that these new values are perhaps more meaningful than previously published results.

There are many areas of uncertainty remaining in the Bryson-Dittberner model. These include means to improve the parameterization of the effects of external factors such as solar variation, anthropogenic factors and volcanic influences, inclusion of more feedback mechanisms such as between albedo and temperature, latent heat and cloudiness, and cross-equatorial heat transport, other anthropogenic heat input, etc. It is our hope that further thoughtful consideration of these problems will lead to improvements in realistic, physically-based climate models. We cannot have much confidence in modeled future climates unless the model realistically simulates the past.

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Global-Scale Disturbances and Dynamic Similarity

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ABSTRACT

Terms in the linearized primitive equations for a generally baroclinic atmosphere are evaluated for their significance in maintaining balance for global-scale disturbances. For gravity waves, the linearized advection term $\mathbf{v}'\cdot\nabla$ reduces to the vertical component, but for Rossby waves, apparently both components are of primary order. Both wind shear terms are shown to be small for the Rossby case. As a by-product of the scaling developed, the traditional viscous and thermal diffusion terms are reduced to simple forms.

The disturbance energy equation is developed for the general basic state, and the influence that the approximations have on its balance is evaluated.

1. Introduction

A number of earlier papers have examined the relative importance of terms in the primitive equations for large-scale atmospheric flow. Charney (1948) addressed flows of moderately large dimension, where the ratio of the horizontal scale L to the planetary radius a was small [termed type I motions by Phillips (1963)]. Burger (1958) considered motions of planetary scale where L/a is O(1) (type II motions). Dickinson (1968) discussed yet a third scaling appropriate to motions whose zonal dimension was of the same order as the planetary radius, but whose meridional scale was smaller than a. These studies focused on the nonlinear equations, and hence, the influence of basic state gradients on linear disturbances was not explicitly evaluated.

Because of the reduced meridional scales, Charney and Dickinson were able to consider the equations in β-plane geometry. In general, when this is not the case, spherical geometry plays an important role. Lindzen (1967) showed that much of this effect can be approximated by both an equatorial and a midlatitude β-plane, but that, in general, neither will suffice alone. Longuet-Higgins (1964) demonstrated that the simple two-dimensional Rossby solutions on a sphere tend to those on a β-plane in the limit of large total horizontal wavenumber. For global-scale modes, however, the error in going to the β-plane remains significant.

When the primitive equations are linearized about

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a generally baroclinic atmosphere in spherical coordinates, additional terms are generated by nonlinear, metric and advection forms due to gradients in the basic state. The advection terms produced are of the form $\mathbf{v}'\cdot\nabla$ and arise from particle motion across surfaces of constant pressure, temperature, etc. Since the ratios of vertical to horizontal velocities and horizontal to vertical gradients are both small, it is not apparent that either term dominates over the other. Some of the advection terms may be eliminated by proper choice of coordinates. For example, pressure coordinates consolidate terms due to motion across isobaric surfaces. For a baroclinic atmosphere, however, no single choice of coordinates will eliminate all of the advection terms, since no one set of surfaces completely describes the basic state.

We shall consider linear global-scale disturbances, where the total horizontal wavenumber, nondimensionalized by the earth's radius, is of order unity. The time scale will be restricted to be of the order of a day or larger. With the exception of these scales, no assumptions are placed on the unknown disturbances, and the "traditional approximation" (Eckart, 1960) is not invoked *a priori*. The primitive equations in spherical geometry are linearized about a generally baroclinic atmosphere in zonal motion. Since there is no complete advantage to choosing any one thermodynamic variable as a vertical coordinate, we use geometric height.

