Radiative Transfer through Media with Uncertain or Variable Parameters

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ABSTRACT

While it is now possible to compute the optical transmission, reflection and absorption of a homogeneous horizontal layer of known parameters to great accuracy, the input parameters (optical depth, scattering phase function and single-scattering albedo) are in general neither precisely known nor exactly constant over a layer. In this paper several simple representations of the distributions of input parameters are used to compute the mean values and standard deviations of layer transmission, reflection and absorption. The effects of variability in input parameters depend on the mean-layer properties; under certain conditions this variability induces errors on the order of 5% in the derived optical properties. This magnitude is comparable to the differences between those obtained by the Eddington, delta-Eddington, and discrete ordinate four-stream approximations, and a more precise 20-stream doubling method.

1. Introduction

Discussions of radiative transfer have generally dealt with the description of methods employed to calculate the transmission, reflection and absorption of a horizontal layer homogeneous in its optical properties. Real planetary atmospheres are composed of materials characterized by a range of optical properties, whose distributions are often poorly known. For calculation, each of the variable optical parameters in a layer is usually represented by a single number, its mean value. This paper will investigate the effects on transmission, reflection and absorption resulting from variability (or uncertainty) in input optical parameters.

The methods and techniques employed in radiative transfer are many and they vary in their application and validity. Excellent reviews of the field may be found in articles by Hunt (1971) and by Irvine (1975). The Radiation Commission of the International Association of Meteorology and Atmospheric Physics (IAMAP) has produced a volume edited by Lenoble (1977) which includes a discussion and references to virtually every method of radiative transfer yet conceived. It also contains a computed comparison between values produced by some of the more accurate methods.

The consideration of inhomogeneities in radiative transfer calculations has been limited to a relatively small number of papers. Theoretical treatments of atmospheres in which the single-scattering albedo decreases exponentially with optical depth were performed by Chamberlain and McElroy (1966) and Kanal (1973). Fymat and Abhyankar (1969, 1970) in a series of four papers considered in considerable detail a perturbation method in which they assume the single-scattering albedo differs from a constant value by a small amount throughout the atmosphere. However, they apply their method only to a semi-infinite atmosphere, and it is difficult to use their results.

Van Blerkom (1971) has studied the case in which horizontal striations are superimposed, much like a medieval battlement, on the top of a homogeneous cloud layer. He employed a Monte Carlo method with isotropic scattering and showed a decrease in reflection relative to a plane-parallel cloud layer, with a large dependence on the incident angle and the degree of striation.

An atmosphere consisting of three homogeneous layers (clear, cloud and aerosol) over snow or ocean surfaces was considered by Shettle and Weinman (1970). They used the Eddington approximation to calculate the irradiance within the layers for three wavelengths. Wiscombe (1977) derived the equations for a multi-level delta-Eddington but only considered cases where the layers are all homogeneous.

By dividing the atmosphere into 10 layers and specifying a vertical distribution of aerosols, Liou (1975) studied the effects of a vertically "inhomogeneous" atmosphere. He found a decrease in reflection and an increase in transmission for his inhomogeneous atmosphere, compared to a homogeneous atmosphere. The magnitude of the effect was on the order of 10–15% change.

Thus in all these cases inhomogeneities in atmospheric optical properties are assumed to be distributed in some specified fashion, usually in superimposed horizontal, uniform slabs. In the real

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atmosphere, particularly in the presence of clouds, this picture is far from realistic; random, highly nonuniform distributions of "blobby" elements contribute to the measured average fluxes. To solve a truly realistic problem would require extensive analytic and/or computational efforts, and it is not clear *a priori* at what degree of nonuniformity the results to be derived differ significantly from those derived using standard methods. It is not even clear *a priori* how to *define* the nonuniformity or the significance of the differences.

In this paper we compute the average albedo, transmission and absorption in visible wavelengths of horizontal layers in which the single-scattering albedo $\tilde{\omega}$, optical depth τ and asymmetry factor g are considered to be random variables with known probability distributions of simple analytic form. In Sections 2a and 2b we describe the problem in general terms and in Section 2c we show the computational methods we have used. Interpretation and use of our results are considered from several viewpoints in Section 3.

2. Description of the method

The intensity $I_i(z)$ of radiation propagating in a direction making an angle θ_i with the vertical at a depth z within a horizontally infinite layer is described by an equation of the form

$$\frac{dI_i}{dz}(z) = M_{ij}I_j(z) + F_i(z), \quad (1)$$

where $F_i(z)$ is the contribution due to a direct beam, if any.

The matrix M_{ij} is n dimensional, where n is as large as is computationally desirable, and its elements are functions of the moments of the scattering phase function, $p(\theta)$. The particular forms of the functions vary with the method and are well explained in the references cited. In general, the M_{ij} and the F_i are assumed constant over a layer, and it is this assumption we examine here.

We wish to choose simple models for the variability in the M_{ij} and F_i and from these to compute the mean values and standard deviations of the reflection, transmission and absorption of the layer. Three-dimensional representations of the variability are extremely complicated, and therefore we have restricted this study to one-dimensional models, i.e., we examine variability which occurs either solely in the horizontal or solely in the vertical. We now examine each of these in turn.

a. Horizontal variability

We consider here neighboring columns, of differing optical properties, where the horizontal spatial scales are much larger than the thicknesses. This system may be visualized as a patchy cloud layer where the cloud size and spacing between the clouds is larger than the cloud thickness. Each photon passing through the layer passes through a single homogeneous column but the optical properties of the columns are different. The variations between columns may be viewed as either temporal or spatial, as long as the incident photon experiences only one set of optical properties on its path to the detector.

In this model, then, each patch is an "experiment" and the averaging over patches corresponds to a statistical averaging over many experiments with identical boundary conditions. The distributions of optical parameters among the patches are described in detail in Section 3.

b. Vertical variability

In this case we assume variation of radiative parameters occurs only in the vertical or z direction. In principle there is an analytic method for calculating the moments of $I_i(z)$ if we assume the radiative transfer process is Markov, i.e., the future of any photon at depth z is independent of its trajectory from 0 to z.

This derivation of the moments is outlined in the Appendix. We have not yet followed the derivation with numerical computation, although it would be straightforward to do so. Instead, we consider a horizontally infinite layer of total vertical depth τ, divided into N horizontal sublayers. The optical properties of each sublayer are constant throughout the layer but the sublayers differ among themselves. The arrangement of the layers also is random.

c. Computational methods

The matrix elements M_{ij} in Eq. (1) can be described in terms of three functions: the optical depth τ, the single-scattering albedo $\tilde{\omega}$, and the scattering phase function $p(\theta)$. The phase function employed throughout this paper was first described by Henyey and Greenstein (1941) for scattering by interstellar dust and is given by

$$p(\theta) = \frac{\tilde{\omega}(1 - g^2)}{(1 + g^2 - 2g\cos\theta)^{3/2}}, \quad (2)$$

where $\tilde{\omega}$ is the single-scattering albedo, g the asymmetry factor and θ the scattering angle.

