NOTES AND CORRESPONDENCE

Baroclinic Instability of Flows Along Sloping Boundaries

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ABSTRACT

In the two-layer quasi-geostrophic model with boundaries sloping perpendicular to the basic flow, the ratios of the slopes of the bottom and the top to that of the interface between the fluid layers in the basic state are important parameters in the expression of the growth rate of unstable waves. When Eady's (1949) model is extended to include sloping bottom and top boundaries, the growth rates of unstable waves depend on the ratios of the slopes of the bottom and the top to that of the isentropes of the basic state. For the Eady model with sloping bottom, an important parameter characterizing the instability is the ratio between the vertical and horizontal heat transports by the wave divided by the slope of the isentropes of the basic state. An interpretation of these ratios and their relations clarifies the stabilization of the system for large slopes, the variation of the wavelength of the most unstable wave with the bottom slope, and the destabilization of some short waves for negative bottom slopes. It is found that the most unstable wave of the system has zero vertical energy flux convergence at the sloping bottom.

1. Introduction

The prevailing atmospheric or oceanic characteristics in several geographical regions indicate that the local mean flow is along a topographic slope. Such is the case for the strong westerlies over the coast of East Antarctica and for the Gulf Stream near the continental shelf of North America. This has motivated several studies in which the stability of a flow along a topographic slope is analyzed. Blumsack and Gierasch (1972, hereafter called BG) introduced a shallow topography as the lower boundary condition to Eady's (1949) model. Davies (1975), Tang (1976) and Davey (1977) considered two-layer quasi-geostrophic models with shallow topography. The steep topography case has been studied by Orlanski (1969) using a two-layer primitive equation model, and by Orlanski and Cox (1973) using a full nonlinear multi-level ocean model.

In this note we restrict our consideration to the case of shallow sloping boundaries with no β effect. We analyze the two-layer quasi-geostrophic model and Eady's model with bottom and top boundaries sloping perpendicular to the basic flow. The growth rates of the unstable waves in the two-layer model depend on

$$\delta_B = \frac{S_B}{S_I}, \quad \delta_T = \frac{S_T}{S_I}, \tag{1.1}$$

where S_B and S_T are the slopes of the bottom and top boundaries respectively, and S_I the slope of the interface between the two fluid layers in the basic state. A similar conclusion is obtained for Eady's model, provided S_I in (1.1) is interpreted as the slope of the isentropes of the basic state.

The case with $\delta_T = 0$, $\delta_B = \delta$ was studied in BG, where it was established that if the absolute value of δ is sufficiently large, waves that are unstable on a horizontal bottom become stable on the slope, and that when δ is negative some waves shorter than the short-wave cutoff present in the horizontal bottom case become unstable. In order to understand the relevance of δ in determining the stability of the flow, we analyze the energetics in BG. Necessary conditions for instability bound δ and Δ, which is the ratio between the vertical and horizontal heat transports by the wave, divided by the slope of the isentropes of the basic state. We derive the expression for Δ and through its dependence on δ we show how the wavelength of the most unstable wave varies with the bottom slope. Finally, we discuss the influence of the sloping bottom on the vertical energy flux.

2. Two-layer model with sloping bottom and top boundaries

The equations for the quasi-geostrophic two-layer model with $\delta_B = \delta_T$ are given in Bretherton (1966) and are easily extended to the case $\delta_B \neq \delta_T$. We consider fluid layers with depth H in the absence of relative motions, and a basic state in which the velocities in both layers are $U_1 = -U_2 = U$, where

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the subscripts 1 and 2 refer to the upper and lower layers, respectively.