2. The background state

The momentum, continuity and thermodynamic equations are

$$\frac{du}{dt} - 2\Omega(v\sin\phi - w\cos\phi) = \frac{-1}{\rho a\cos\phi}\frac{\partial p}{\partial\lambda} + \frac{uv\tan\phi}{a} - \frac{uw}{a} + \frac{1}{\rho}F_\lambda, \quad (1.1)$$

$$\frac{dv}{dt} + 2\Omega u\sin\phi = \frac{-1}{\rho a}\frac{\partial p}{\partial\phi} - \frac{u^2\tan\phi}{a} - \frac{vw}{a} + \frac{1}{\rho}F_\phi, \quad (1.2)$$

$$\frac{dw}{dt} - 2\Omega u\cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \frac{u^2+v^2}{a} + \frac{1}{\rho}F_z, \quad (1.3)$$

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0, \quad (1.4)$$

$$\frac{dp}{dt} = c^2\frac{d\rho}{dt} + (\gamma-1)[\nabla\cdot(k\nabla T) + \rho q], \quad (1.5)$$

where

$$\left.\begin{aligned}\frac{d}{dt} &= \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z}\\ \nabla\cdot\mathbf{v} &= \frac{1}{a\cos\phi}\frac{\partial u}{\partial\lambda} + \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi v) + \frac{\partial w}{\partial z}\end{aligned}\right\}, \quad (1.6)$$

ϕ, λ and z are the latitude, longitude and geometric height, respectively, c^2 is the adiabatic sound speed, and Ω, a, γ, k and R are the earth's angular speed and radius, the ratio of specific heats, the thermal conductivity and the specific gas constant, respectively. The remaining variables are in standard notation. The vector

$$\mathbf{F} = \nabla\left(\frac{\mu}{3}\nabla\cdot\mathbf{v}\right) + \nabla\cdot(\mu\nabla\mathbf{v}) \quad (1.7)$$

is the divergence of the viscous stress tensor (Aris, 1962) and μ is the coefficient of viscosity. The viscous and conduction terms have been retained in their traditional forms, however, the molecular coefficients may be replaced by their eddy counterparts to represent more realistic diffusion. The quantity q is the heating rate per unit mass.

The zero order or background state is taken to be independent of longitude and in zonal motion. Since this study is concerned primarily with disturbance quantities, as is typically done, we will assume that the basic state is prescribed, but subject to satisfaction of the adiabatic, inviscid equations. Primes indicate perturbation quantities and zero subscripts background state variables. The local angular velocity of the background state is then

$$A(\phi,z) = \frac{u_0(\phi,z)}{a\cos\phi}. \quad (2)$$

The first momentum, continuity and energy equations are satisfied identically for this background configuration. The second and third equations of motion become

$$Aa\cos\phi\sin\phi(2\Omega + A) = -\frac{1}{\rho_0 a}\frac{\partial p_0}{\partial\phi}, \quad (3)$$

$$Aa\cos^2\phi(2\Omega + A) = -\frac{1}{\rho_0}\frac{\partial p_0}{\partial z} - g. \quad (4)$$

Insertion of typical atmospheric values easily reduces (4) to the hydrostatic relation

$$-\frac{1}{\rho_0}\frac{\partial p_0}{\partial z} - g = 0. \quad (5)$$

By using (5) and the equation of state

$$p_0 = \rho_0 R T_0, \quad (6)$$

it follows that

$$p_0(z,\phi) = p_0(0,\phi)e^{-\xi(z,\phi)}, \quad \xi(z,\phi) = \int_0^z \frac{dz'}{H(z',\phi)},$$

where

$$H(z,\phi) = \frac{RT_0(z,\phi)}{g} = \frac{p_0}{\rho_0 g} \tag{7}$$

is the local pressure scale height. Now the mean sea level pressure varies by only a few percent (Charney, 1970); hence, it will be assumed that the surface pressure is independent of latitude. We define the O(1) dimensionless forms

$$\left.\begin{aligned} \tilde{H} = H/\bar{H}, \quad \tilde{\xi} = \int_0^\zeta \frac{d\zeta'}{\tilde{H}}, \\ \zeta = z/\bar{H}, \quad \tilde{A} = A/\Lambda, \end{aligned}\right\}, \tag{8}$$

where $\bar{H}$ and Λ represent mean values. Then (3) becomes

$$\eta\beta\tilde{A}\cos\phi\sin\phi(1 + \beta\tilde{A}) = \tilde{H}\frac{\partial\tilde{\xi}}{\partial\phi}, \tag{9}$$