This phase function has become a much used analytic representation of actual phase functions (van de Hulst and Grossman, 1968; Hansen, 1969). It is mathematically well-behaved and can be made to represent very different scattering media merely by altering g. For isotropic scattering $g = 0$, while for purely forward or backward scattering $g = +1$ or $g = -1$, respectively.

One of the goals of this work was to compare variability in computed layer properties due to differences in computational method for the same input parameters with the variability induced by fluctuations in layer parameters. Therefore, the transfer of radiation through deterministic layers with the properties given above was first performed using one "precise" method and three approximate methods. The precise method employed was a 20-stream matrix formulation of a doubling method as described by Twomey *et al.* (1966). A doubling method calculates the properties of a very thin layer in which single scattering may be assumed, and by combining two thin layers, calculates the properties for a layer of optical thickness twice that of the original layer. This doubling of layers is continued until the layer of desired optical depth is reached. Further discussions of the doubling or adding method may be found in van de Hulst and Grossman (1968) and Hansen (1969).

The approximate methods consisted of two two-stream models, the Eddington and delta-Eddington approximations; and a four-stream solution of the discrete ordinate method. The Eddington approximation assumes that the radiance $I(\tau,\mu)$ can be given by

$$I(\tau,\mu) = I_0(\tau) + I_1(\tau)\mu, \quad (3)$$

where μ is the cosine of the angle between the propagation direction and the vertical. We used the formulation of Shettle and Weinman (1970), though other two-stream formulations may be found in Chandrasekhar (1950), Sagan and Pollack (1967), Huang (1968), Liou (1974), Lenoble (1975) and Coakley and Chýlek (1975).

A simple yet computationally fast modification of the Eddington approximation, the delta-Eddington, results in a significant increase in accuracy (Joseph *et al.*, 1976). By expanding the phase function into a Dirac delta-function forward-scatter peak plus a two-term expansion of the remaining phase function, Joseph *et al.* (1976) have derived a set of transformations which when applied to the Eddington approximation result in improved accuracy. These transformations for the case of the Henyey-Greenstein phase function are given as

$$\left.\begin{aligned} g' &= \frac{g}{1+g}, \quad \tilde{\omega}' = \frac{(1-g^2)\tilde{\omega}}{(1-g^2\tilde{\omega})} \\ \tau' &= (1-\tilde{\omega}g^2)\tau \end{aligned}\right\}, \quad (4)$$

where the primes are the transformed quantities. A discussion of the accuracy of the Eddington and delta-Eddington approximations may be found in Wiscombe and Joseph (1977).

The four-stream approximation which we employed is based on the discrete ordinate method originally introduced by Chandrasekhar (1950). Liou (1973) presents a theoretical discussion of the method and helpful numerical procedures, and he compares the discrete ordinate method with doubling. In a later paper Liou (1974) derives analytic solutions for the two- and four-stream cases. It was this four-stream solution that we employed for this study. However, if the four-stream solution is applied exactly as described in the 1974 paper, the results become unstable for $\tilde{\omega} > 0.99$. When an iterative scheme for the calculation of Liou's W functions is employed, as described in Section 3c of his 1973 paper, the results become stable and agree with those obtained by doubling.

The choice of approximations is often dictated by the need for computational rapidity. For a single value of incident angle, optical depth, asymmetry factor and single-scattering albedo, the times required were roughly in the ratio $1:1:20:10^4$ for the Eddington, delta-Eddington, four-stream and doubling methods, respectively. All computations were performed on a PRIME 400 computer. Of course, the doubling method produced angular dependent flux data for approximately 15 intermediate optical depths. Much of this information is superfluous but cannot be avoided in order to reach the required optical depth. The results of the three approximations were compared with the doubling method over a wide range of single-scattering albedos and optical depths. In most cases the four-stream method produces slightly superior accuracy to the two-stream approximations. However, for this study, the factor of 20 in computation time saved by the delta-Eddington was judged more valuable than the greater accuracy of the four-stream method. Therefore, we have used only the delta-Eddington method to evaluate the effects of variability in input parameters.

We assume in each calculation that two of the input parameters ($\tilde{\omega}$, τ and g) are constant and the third is variable, with a modified Gaussian probability density function $p_\sigma(\xi)$:

$$p_\sigma(\xi) = \sigma^{-1}(2\pi)^{-1/2}\exp\{-\tfrac{1}{2}[(\xi-\xi_0)/\sigma]^2\}, \quad (5)$$

where

$\xi = \xi_1 = \ln[(1+g)/(1-g)]$ for the distribution in g,

$\xi = \xi_2 = \ln[(1-\tilde{\omega})/\tilde{\omega}]$ for the distribution in $\tilde{\omega}$,

$\xi = \xi_3 = \ln\tau$ for the distribution in τ,

and ξ_0 is the peak value of the distribution and σ^2 its variance. These ξ transformations allow the distributions to fall to zero at the end points of the ranges of single-scattering albedo ($\tilde{\omega} = 0$ and $\tilde{\omega} = 1$) and asymmetry factor ($g = -1$ and $g = +1$). The peak value (ξ_0) of the g and $\tilde{\omega}$ distributions corresponding to given mean values ($\bar{g}$ and $\bar{\tilde{\omega}}$ were calculated numerically through the use of a "look-up" table. The peak value of the distribution in ξ_3 was

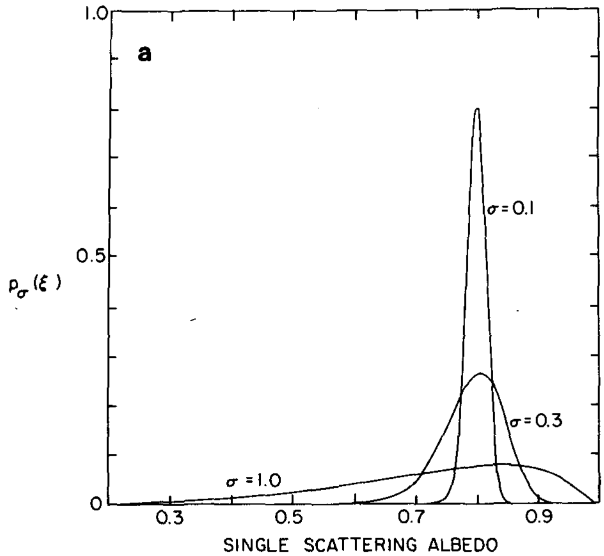

FIG. 1a. Three distributions $p_\sigma(\xi)$ of single-scattering albedo having the same mean value $\bar{\omega}_0 = 0.80$ (case I). The value of σ determines the width of the distribution.

calculated analytically from the mean by the formula $\xi_0 = \bar{\xi}_3 e^{-\sigma^2/2}$. Three values of σ (0.1, 0.3, 1.0) were chosen to define narrow, medium and broad distributions. Examples of the shape of these distributions are shown in Figs. 1a and 1b for distributions in single-scattering albedo. The density $p_\sigma(\xi)$ defines the distribution of the input variable (or transformed variable ξ) over horizontal distributions of vertically uniform patches in the first model and over vertical distributions of horizontally uniform sublayers in the second. These configurations are shown schematically in Fig. 2.