The basic-flow potential vorticity gradients are

$$\frac{\partial \bar{q}_1}{\partial y} = \frac{fS_I}{H}(1 - \delta_T), \quad \frac{\partial \bar{q}_2}{\partial y} = -\frac{fS_I}{H}(1 - \delta_B),$$

where $\bar{q}_i$ is the basic flow potential vorticity in the ith layer and f the Coriolis parameter. The necessary condition for instability (Pedlosky, 1964) requires that $(1 - \delta_B)$ and $(1 - \delta_T)$ have the same sign. If q_i' is the perturbation potential vorticity in the ith layer, we can assume

$$q_i' = \mathrm{Re}[Q_i(t)e^{ikx} \sin ly].$$

We introduce the Rossby radius of deformation

$$\lambda = \frac{NH}{f},$$

where N is the Brunt-Väisälä frequency, and the nondimensional expressions

$$\left.\begin{aligned} \mu &= \lambda^2(k^2 + l^2) \\ A &= -\frac{\mu + 1}{\mu(\mu + 2)}, \quad C = \frac{1}{\mu(\mu + 2)} \end{aligned}\right\}.$$

From the linearized potential vorticity equations, we easily obtain the expression

$$\frac{d^2 Z_1}{dt^2} + U^2k^2G(\mu,\delta_B,\delta_T)Z_1 = 0, \tag{2.1}$$

where

$$Q_1 = Z_1 \exp[-ikUA(\delta_B - \delta_T)t],$$

$$G = 1 + 2A[(1 - \delta_B) + (1 - \delta_T)] + A^2(\delta_B - \delta_T)^2 + 4C(1 - \delta_B)(1 - \delta_T).$$

Eq. (2.1) has solutions that grow exponentially with time if, and only if, the coefficient of the second term is negative. Taking into account the signs of A and C, this condition implies that for instability

$$\delta_B < 1, \quad \delta_T < 1.$$

Therefore, for the flow to be unstable, both the bottom and top slopes have to be smaller than that of the interface between the two fluid layers in the basic state.

In the particular case with $\delta_T = 0$ and $\delta_B = \delta$ the growth rate σ of the unstable waves can be written as

$$\frac{\sigma}{kU} = \left\{\frac{2 - \mu}{2 + \mu} - \left[\left(\frac{1 + \mu}{\mu(2 + \mu)}\right)^2 \delta^2 - 2\frac{1 - \mu}{\mu(2 + \mu)}\delta\right]\right\}^{1/2}. \tag{2.2}$$

The criterion of marginal instability is found by setting the right-hand side of (2.2) equal to zero. When the bottom is horizontal ($\delta = 0$), this criterion does not depend upon the vertical shear and provides the short wave cutoff: $\mu < 2$. When the bottom is not horizontal the criterion, which involves δ, is

$$\delta = \frac{\mu(\mu + 2)(1 - \mu)}{(\mu + 1)^2} \times \left\{1 \pm \left[1 + \left(\frac{\mu + 1}{1 - \mu}\right)^2\left(\frac{2 - \mu}{\mu + 2}\right)\right]^{1/2}\right\}.$$

The two branches are represented by the heavy solid lines in Fig. 1. There is also a condition for a wave to have the same growth rate in the horizontal bottom case and in the case when the bottom boundary is sloping. From (2.2) this condition is given by

$$\delta = 2\frac{\mu(\mu + 2)(1 - \mu)}{(\mu + 1)^2}$$

and is represented by the dashed line in Fig. 1. The shaded areas in Fig. 1 correspond to cases where the growth rates for the waves on the slope are larger than on the horizontal bottom. An inspection of Fig. 1 reveals that no instabilities are possible if $\delta > 1$, and, in general, that all unstable waves in the horizontal bottom case are stable for a sufficiently large value of the bottom slope. It also reveals that, when compared to the horizontal bottom case, the instability of the longer waves ($\mu < 1$) is enhanced by small positive slopes while that of the shorter waves ($\mu > 1$) is enhanced by small negative slopes.