where

$$\beta = \Lambda/2\Omega \ll 1 \tag{10}$$

is a background Rossby number, and is $O(10^{-1})$ (see Kantor and Cole, 1964), and

$$\eta = \frac{(2\Omega)^2 a^2}{g\bar{H}} \gg 1 \tag{11}$$

is Lamb's parameter (Longuet-Higgins, 1968) and is O(10) for the earth's atmosphere. Since the left hand side of (9) is O(1), $\partial\tilde{\xi}/\partial\phi$ is O(1), and hence, the ϕ coordinate is already scaled appropriately for the background state. With the neglect of terms $O(\beta)$, Eq. (3) reduces to

$$2\Omega a\cos\phi\sin\phi A = -\frac{1}{\rho_0 a}\frac{\partial p_0}{\partial\phi}, \tag{12}$$

which implies the following expression of thermal balance:

$$\frac{\partial A}{\partial z} = -\frac{g}{2\Omega a^2\cos\phi\sin\phi}\frac{1}{T_0}\frac{\partial T_0}{\partial\phi} + \frac{A}{T_0}\frac{\partial T_0}{\partial z}. \tag{13}$$

3. The linearized equations and dynamic similarity

The linearized equations resulting from (1) can be consolidated into a convenient form, if in the expression for A we replace the radial distance by the exact value, $a + z$, before differentiation, and then evaluate it at $z = 0$. It will turn out that any terms generated by this procedure are small, and hence, of no consequence in the final result. Then

$$\left.\begin{aligned} \frac{\partial u_0}{\partial\phi} &= \frac{\partial A}{\partial\phi}a\cos\phi - Aa\sin\phi \\ \frac{\partial u_0}{\partial z} &= \frac{\partial A}{\partial z}a\cos\phi + A\cos\phi \end{aligned}\right\},$$

and the perturbation equations may be written

$$\frac{Du'}{Dt} - 2(\Omega + A)[\sin\phi v' - \cos\phi w'] + \left[\frac{\partial A}{\partial\phi}v' + a\frac{\partial A}{\partial z}w'\right]\cos\phi = -\frac{1}{\rho_0 a\cos\phi}\frac{\partial p'}{\partial\lambda} + \left(\frac{F_\lambda}{\rho}\right)', \tag{14.1}$$

$$\frac{Dv'}{Dt} + 2(\Omega + A)\sin\phi u' = \frac{-1}{\rho_0 a}\frac{\partial p'}{\partial\phi} + \frac{1}{a\rho_0^2}\frac{\partial p_0}{\partial\phi}\rho' + \left(\frac{F_\phi}{\rho}\right)', \tag{14.2}$$

$$\frac{Dw'}{Dt} - 2(\Omega + A)\cos\phi u' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{1}{\rho_0^2}\frac{\partial p_0}{\partial z}\rho' + \left(\frac{F_z}{\rho}\right)', \tag{14.3}$$

$$\frac{D\rho'}{Dt} + \frac{1}{a}\frac{\partial\rho_0}{\partial\phi}v' + \frac{\partial\rho_0}{\partial z}w' + \rho_0\nabla\cdot\mathbf{v}' = 0, \tag{14.4}$$

$$\frac{Dp'}{Dt} + \frac{1}{a}\frac{\partial p_0}{\partial\phi}v' + \frac{\partial p_0}{\partial z}w' = c_0^2\left\{\frac{D\rho'}{Dt} + \frac{1}{a}\frac{\partial\rho_0}{\partial\phi}v' + \frac{\partial\rho_0}{\partial z}w'\right\} + (\gamma - 1)[\nabla\cdot(k\nabla T') + \rho_0 q'], \tag{14.5}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + A\frac{\partial}{\partial\lambda},$$

$$c_0^2 = \gamma RT_0 = \gamma gH.$$

The total angular velocity of the background state is

$$\hat{\Omega}(z,\phi) = \Omega + A(z,\phi).$$

Eqs. (14) are separable in t and λ, and hence "steady" disturbances may be assumed of the form $e^{i(m\lambda-\sigma t)}$. We will use σ and m to denote $\partial/\partial t$ and $\partial/\partial\lambda$. Then the Stokes operator becomes

$$\frac{D}{Dt} = -i\sigma + iA(z,\phi)m = -i\omega(z,\phi), \tag{15}$$

where the intrinsic frequency $\omega(z,\phi)$ has been defined.