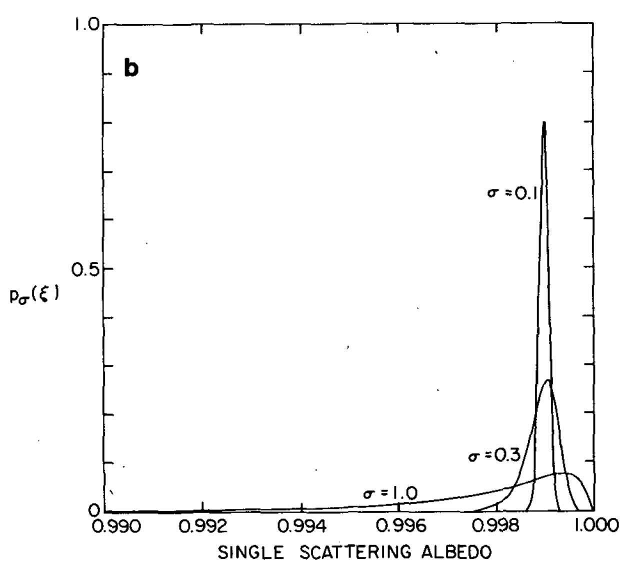

FIG. 1b. As in 1a except $\bar{\omega}_0 = 0.999$ (case III).

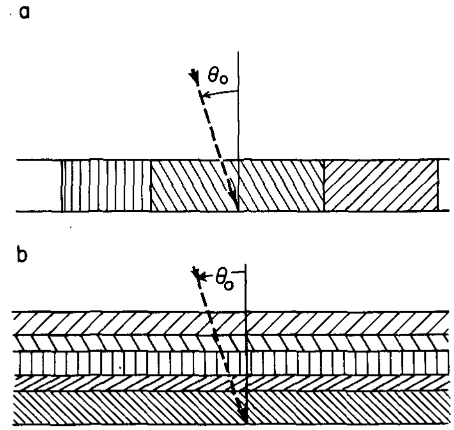

FIG. 2. Schematic diagram of layer geometries employed for this study. The incident beam is represented by the dotted line. (a) Vertically uniform, horizontally inhomogeneous layer; (b) horizontally uniform, vertically inhomogeneous layer (only five sublayers shown for simplicity).

In the model investigating horizontal variability, the layers' optical properties are obtained by calculating the patches' optical properties as functions of ξ, multiplying by $p_\sigma(\xi)$ and then summing over ξ. For example, if $T(\xi)$ is the transmission of a layer in which the value of the transformed optical parameter is between $\xi_{\min}$ and $\xi_{\max}$, then the average transmission $\bar{T}$ is

$$\bar{T} = \frac{\sum_{\xi=\xi_{\min}}^{\xi=\xi_{\max}} T(\xi) p_\sigma(\xi) d\xi}{\sum_{\xi=\xi_{\min}}^{\xi=\xi_{\max}} p_\sigma(\xi) d\xi} . \quad (6)$$

The summation limits were chosen to be $\xi_{\min} = \xi_0 - 5\sigma$ and $\xi_{\max} = \xi_0 + 5\sigma$, which encompass more than 0.9999 of the probability function as given by Eq. (5).

For the vertical variability model a set of random numbers distributed with a density given by Eq. (5) was constructed for each calculation. The total layer was divided into ten sublayers, whose properties were then assigned by a Monte Carlo technique using a random number generator. The system optical properties were then calculated using a 10-layer version of a multi-level delta-Eddington program developed by Wiscombe (1977). This procedure was repeated 100 times and the mean values and standard deviations of the layer properties computed for these 100 trials.

The calculations were performed for pairs of single-scattering albedos and asymmetry factors corresponding to three distinct physical situations.

The cases considered were the following:

Case I: Martian dust ($\bar{\omega} = 0.8$, $g = 0.75$).

This is that test case used by Liou (1973) with

TABLE 1. Comparison of the transmission as calculated by the Eddington, delta-Eddington, four-stream and doubling methods (column 0.0) versus the transmission as calculated by the delta-Eddington for narrow, medium and broad distributions in optical depth (σ = 0.1, 0.3, 1.0). Case I, $\tilde{\omega}$ = 8, g = 0.75. Results are shown for three zenith angles (μ = 0.95, 0.55, 0.15).

μ_0	0.15				0.55				0.95			
σ	0.0	0.1	0.3	1.0	0.0	0.1	0.3	1.0	0.0	0.1	0.3	1.0
$\bar{\tau}$ = 0.25												
EDD	0.54170				0.85661				0.94805			
DELTA	0.59647	0.59749	0.60555	0.68038	0.86860	0.86869	0.86949	0.88070	0.93019	0.93020	0.93029	0.93245
ST DEV		0.02600	0.07548	0.19414		0.01223	0.03645	0.11533		0.00713	0.02108	0.07746
FOUR	0.55417				0.87765				0.92665			
DBL	0.53230				0.86417				0.93245			
$\bar{\tau}$ = 1.00												
EDD	0.31126				0.56295				0.75547			
DELTA	0.26523	0.26654	0.27712	0.39364	0.57732	0.57821	0.58533	0.65866	0.73543	0.73568	0.73787	0.77040
ST DEV		0.01790	0.05629	0.20699		0.03073	0.08964	0.22879		0.02421	0.07193	0.19780
FOUR	0.25974				0.56254				0.73279			
DBL	0.23770				0.54996				0.73648			
$\bar{\tau}$ = 4.00												
EDD	0.07119				0.12551				0.20883			
DELTA	0.05742	0.05852	0.06699	0.15452	0.12447	0.12698	0.14637	0.31514	0.24151	0.24432	0.26579	0.43510
ST DEV		0.01128	0.03427	0.14040		0.02492	0.07643	0.25104		0.03865	0.11151	0.28346
FOUR	0.05005				0.11761				0.23945			
DBL	0.05157				0.11841				0.23882			
$\bar{\tau}$ = 16.00												
EDD	0.00020				0.00035				0.00061			
DELTA	0.00016	0.00021	0.00093	0.03233	0.00034	0.00046	0.00198	0.07015	0.00111	0.00145	0.00547	0.11731
ST DEV		0.00017	0.00202	0.06040		0.00037	0.00432	0.12861		0.00111	0.01067	0.18250
FOUR	0.00030				0.00070				0.00152			
DBL	0.00031				0.00069				0.00157			

parameters closest to those currently proposed for Martian dust ($\tilde{\omega} \approx 0.86$, $g \approx 0.79$; Pollack *et al.*, 1979).

Case II: Urban aerosol ($\tilde{\omega} = 0.62$, $g = 0.75$). These values are representative of those measured in an urban environment of the eastern United States (Weiss *et al.*, 1976).

Case III: Water clouds ($\tilde{\omega} = 0.999$, $g = 0.844$).

For each case, the computations were repeated for optical depths τ of 0.25, 1.0, 4.0 and 16.0 for cases I and II, and τ = 4.0, 16.0, 32.0 and 64.0 for case III. Incident beam geometries ranging from oblique to nearly vertical were considered as the cosine μ_0 of the incident angle θ_0 was taken to be 0.15, 0.55 and 0.95. In all cases the surface albedo was taken to be zero.