3. Eady's model with sloping bottom and top boundaries

In the notation used in BG which is adopted in what follows, the conservation of potential vorticity gives the nondimensional expression

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)\nabla^2\phi - \frac{\partial w}{\partial z} = 0. \tag{3.1}$$

The kinetic energy equation is easily obtained after multiplying (3.1) by ϕ. If an overbar indicates average in the x direction, along the slope, such equation is

$$\frac{\partial}{\partial t}\int_{z_1}^{z_2} \frac{\overline{(\nabla\phi)^2}}{2}dz = \int_{z_1}^{z_2} \overline{wT}dz - \overline{w\phi}\Big|_{z_1}^{z_2}. \tag{3.2}$$

The temperature equation is, in turn,

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)\frac{\partial\phi}{\partial z} - \frac{\partial\phi}{\partial x} + \frac{1}{F}w = 0, \tag{3.3}$$

where F is the Froude number (the square of the ratio between the horizontal length and the Rossby radius of deformation). The potential energy equation

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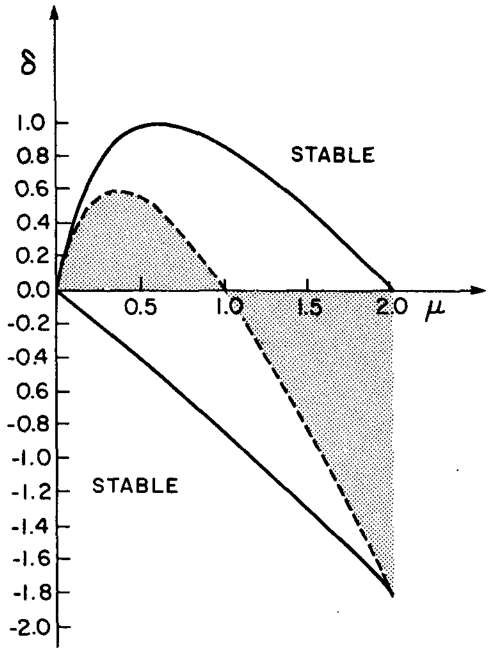

FIG. 1. Criteria of marginal stability and for the wave on the slope to have the same growth rate as in the horizontal bottom case. The symbols are explained in the text.

is obtained after multiplying (3.3) by $FT = F\partial\phi/\partial z$, yielding

$$\frac{\partial}{\partial t}\int_{z_1}^{z_2} F\,\frac{\overline{T^2}}{2}\,dz = F\int_{z_1}^{z_2}\overline{vT}dz - \int_{z_1}^{z_2}\overline{wT}dz. \quad (3.4)$$

The energy equation is obtained after adding (3.2) and (3.4). The result is

$$\frac{\partial}{\partial t}\int_{z_1}^{z_2}\overline{\left(\frac{(\nabla\phi)^2}{2} + F\,\frac{T^2}{2}\right)}dz = F\int_{z_1}^{z_2}\overline{vT}dz - \overline{w\phi}\Big|_{z_1}^{z_2}. \quad (3.5)$$

Elimination of the vertical velocity between (3.1) and (3.3) leads to the relation

$$\left(\frac{\partial}{\partial t} + z\,\frac{\partial}{\partial x}\right)\left(\nabla^2\phi + F\,\frac{\partial^2\phi}{\partial z^2}\right) = 0. \quad (3.6)$$

The appropriate set of boundary conditions is

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\frac{\partial\phi}{\partial z} + (\delta_T - 1)\,\frac{\partial\phi}{\partial x} = 0,\quad z = 1, \quad (3.7)$$

$$\left(\frac{\partial}{\partial t}\right)\frac{\partial\phi}{\partial z} + (\delta_B - 1)\,\frac{\partial\phi}{\partial x} = 0,\quad z = 0, \quad (3.8)$$

where δ_B and δ_T are defined according to (1.1).[1]

[1] The mathematical problem treated by Blumen (1978) is formally similar to the problem defined by (3.6)–(3.8) when the slopes of the top and bottom boundaries are the same.