The relative importance of the terms in (14) will be exposed by a scaling appropriate to the disturbances under consideration. With the exception of the horizontal dimensions and time, these scales are unknown, and must be determined from the equations. We will assume that the dissipative terms do not dictate the balance of the equations. Since the equations are linear, all of the scales depend proportionally on a single fundamental scale. For the forced problem, this scale may be determined directly from the inhomogeneous (forcing) terms in the equations or boundary conditions. For the homogeneous problem, the solutions are unique only up to a multiplicative constant, and hence, the fundamental scale is arbitrary.

We shall proceed with the homogeneous equations and take the fundamental scale to be that of the horizontal velocity $\bar{u}$. Results for the inhomogeneous problem follow by choosing $\bar{u}$ such that the dimensionless forcing is O(1).

We define

$$\left.\begin{aligned} \tilde{u} &= u'/\bar{u} = O(1) \\ \tilde{v} &= v'/\bar{u} = O(1) \\ \tilde{w} &= w'/\bar{w} \leqslant O(1) \\ \tilde{\omega} &= \omega/2\Omega \leqslant O(1) \end{aligned}\right\},$$

where tildes indicate nondimensional quantities. Just as the zonal wavenumber m represents the inverse longitudinal scale $\partial/\partial\lambda$, we let ν denote the inverse meridional scale $\partial/\partial\phi$. For this discussion, the zonal and meridional scales will be restricted to be O(1), but practically, we will allow $(m^2 + \nu^2)^{1/2} \leqslant 4$.

Eq. (14.1) becomes

$$\underbrace{-i\tilde{\omega}\tilde{u}}_{O(\tilde{\omega}) \leqslant O(1)} - \underbrace{(1+2\beta\tilde{A})\left[\sin\phi\tilde{v} - \left(\frac{\bar{w}}{\bar{u}}\right)\cos\phi\tilde{w}\right]}_{O(1)} = -\frac{img\epsilon}{(2\Omega)\bar{u}p_0}\frac{\tilde{H}}{\cos\phi}p'$$

$$-\left[\underbrace{\beta\frac{\partial\tilde{A}}{\partial\phi}\tilde{v}}_{O(\beta)<O(1)} + \underbrace{\frac{\beta}{\epsilon}\left(\frac{\bar{w}}{\bar{u}}\right)\frac{\partial\tilde{A}}{\partial\zeta}\tilde{w}}_{O\left(\frac{\beta}{\epsilon}\left(\frac{\bar{w}}{\bar{u}}\right)\right)}\right]\cos\phi + \frac{1}{2\Omega\bar{u}}\left(\frac{F_\lambda}{\rho}\right)', \quad (16)$$

where

$$\epsilon = \bar{H}/a \ll 1$$

is the atmosphere's shallowness and is O(10^{-3}).

The pressure term in (16) must balance the largest of the acceleration terms on the left-hand side. Then $g\epsilon p'/(2\Omega\bar{u}p_0)$ is O(1), and we take

$$\tilde{p} = \frac{g\epsilon}{2\Omega\bar{u}p_0(\zeta,\phi)}p' = O(1). \quad (17)$$

Eq. (14.3) becomes

$$\underbrace{-i\left(\frac{\bar{w}}{\bar{u}}\right)\tilde{\omega}\tilde{w}}_{O(\tilde{\omega}\cdot\bar{w}/\bar{u}) \leqslant O(1)} - \underbrace{(1+2\beta\tilde{A})\cos\phi\tilde{u}}_{O(1)} = \underbrace{\frac{1}{\epsilon}\left(\tilde{p} - \tilde{H}\frac{\partial\tilde{p}}{\partial\zeta}\right)}_{O(1/\epsilon)} - \frac{g^2\bar{H}}{(2\Omega)\bar{u}p_0}\tilde{H}\rho' + \frac{1}{2\Omega\bar{u}}\left(\frac{F_z}{\rho}\right)'.$$