3. Results

a. Horizontal variability

In Table 1 we show an example of the results we obtained from comparison of the four computational methods (using the mean values of the input parameters) and the results of horizontal variability. The table shows computed transmission of a layer with $\tilde{\omega} = 0.8$ and $g = 0.75$ for distributions in optical depth at four mean optical depths $\bar{\tau}$; σ is the width of the distributions of $\ln\tau$. For $\sigma = 0$ (e.g., the non-random, deterministic limit) we compare the four computational methods, labeled EDD (Eddington), DELTA (delta-Eddington), FOUR (Liou four-stream) and DBL (doubling method). It may be seen, for example, that at $\bar{\tau} = 1$, $\mu_0 = 0.15$, the transmission calculated by the Eddington method is 0.3113 and that calculated by the more accurate doubling method is 0.23770. That is, for this example the Eddington method was high by 31%. The other rows labeled DELTA show computed mean transmission for three values (σ = 0.1, 0.3, 1.0) of the width of the distribution in τ. The lines labeled ST DEV show the standard deviation of the computed transmission, $\sigma_T = (\bar{T^2} - (\bar{T})^2)^{1/2}$. For $\bar{\tau} = 1$ again it may be seen that at $\sigma = 0.3$ the computed mean transmission is 0.27712 and the standard deviation $\sigma_T = 0.056$, or 20%. The extremely large relative changes in transmission at large optical depths (over 130% at $\bar{\tau} = 16$, $\sigma = 0.3$) reflect the fact that for these cases a few cells with slightly smaller optical depths transmit substantially more radiation than the remainder of the sky.

Table 2 shows the analogous effects on transmission of variability in single-scattering albedo. As can be seen in Table 2, the tendency, particularly at larger values of τ, is for an increase in transmission even for a narrow distribution ($\sigma = 0.1$) of single-

TABLE 2. As in Table 1 except for a distribution in single-scattering albedo.

μ_0	0.15				0.55				0.95			
σ	0.0	0.1	0.3	1.0	0.0	0.1	0.3	1.0	0.0	0.1	0.3	1.0
$\tau = 0.25$												
EDD	0.54170				0.85661				0.94805			
DELTA	0.59647	0.59656	0.59735	0.60436	0.86860	0.86862	0.86876	0.87008	0.93019	0.93020	0.93025	0.93096
ST DEV		0.01209	0.03551	0.10189		0.00578	0.01635	0.04851		0.00394	0.01102	0.03252
FOUR	0.55417				0.87765				0.92665			
DBL	0.53230				0.86417				0.93245			
$\tau = 1.00$												
EDD	0.31126				0.56295				0.75547			
DELTA	0.26523	0.26563	0.26871	0.29429	0.57732	0.57753	0.57913	0.59307	0.73543	0.73555	0.73652	0.74513
ST DEV		0.01504	0.04397	0.11933		0.01499	0.04396	0.12305		0.01207	0.03547	0.10097
FOUR	0.25974				0.56254				0.73279			
DBL	0.23770				0.54996				0.73648			
$\tau = 4.00$												
EDD	0.07119				0.12551				0.20883			
DELTA	0.05742	0.05798	0.06228	0.09791	0.12447	0.12523	0.13107	0.17897	0.24151	0.24227	0.24815	0.29655
ST DEV		0.00794	0.02412	0.07808		0.01341	0.03996	0.11978		0.01740	0.05141	0.14754
FOUR	0.05005				0.11761				0.23945			
DBL	0.05157				0.11841				0.23882			
$\tau = 16.00$												
EDD	0.00020				0.00035				0.00061			
DELTA	0.00016	0.00018	0.00036	0.00551	0.00034	0.00038	0.00070	0.00895	0.00111	0.00118	0.00179	0.01421
ST DEV		0.00008	0.00052	0.01216		0.00016	0.00094	0.01889		0.00039	0.00184	0.02710
FOUR	0.00030				0.00070				0.00152			
DBL	0.00031				0.00069				0.00157			

scattering albedo. The transmission as calculated using the mean value of the single-scattering distribution ($\bar{\tilde{\omega}}$) is thus seen to underestimate the transmission of an atmosphere having a distribution of single-scattering albedos. One must caution the potential user of this table that even though the percentage change in transmission is large for thick atmospheres, the actual transmission remains quite small.

We have produced similar tables for the three cases mentioned in Section 2 (Martian dust, urban haze, water cloud). For each case we considered distributions in single-scattering albedo, asymmetry factor, and optical depth and calculated transmission, reflection, and absorption as a function of the cosine of the incident angle. We have also constructed graphical representations of the results, examples of which are shown in Figs. 3 and 4. The percentage difference of the approximations from the doubling method are shown for transmission, reflection and absorption in Figs. 3a, 3b and 3c respectively. The effects of including horizontal variability are also shown for medium and broad distributions ($\sigma = 0.3$ and 1.0) of single-scattering albedo. Fig. 3d shows the actual transmission, reflection and absorption as calculated by the doubling method. For this case, the four-stream method is slightly superior in accuracy to the delta-Eddington (at a twenty-fold cost in computing time) while the Eddington has substantial errors. However, the effects of variability in $\tilde{\omega}$ cause differences from the doubling results equal to or larger than those induced by the use of an approximation. Transmission shows the largest relative differences as the actual transmission is quite small while absorption has the largest absolute differences. Analogous curves (Fig. 4) are shown for the "cloud" case ($\tilde{\omega} = 0.999$) for a distribution in optical depth. In this case the approximations are of comparable accuracy. The effect of considering a distribution in optical depth is shown in Fig. 4 ($\sigma = 0.3$ and $\sigma = 1.0$ curves) to be considerable. In view of the extremely variable optical density within clouds, the neglect of this effect may introduce errors in computations of the radiative behavior of clouds.

Complete tabular and graphical output from these computations is available from the authors.

In general, the effect of variability in optical depth is to increase the transmission of a layer, while reducing the reflection and absorption, over those values calculated for a homogeneous layer. At large optical depths the transmission is very small ($\sim 10^{-3}$) so that the large percentage change in transmission (factors of 5–100) is attributable to the inclusion of a few cells of increased transmission which effectively account for the entire transmission of the layer. For small optical depths, the transmission is large so the inclusion of cells with slightly differing

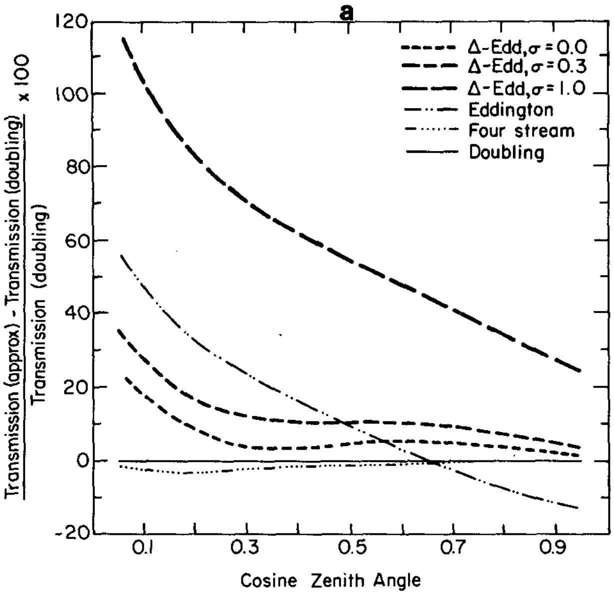

FIG. 3a. Percentage difference in transmission of approximations, compared with doubling, for the case $\tau = 4.0$, $\bar{\omega} = 0.8$, $g = 0.75$. The effects of a medium and broad distribution in $\bar{\omega}$ are shown in the curves for $\sigma = 0.3$ and $\sigma = 1.0$. Positive differences indicate that the approximation transmission is greater than that calculated by doubling.

