi'^4\rangle \sim \tfrac{3}{8}\beta^2, \quad (4.4d)$$

all within $1 + O(\epsilon)$. The corresponding approximations to k, c and a_n, as determined from (3.8b) and (3.11b), are

$$kl = \tfrac{1}{3}(2n\pi a U_0'/U_0)^{1/2}, \quad c - c_n = 8(9n\pi)^{-1}U_0', \quad (4.5a,b)$$

$$a_n/l = 16(3n\pi)^{-3}(U_0'l/U_0) \quad (n = 1,3,\ldots), \quad (4.5c)$$

wherein $\mathrm{sgn}a = \mathrm{sgn}a_n = \mathrm{sgn}U_0'$. If n is even, U_0' must be replaced by $[(7/8)(U_0'^2/U_0) - \tfrac{1}{2}U_0'']l$ in each of (4.5a,b,c). It follows from (4.5c) and the underlying assumption of weak shear that $|a_n| \ll l$, in consequence of which the implicit restriction $|a| \ll |a_n|$ may be significant (especially for even n); see (3.10) and (3.11).

The approximations (4.5a,b) agree with the corresponding approximations of Long (1964), as modified by Larsen (1965), after allowing for the fact that Long and Larsen assume that $c = 0$ and expand U about $y = y_1$.

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