The first term on the right-hand side is O($1/\epsilon$) and cannot be balanced by either term on the left-hand side. Therefore $g^2\bar{H}\rho'/2\Omega\bar{u}p_0$ is O($1/\epsilon$) and we define

$$\tilde{\rho} = \frac{g^2\bar{H}\epsilon}{2\Omega\bar{u}p_0}\rho' = O(1). \quad (18)$$

The third equation of motion then reduces to the hydrostatic relation to O(ϵ). The continuity and thermodynamic equations become

$$\underbrace{-i\eta\tilde{\omega}\tilde{p}}_{O(\eta\tilde{\omega})} - \underbrace{\frac{1}{\tilde{H}^2}\left(\tilde{H}\frac{\partial\tilde{\xi}}{\partial\phi} + \frac{\partial\tilde{H}}{\partial\phi}\right)\tilde{v}}_{O(1)} - \underbrace{\frac{1}{\epsilon}\left(\frac{\bar{w}}{\bar{u}}\right)\frac{1}{\tilde{H}^2}\left(\frac{\partial\tilde{H}}{\partial\zeta}+1\right)\tilde{w}}_{O[\epsilon^{-1}(\bar{w}/\bar{u})]}$$

$$+\frac{1}{\tilde{H}}\left[\underbrace{\frac{im}{\cos\phi}\tilde{u}}_{m} + \underbrace{\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\tilde{v})}_{\nu} + \underbrace{\frac{1}{\epsilon}\left(\frac{\bar{w}}{\bar{u}}\right)\frac{\partial\tilde{w}}{\partial\zeta}}_{O[\epsilon^{-1}(\bar{w}/\bar{u})]}\right] = 0, \quad (19)$$

$$\underbrace{-i\eta\tilde{\omega}\tilde{p}}_{O(\eta\tilde{\omega})} + \underbrace{\left[\frac{\gamma}{\tilde{H}}\frac{\partial\tilde{H}}{\partial\phi} + (\gamma-1)\frac{\partial\tilde{\xi}}{\partial\phi}\right]}_{O(1)}\tilde{v} + \frac{1}{\epsilon}\left(\frac{\tilde{w}}{\tilde{u}}\right)\frac{1}{\tilde{H}}\underbrace{\left[\gamma\frac{\partial\tilde{H}}{\partial\zeta} + (\gamma-1)\right]}_{O[\epsilon^{-1}(\tilde{w}/\tilde{u})]}\tilde{w}$$

$$= \underbrace{-i\eta\tilde{\omega}\gamma\tilde{H}\tilde{\rho}}_{O(\eta\tilde{\omega})} + \frac{a(\gamma-1)}{p_0\tilde{u}}\nabla\cdot(k\nabla T'). \quad (20)$$

The balance maintained in (19) and (20) differs for two frequency regimes. Once the intrinsic frequency $\tilde{\omega} = \tilde{\sigma} - \beta m\tilde{A}$ is specified, the vertical velocity scale can be obtained and the relative importance of the remaining terms evaluated.

a. Gravity regime $\tilde{\sigma} = O(1)$

It follows that $\tilde{\omega}$ is O(1). In order that (19) be balanced, the order of the terms involving $\tilde{w}$ can be no larger than the largest of those remaining. Thus $O[\epsilon^{-1}(\tilde{w}/\tilde{u})] \leqslant \eta$, and we take

$$\frac{\tilde{w}}{\tilde{u}} = \eta\epsilon \approx 10^{-2}, \quad (21.1)$$

with $\tilde{w} \leqslant O(1)$. Then the meridional advection terms in (19) and (20) are $O(1/\eta)$ relative to the largest terms and may be neglected. The meridional shear term in (16) is $O(\beta)$, but the vertical shear term remains O(1).

b. Rossby regime $\tilde{\sigma} \leqslant O(1/\eta)$

Here $\eta\tilde{\omega}$ is O(1). Since the order of the term involving $\tilde{w}$ in (20) can be no larger than the others, we take

$$(\tilde{w}/\tilde{u}) = \epsilon \approx 10^{-3}. \quad (21.2)$$

This is equivalent to expressions derived by Charney and Burger.