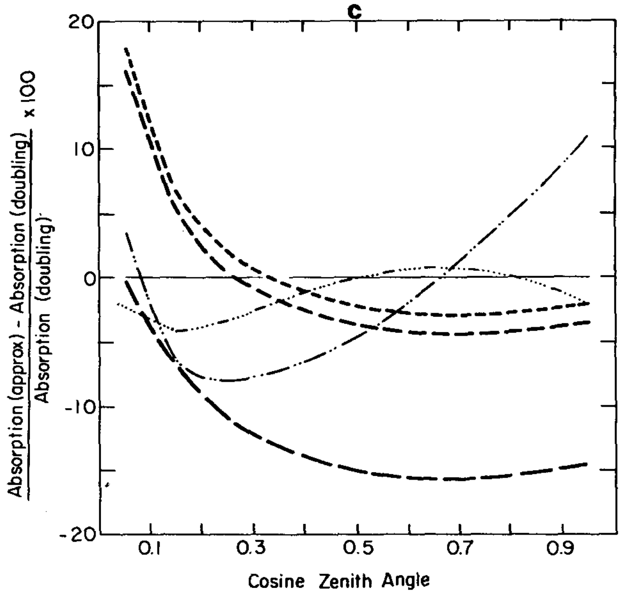

FIG. 3c. As in Fig. 3a except for absorption.

transmission does not greatly affect the total transmission. However, when the incident angle is large the optical path is large, and the effect of a distribution in optical depth causes up to a 25% increase in transmission. The reflection is seen to decrease 10–20% with the inclusion of cells of varying optical depth, the major effect appearing at small optical depths where the total reflection is quite small (<10%). Thus the inclusion of a few highly transmitting cells reduces the reflection considerably. For case III ($\bar{\omega} = 0.999$, $g = 0.844$), where the absorption is extremely small, the increase in transmission within a few cells is balanced by a substantial reduction in reflection. For example, for case III, $\mu_0 = 0.15$ and $\bar{\tau} = 16$, the change in reflection for a broad distribution of optical depths is from 0.56 to 0.43 or −25%. Absorption is decreased 10–25% by the inclusion of cells of varying optical

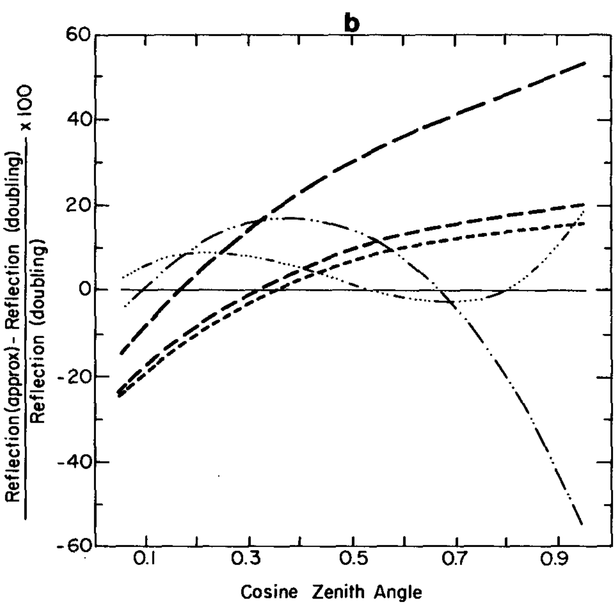

FIG. 3b. As in Fig. 3a except for reflection.

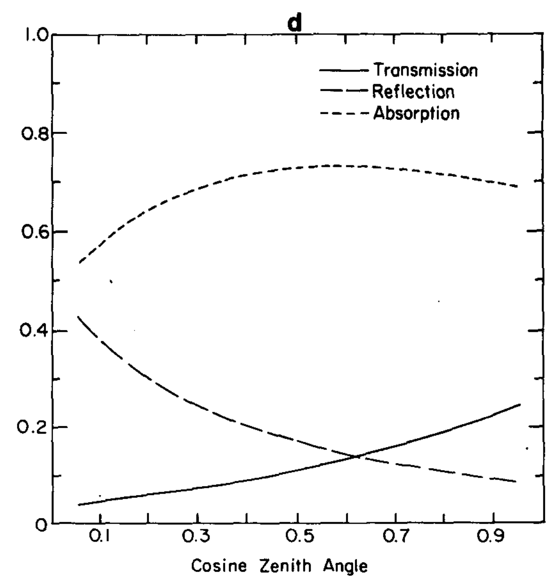

FIG. 3d. Actual transmission, reflection and absorption as calculated by the doubling method. $\sigma = 0$.

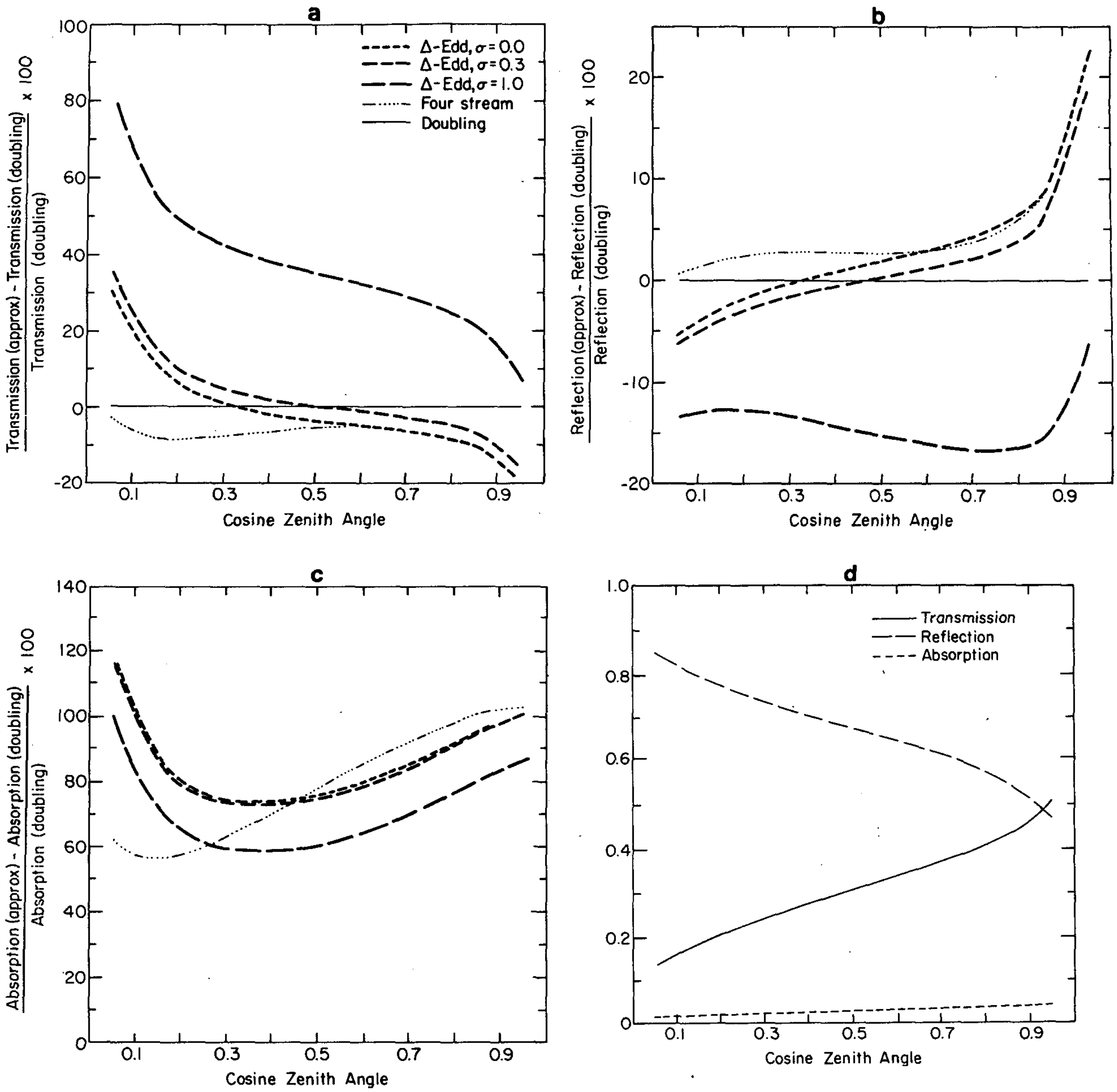

FIG. 4. As in Fig. 3 except for $\bar{\tau} = 16.0$, $\div\bar{\omega} = 0.999$, $g = 0.844$. The distribution is over optical depth τ. For this case the Eddington approximation is essentially identical to delta-Eddington ($\sigma = 0$) and is not shown.

depth, with the maximum effect appearing for case II ($\bar{\omega} = 0.62$, $g = 0.75$) at intermediate mean optical depths.

For a nonhomogeneous atmosphere with variable single-scattering albedo, the transmission is increased, the reflection increased, and the absorption decreased over the homogeneous case. The magnitude of the effect is negligibly small for small optical depths and large mean single-scattering albedos (case III). Only for large optical paths is the effect considerable. The large relative change in transmission is again attributable to the smallness of the actual transmission value at large optical depths. Reflection, for a broad distribution, increases by 12–33%. In cases I and II, a broad distribution ($\sigma = 1$) contains an appreciable number of scatterers having $\tilde{\omega} = \bar{\tilde{\omega}} \pm 0.33\bar{\tilde{\omega}}$. The limits at half-height of the broad distribution with $\bar{\tilde{\omega}}$ of 0.62 are ~0.40 and ~0.85. This may be an unrealistically broad distribution for the actual atmosphere. A reduction in the effect to less than 5% accompanies narrowing the distribution to $\sigma = 0.3$ ($\tilde{\omega}$ limits at half-height are 0.54 and 0.70). Thus, in most cases, the effects of a distribution of single-scattering

TABLE 3. This table summarizes the results for horizontal variability in asymmetry factor.

Layer property	Case I ($\bar{\omega} = 0.8$, $\bar{g} = 0.75$)	Case II ($\bar{\omega} = 0.62$, $\bar{g} = 0.75$)	Case III ($\bar{\omega} = 0.999$, $\bar{g} = 0.844$)
Transmission	increase small τ (1–3%) large τ (10–200%)	increase small τ (~1%) large τ (10–20%)	increase small τ (1%) large τ (20%)
Reflection	decrease large θ_0 (10%) small θ_0 (5%)	increase at small incident angle (2–5%) decrease at large incident angle (4–8%)	decrease small, large θ_0 (5–10%) intermediate θ_0 (13–33%)
Absorption	increase 2–5%	<1% decrease	<1% increase

albedos may be ignored in the real atmosphere unless precision to greater than 5% is required by the problem under consideration.

The effects of variability in scattering asymmetry factor are summarized in Table 3 for horizontal patchiness. Only the broad distribution ($\sigma = 1.0$) shows significant alteration from the homogeneous layer case. Again there is a 10–200% increase in transmission for large optical depths due to small absolute changes in a small actual transmission, resulting in large percentage changes. The reflection decreases from 5–33% depending on incident angle and case. Absorption in general has a small increase or negligible change, with respect to calculations employing the mean scattering asymmetry. As with the distribution in single-scattering albedo, the broad distribution ($\sigma = 1$) may be unrealistically wide and thus serve as an upper limit on the effect. The g distribution with $\sigma = 0.3$ has a maximum effect of ±5% and tends to be more representative of an actual particle distribution.

To summarize, if optical depths, single-scattering

TABLE 4. Transmission, reflection and absorption of vertically inhomogeneous layers computed using the delta-Eddington method for no, narrow, medium and broad distributions of optical depth ($\sigma = 0.0, 0.1, 0.3, 1.0$). Results are shown for cosine zenith angles of 0.15, 0.55 and 0.95. Case I, $\bar{\omega} = 0.8$, $g = 0.75$.