With the neglect of terms $O(\epsilon)$ and $O(1/\eta)$, the diffusion forms reduce to the vertical derivatives of horizontal velocity and temperature. The approximate system of equations to order ϵ, β and $1/\eta$ in dimensional form is then

$$\frac{D\mathbf{v}_h'}{Dt} + 2\hat{\Omega}\sin\phi\mathbf{k}\times\mathbf{v}_h' = -\frac{1}{\rho_0}\nabla p' + \frac{\nabla p_0}{\rho_0^2}\rho' - \delta_G\mathbf{v}'\cdot\hat{\nabla}\mathbf{v}_0 + \frac{1}{\rho_0^2}\frac{\partial}{\partial z}\left(\mu\frac{\partial\mathbf{v}_h'}{\partial z}\right), \quad (22.1)$$

$$\frac{D\rho'}{Dt} + \mathbf{v}'\cdot\hat{\nabla}\rho_0 = -\rho_0\nabla\cdot\mathbf{v}', \quad (22.2)$$

$$\frac{Dp'}{Dt} + \mathbf{v}'\cdot\hat{\nabla}p_0 = c_0^2\left(\frac{D\rho'}{Dt} + \mathbf{v}'\cdot\hat{\nabla}\rho_0\right) + (\gamma-1)\left[\frac{\partial}{\partial z}\left(k\frac{\partial T'}{\partial z}\right) + \rho_0 q'\right], \quad (22.3)$$

$$\hat{\nabla} = \delta_R\frac{1}{a}\frac{\partial}{\partial\phi}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}, \quad (22.4)$$

where $\delta_G(\delta_R)$ is one for the gravity (Rossby) regime and zero otherwise, and subscript h denotes horizontal component. The advection operator $\mathbf{v}'\cdot\nabla$ has reduced to $\mathbf{v}'\cdot\hat{\nabla}$. Strictly speaking, we have replaced terms $O(\beta)$ or $O(1/\eta)$ by δ_R, $\delta_G = 0$. For the Rossby case, both background shear terms are $O(\beta)$. The equivalent of the vertical shear term was shown to be small by Charney for smaller scale motions.

If we allow η to vanish, the equations reduce to a special subset for the motionless, isothermal case. Eq. (20) implies that w is zero. The continuity equation (19) then reduces to a statement of horizontal nondivergence, and the solutions may be represented by a streamfunction. The horizontal momentum equations are then equivalent to a statement of conservation of absolute vorticity. The solutions of this set were first obtained on the sphere by Haurwitz (1940) and are named after him. They form the limiting (divergenceless) solutions of the "second class" of Laplace's tidal equation, for the Lamb's parameter becoming small relative to the frequency (Longuet-Higgins, 1968). Here the factor $\eta\tilde{\omega}$ is a measure of the horizontal divergence, which Burger first demonstrated was not negligible for motions of planetary scale. Eqs. (19) and (20) confirm that terms $O(\eta\tilde{\omega})$ are significant for realistic values of η.

4. Energy balance

To gage the effect of the approximations developed in the previous section, we shall examine the balance of disturbance energy resulting from (22). These may be utilized with the approximate form of the Second Law

$$\frac{Ds'}{Dt} + \mathbf{v}'\cdot\hat{\nabla}s_0 = \frac{1}{T_0}\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(k\frac{\partial T'}{\partial z}\right) + q'\right], \quad (23)$$

to yield

$$\frac{D}{Dt}\left[\frac{\rho_0|\mathbf{v}_h'|^2}{2} + \frac{p'^2}{2\gamma p_0} + \frac{\rho_0 g}{2\theta_0(\partial\theta_0/\partial z)}\theta'^2\right]$$
$$= -\nabla\cdot(p'\mathbf{v}') + \delta_R\rho_0|\nabla p_0 \times \nabla(1/\rho_0)|v'\theta'$$
$$- \delta_G\rho_0 u'(\mathbf{v}'\cdot\hat{\nabla}\mathbf{v}_0) + \mathbf{v}_h'\cdot\frac{\partial}{\partial z}\left(\mu\frac{\partial\mathbf{v}_h'}{\partial z}\right) + \frac{1}{c_pT_0}$$
$$\times\left(p' + \frac{\rho_0 g}{\partial\theta_0/\partial z}\theta'\right)\times\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(k\frac{\partial T'}{\partial z}\right) + q'\right].$$

Here s is the entropy and θ = constant $\times \exp(s/c_p)$ is the potential temperature. The left-hand side of this relation is the time rate of change of total disturbance energy: kinetic, plus elastic and buoyancy potential energies.