μ_0	0.15				0.55				0.95			
σ	0.0	0.1	0.3	1.0	0.0	0.1	0.3	1.0	0.0	0.1	0.3	1.0
$\bar{\tau} = 0.25$												
TRANS	0.59647	0.59703	0.59774	0.62755	0.86862	0.86883	0.86883	0.87726	0.93027	0.93038	0.93034	0.93434
ST DEV		0.00000	0.00000	0.00899		0.00456	0.00306	0.02596		0.01229	0.00724	0.09034
REFL	0.16641	0.16619	0.16585	0.15351	0.04398	0.04391	0.04389	0.04079	0.01650	0.01647	0.01647	0.01539
ST DEV		0.00334	0.00967	0.03283		0.00128	0.00373	0.01428		0.00052	0.00152	0.00619
ABS	0.23711	0.23677	0.23641	0.21894	0.08740	0.08725	0.08728	0.08195	0.05323	0.05314	0.05318	0.05026
ST DEV		0.00561	0.01628	0.05768		0.00287	0.00842	0.03618		0.00184	0.00540	0.02479
$\bar{\tau} = 1.00$												
TRANS	0.26523	0.26401	0.26769	0.28139	0.57733	0.57506	0.58011	0.58890	0.73548	0.73365	0.73720	0.74002
ST DEV		0.00000	0.00000	0.00563		0.00986	0.00789	0.01850		0.03139	0.02465	0.06498
REFL	0.26541	0.26560	0.26479	0.26038	0.11659	0.11700	0.11586	0.11247	0.05164	0.05189	0.05129	0.05001
ST DEV		0.00108	0.00338	0.01396		0.00188	0.00604	0.01906		0.00112	0.00358	0.01111
ABS	0.46936	0.47038	0.46752	0.45823	0.30608	0.30793	0.30402	0.29863	0.21289	0.21446	0.21151	0.20997
ST DEV		0.00475	0.01520	0.05141		0.00795	0.02535	0.07903		0.00662	0.02103	0.06634
$\bar{\tau} = 4.00$												
TRANS	0.05742	0.05733	0.05690	0.07111	0.12447	0.12428	0.12338	0.15569	0.24152	0.24110	0.23882	0.27753
ST DEV		0.00000	0.00000	0.00377		0.00832	0.01307	0.01053		0.02322	0.03642	0.03836
REFL	0.28389	0.28388	0.28387	0.28308	0.16531	0.16531	0.16527	0.16305	0.09178	0.09178	0.09176	0.08926
ST DEV		0.00066	0.00068	0.00175		0.00043	0.00082	0.00463		0.00045	0.00113	0.00526
ABS	0.65869	0.65878	0.65923	0.64581	0.71022	0.71041	0.71135	0.68125	0.66670	0.66712	0.66942	0.63321
ST DEV		0.00388	0.01032	0.03681		0.00819	0.02252	0.08138		0.01273	0.03533	0.11890
$\bar{\tau} = 16.00$												
TRANS	0.00016	0.00016	0.00024	0.00206	0.00034	0.00034	0.00051	0.00440	0.00111	0.00109	0.00163	0.01122
ST DEV		0.00000	0.00000	0.00003		0.00007	0.00023	0.00018		0.00038	0.00114	0.00464
REFL	0.28470	0.28470	0.28470	0.28470	0.16725	0.16725	0.16725	0.16724	0.09514	0.09513	0.09513	0.09510
ST DEV		0.00059	0.00057	0.00057		0.00034	0.00033	0.00033		0.00016	0.00014	0.00023
ABS	0.71514	0.71514	0.71505	0.71324	0.83240	0.83241	0.83223	0.82836	0.90375	0.90377	0.90324	0.89367
ST DEV		0.00161	0.00146	0.00485		0.00189	0.00192	0.01009		0.00223	0.00224	0.02236

TABLE. 5. Percentage change in transmission, reflection, and absorption for a 1% change in τ, $\tilde{\omega}$, g or μ_0. Case I is $\tilde{\omega} = 0.8, g = 0.75$; case II is $\tilde{\omega} = 0.62, g = 0.75$; case III is $\tilde{\omega} = 0.999, g = 0.84$. Calculated for four values of optical depth (τ = 0.25, 1.0, 4.0, 16.0) at an incident angle of 18° (μ = 0.95).

		Transmission				Reflection				Absorption			
y	Case	0.25	1.0	4.0	16.0	0.25	1.0	4.0	16.0	0.25	1.0	4.0	16.0
τ	I	−0.07	−0.33	−1.60	−7.51	0.92	0.68	0.13	0.00	1.01	0.97	0.57	0.01
	II	−0.12	−0.51	−2.37	−10.64	0.84	0.46	0.02	−0.00	0.98	0.85	0.32	0.00
	III	−0.01	−0.06	−0.26	−0.70	1.01	1.02	0.90	0.45	1.17	1.06	1.23	1.13
$\tilde{\omega}$	I	0.20	0.81	3.58	16.22	1.38	2.32	4.10	4.44	−3.76	−3.30	−1.81	−0.48
	II	0.15	0.60	2.40	8.87	1.29	1.92	2.62	2.64	−1.50	−1.19	−0.44	−0.11
	III	0.25	1.08	5.30	35.83	1.50	2.96	8.23	25.24	−230.88	−236.36	−234.44	−215.68
g	I	0.07	0.31	1.62	7.61	−3.40	−3.33	−3.02	−2.93	−0.09	−0.25	−0.16	0.30
	II	0.05	0.27	1.56	8.52	−3.43	−3.47	−3.46	−3.46	−0.07	−0.17	−0.04	0.15
	III	0.08	0.34	1.39	3.53	−5.82	−5.72	−4.84	−2.43	0.15	−0.46	−1.11	−0.70
μ_0	I	0.10	0.36	1.21	3.43	−2.39	−2.04	−1.52	−1.44	−0.92	−0.74	−0.22	0.15
	II	0.13	0.51	1.92	7.83	−2.45	−2.23	−1.99	−1.98	−0.91	−0.69	−0.16	0.08
	III	0.04	0.13	0.37	0.59	−2.67	−2.24	−1.32	−0.43	−1.00	−0.86	−0.51	0.15

albedos or asymmetry factors are either uncertain or known to be fluctuating, with relative standard deviations of ~10% or greater, then little real benefit is added by the use of computationally precise, but costly, many-stream radiation-transfer algorithms.

b. Vertical variability

We have repeated this analysis for the case of vertical variability discussed in Section 2b. A sample of the results for distributions in optical depth is shown in Table 4 for the parameters of case I. All the computations use the delta-Eddington method and show, for each value of $\bar{\tau}$, the mean values and standard deviations of transmission, reflection and absorption of the random 10-layer system. The variation in mean values with vertically random sublayers is less than that with the horizontally patchy model for all cases and distributions that we considered. Again, tabular results of this study are available from the authors.

4. Sensitivity analysis

To examine the consequences of an *error* in a determination of an input parameter for a radiative transfer calculation (as contrasted with a potentially precise determination of the probability densities), we performed a sensitivity analysis. The sensitivity S is defined as

$$S = \frac{d \ln x}{d \ln y}, \tag{7}$$

where x is the transmission, reflectance or absorbance, and y the optical depth, single-albedo, asymmetry factor or zenith angle. The sensitivity is thus approximately the percentage change in x for a 1% change in y.

We calculated S for the three cases of horizontal patchiness described in Section 2c. The results are shown in Tables 5 and 6, for near vertical and oblique incident angles respectively. The sensitivity is calculated for transmission, reflection and absorption at each of four optical depths. We employed the delta-Eddington approximation for the production of these tables for the reasons outlined in Section 2c.

Through examination of these tables, we find that radiative transfer calculations are most sensitive to small errors in $\tilde{\omega}$ and less sensitive to small errors in optical depth, asymmetry factor and incident angle. The transmission can decrease by 15–35% for a single percent diminution in single-scattering albedo at large optical depths. The percent increase in reflection is generally about twice the percent change in $\tilde{\omega}$. Absorption is seen to increase rapidly for a decrease in $\tilde{\omega}$, especially for case III where the absorption is quite small. Of the other independent parameters, transmission is most sensitive to variations in optical depth while reflection is most sensitive to asymmetry factor changes. By comparing Tables 5 and 6 we find that, in general, transmission is more sensitive to changes in the four independent parameters at oblique incidence, while reflection and absorption are more sensitive at near-vertical incidences.