The energy flux is given by $\mathbf{I} = -\boldsymbol{\tau}'\cdot\mathbf{v}'$, where $\boldsymbol{\tau}$ is the total stress tensor. This is not equal to $p'v'$ for $\mu \neq 0$. Since the right-hand side of the momentum equation has the form of the divergence of $\boldsymbol{\tau}$, it follows from (22.1) and the symmetry of $\boldsymbol{\tau}$ that

$$\boldsymbol{\tau}' = \begin{bmatrix} -p & 0 & \mu\dfrac{\partial u'}{\partial z} \\ 0 & -p & \mu\dfrac{\partial v'}{\partial z} \\ \mu\dfrac{\partial u'}{\partial z} & \mu\dfrac{\partial v'}{\partial z} & -p \end{bmatrix}. \tag{24}$$

The energy flux is then

$$\mathbf{I} = p'\mathbf{v}' - \mathbf{v}_h'\cdot\mu\frac{\partial\mathbf{v}_h'}{\partial z}\mathbf{k} \tag{25}$$

and for the energy balance, we have

$$\frac{D}{Dt}\left[\frac{\rho_0|\mathbf{v}_h'|^2}{2} + \frac{p'^2}{2\gamma p_0} + \frac{\rho_0 g}{2\theta_0(\partial\theta_0/\partial z)}\theta'^2\right] + \nabla\cdot\mathbf{I}$$
$$= \delta_R\frac{\rho_0}{(\partial\theta_0/\partial z)}|\nabla p_0 \times \nabla(1/\rho_0)|v'\theta'$$
$$- \delta_G\rho_0 u'(\mathbf{v}'\cdot\hat{\nabla}\mathbf{v}_0) - \mu\left|\frac{\partial\mathbf{v}_h'}{\partial z}\right|^2$$
$$+ \frac{1}{c_pT_0}\left[p' + \frac{\rho_0 g}{(\partial\theta_0/\partial z)}\theta'\right]$$
$$\times\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(k\frac{\partial T'}{\partial z}\right) + q'\right]. \tag{26}$$

Terms on the right-hand side of (26) appear as local sources and sinks of disturbance energy. The first two arise from the baroclinic nature of the basic state. The first of these, which is $O(1/\eta)$ for gravity waves, is directly proportional to the basic state baroclinicity and the heat flux along the meridional temperature gradient. The second is proportional to the disturbance momentum flux along the background shear, and is usually associated with baroclinic and barotropic instability. This term is $O(\beta)$ for Rossby waves and reduces to the vertical shear term for the gravity class. The third term is the viscous dissipation rate and can only remove energy. Energy can be either introduced or dissipated by the last term, depending on the relative sizes of conduction and heating.

5. Summary and conclusions

Terms generated by spherical metric forms in the linearization process may be combined with Coriolis terms to form the total angular velocity of the basic state. This reduces to the planetary angular velocity with the neglect of terms $O(2\beta)$.

For global-scale gravity waves, the linearized advection operator $\mathbf{v}'\cdot\nabla$ reduces to the vertical component with the neglect of terms $O(1/\eta)$. Thus, the gravity motions are insensitive to the horizontal gradients of the basic state.

For global Rossby waves, both the vertical and the horizontal components are of primary order. In particular, meridional gradients in temperature, etc., are of comparable importance with the vertical gradients in balancing the equations. Both background shear terms, however, may be neglected with error $O(1/\eta)$.

The implications of these approximations to the energy balance involve the neglect of source terms due to meridional gradients for gravity waves, and the neglect of shear source terms for Rossby waves.

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