5. Conclusions

The neglect of inhomogeneities in atmospheric layers for the purpose of radiative transfer calculations is justified providing inaccuracies of ~5% are tolerable. The errors induced by use of the approximate methods in some circumstances are of the same order of magnitude as those errors induced by the assumption of horizontal homogeneity of the atmos-

TABLE 6. As in Table 5 except for oblique incident angle ($\mu_0 = 0.15$).

y	Case	Transmission 0.25	1.0	4.0	16.0	Reflection 0.25	1.0	4.0	16.0	Absorption 0.25	1.0	4.0	16.0
τ	I	−0.44	−0.67	−1.95	−7.80	0.59	0.12	0.01	−0.00	0.69	0.31	0.17	0.00
	II	−0.65	−1.02	−3.11	−12.43	0.50	0.04	0.00	−0.00	0.61	0.19	0.05	0.00
	III	−0.16	−0.27	−0.33	−0.71	0.79	0.40	0.23	0.16	0.84	0.57	0.60	0.82
$\tilde{\omega}$	I	1.01	2.83	6.86	22.90	1.50	2.21	2.64	2.68	−3.50	−2.77	−1.69	−1.05
	II	0.81	2.34	5.30	16.95	1.37	1.77	1.86	1.86	−1.31	−0.85	−0.44	−0.36
	III	1.34	4.10	10.24	42.92	1.79	3.47	6.36	14.18	−236.93	−236.39	−233.16	−216.58
g	I	0.39	−0.01	0.73	5.11	−1.99	−1.42	−1.45	−1.45	0.42	0.82	0.57	0.58
	II	0.25	−0.58	0.03	4.07	−2.14	−1.83	−1.87	−1.87	0.32	0.54	0.37	0.37
	III	0.81	1.32	1.70	3.53	−3.81	−1.96	−1.22	−0.93	0.76	2.05	1.87	0.72
μ_0	I	0.45	0.43	0.26	0.26	−0.72	−0.28	−0.20	−0.20	−0.61	−0.07	0.07	0.08
	II	0.65	0.57	0.31	0.31	−0.67	−0.28	−0.26	−0.26	−0.55	−0.05	0.05	0.05
	III	0.19	0.31	0.19	0.18	−0.89	−0.46	−0.13	−0.05	−0.80	−0.33	0.05	0.13

pheric layer. Both errors are on the order of a few percent with occasional larger errors from each source. There is no general rule as to the direction of the errors; in some instances, they are compensatory and in other instances, additive. If accuracy on the order of 95% is desired, the use of an approximation and the assumption of homogeneity appear justified, with caution exercised for large zenith angles and optical depths. More precise calculations necessitate the use of a multi-stream formulation such as doubling, *and the consideration of nonhomogeneity*. The specifics of the problem will dictate the exact manner in which inhomogeneities of the scattering media must be considered.

The calculations of transmission, reflection and absorption, when a homogeneous layer is assumed, are most sensitive to errors in the prescribed value of the single-scattering albedo. Relative errors of transmission, reflection and absorption resulting from relative errors in optical depths, asymmetry factor and incident angle are generally reduced by the radiative transfer calculations, though they are important in specific cases. If $\tilde{\omega}$, τ and g are assumed to vary independently, then the total variance in computed layer properties due to inhomogeneities may be approximated by the Pythagorean sums of the quantities computed in the tables.

On the whole, horizontal variability appears to be a more important effect than vertical variability in determining mean layer properties.

We note that in studies of the "heating/cooling" effect of pollutants the magnitude of the net effect is often found by subtracting two relatively large "average" global albedos computed for layers assumed to be homogeneous. The results presented here show that if the difference is $\lesssim 5\%$ careful attention should be paid to estimating the influence of variability in the layers.

In this study we have dealt only with radiation at visible wavelengths. The obvious extension is to consider the effects at longer wavelengths, since inhomogeneities in radiative properties in the infrared may have important consequences in interpreting satellite measurements.

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APPENDIX

Markovian Radiative Transfer

If the matrix elements M_{ij} and the inhomogeneous term F_i are functions of a random variable (which might be the density or the chemical composition of the medium for example), then Eq. (1) becomes a stochastic differential equation and its solution depends on the nature of the introduced randomness. We can write

$$M_{ij}(z) = A_{ij}\rho(z),$$
$$F_i(z) = B_i\rho(z),$$

where $\rho(z)$ is the density of the medium at z. We assume for simplicity that $\rho(z)$ varies in such a way that we can rewrite (1) as

$$dI_i = (A_{ij}I_j + B_i)\rho_0 dz + (A_{ij}I_j + B_i)dw, \quad (A1)$$

where ρ_0 is the average value of the density, assumed constant, and dw white noise, with width s, i.e.,

$$E\{dw\} = 0,$$
$$E\{(dw)^2\} = s^2 dz.$$

We have used the notation $E\{\ \}$ to mean expectation value, i.e., an average over all possible photon paths. For this model I_i is a Markov process and thus we can write a Fokker-Planck equation for $p(I_1 \cdots I_n, z)$, the probability density that the intensities have the values I_i at z:

$$\frac{\partial p}{\partial z} = - \sum_{i=1} \frac{\partial}{\partial I_i} (\eta_i p) + \tfrac{1}{2} \sum_{i,j=1} \frac{\partial^2}{\partial I_i \partial I_j} (V_{ij} p), \quad \text{(A2)}$$

where

$$\eta_i \equiv \lim_{\Delta z \to 0} \frac{1}{\Delta z} [E\{(I_i(z + \Delta z) - I_i(z))/I_i(z) = I_i\}], \quad \text{(A3)}$$

$$V_{ij} \equiv \lim_{\Delta z \to 0} \frac{1}{\Delta z} [E\{(I_i(z + \Delta z) - I_i(z))(I_j(z + \Delta z) - I_j(z))/I_i(z) = I_i, I_j(z) = I_j\}]. \quad \text{(A4)}$$

The notation

$$E\{x/y\} \equiv \int_0^\infty \{x\} p(I_1 \cdots I_n, z/y) dI_1 \cdots dI_n$$

and $p(I_1 \cdots I_n, z/y)$ is the conditional probability that the values of the intensities at z are the I_i given the event y.

From Eq. (A1) we can derive non-random equations for the moments of I_i, the $E\{I_i^m(z)\}$, by multiplying Eq. (A1) by I_i^m and integrating over all the I_i.

We first find from Eq. (A1) and the definitions (A3) and (A4) that

$$\eta_i = (A_{ij} E\{I_j\} + B_i)\rho_0, \quad \text{(A5)}$$

$$E\{(A_{ik} T_k + B_i)(A_{jl} I_l + B_j)\} s^2 = V_{ij}. \quad \text{(A6)}$$

Then from Eq. (A1), we have

$$\frac{\partial E}{\partial z} \{I_i\} = \eta_i, \quad \text{(A7)}$$

$$\frac{\partial}{\partial z} E\{I_i^2\} = 2E\{I_i\}\eta_i + V_{ii}. \quad \text{(A8)}$$

From Eqs. (A7) and (A8) we thus have

$$\frac{\partial}{\partial z} \sigma_i^2 = V_{ii}, \quad \text{(A9)}$$

where $\sigma_i^2 = E\{I_i^2\} - (E\{I_i\})^2$, the variance of I_i. From Eq. (A7) we see that the first moment of I_i is unaffected by the noise in this model but that the variance in I_i grows with z.

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