The system (3.6)–(3.8) has solutions of the form

$$\phi(x,z,t) = H(z)e^{i(kx-\sigma t)}, \quad (3.9)$$

with

$$H(z) = \sinh\mu z - \frac{\eta}{1-\delta_B}\cosh\mu z, \quad (3.10)$$

$$\mu = k/F^{1/2},\quad \eta = \sigma/F^{1/2}. \quad (3.11)$$

The real part of η is given by

$$\eta_r = \frac{\mu}{2} + \frac{\delta_T - \delta_B}{2\tanh\mu} \quad (3.12)$$

and its imaginary part by

$$\eta_i = \left[\frac{\mu - (1-\delta_T)\tanh\mu}{\tanh\mu}(1-\delta_B) - \frac{1}{4}\left(\frac{\delta_B - \delta_T}{\tanh\mu} - \mu\right)^2\right]^{1/2}. \quad (3.13)$$

The criterion of marginal stability for the case with $\delta_T = 0$ and $\delta_B = \delta$ is represented by the heavy solid lines in Fig. 2, and the condition for the wave to have the same growth rate on the slope as on the horizontal bottom by the dashed line on the same figure. The shaded areas have the same meaning as in Fig. 1.1. Contours of η_i are included in the unstable region.

The boundary conditions imply that the ratio between the vertical velocity w and the meridional velocity v is independent of x at the top and at the bottom of the region. Thus, w and v are in phase there. But v is 90° out of phase with ϕ at all points. Therefore,

$$\overline{w\phi} = 0,\quad z = 0,\quad z = 1. \quad (3.14)$$

Taking $z_1 = 0$, $z_2 = 1$ in (3.5) gives

$$\int_0^1 \overline{vT}dz > 0. \quad (3.15)$$

for an unstable perturbation.

The northward transport of heat can be computed by using the solution (3.9)–(3.13) with the result

$$\int_0^1 \overline{vT}dz = \tfrac{1}{2}F^{1/2}\mu^2\,\frac{\eta_i}{1-\delta_B}\exp(2F^{1/2}\eta_i t). \quad (3.16)$$

Thus, a necessary condition for instability is that the ratio of the slope of the bottom to that of the isentropes of the basic state must be less than 1. Pedlosky's (1964) necessary condition for instability requires that $(1 - \delta_B)$ and $(1 - \delta_T)$ have the same sign. Therefore, for instability, $\delta_T < 1$.

From (3.2) note that the vertical transport of heat by an unstable wave is positive. From (3.15) and the

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potential energy equation (3.4), we conclude that

$$0 < \frac{\int_0^1 \overline{wT}dz}{\int_0^1 \overline{vT}dz} F^{-1} < 1. \qquad (3.17)$$

We emphasize that all quantities in (3.17) are nondimensional. In the notation used in BG, where H is the depth of the fluid and dimensional variables are indicated by a subscript asterisk, Eq. (3.17) becomes

$$0 < \Delta = \frac{\int_0^H \overline{w_* T_*} dz_*}{\int_0^H \overline{v_* T_*} dz_*} S_I^{-1} < 1, \qquad (3.18)$$

where Δ is the ratio of the slope of the heat transport by the wave to that of the isentropes of the basic state. The relation (3.18) is a necessary condition for instability.

4. The particular case with $\delta_T = 0$ and $\delta_B = \delta$

The value of Δ for this case can be computed from the solution of (3.9) to (3.13). The result is

$$\Delta = \frac{1}{2}\left(\frac{\mu}{\tanh\mu} - 1\right) + \frac{\delta}{2}. \qquad (4.1)$$

First of all we observe that the first term on the right-hand side of (4.1) is positive for positive μ and increases monotonically with μ. We showed in the preceding section that $0 < \Delta < 1$ for the wave to be unstable. Therefore, sufficiently large values of the absolute value of δ will stabilize a wave otherwise unstable on a horizontal bottom. Also, a negative value of δ can destabilize a wave that is stable on a horizontal bottom. Such destabilization will occur for values of μ large enough to make the first term on the right-hand side of (4.1) greater than 1. Thus some short waves, particularly in the short-wave cutoff region, will be destabilized by a negative sloping bottom.

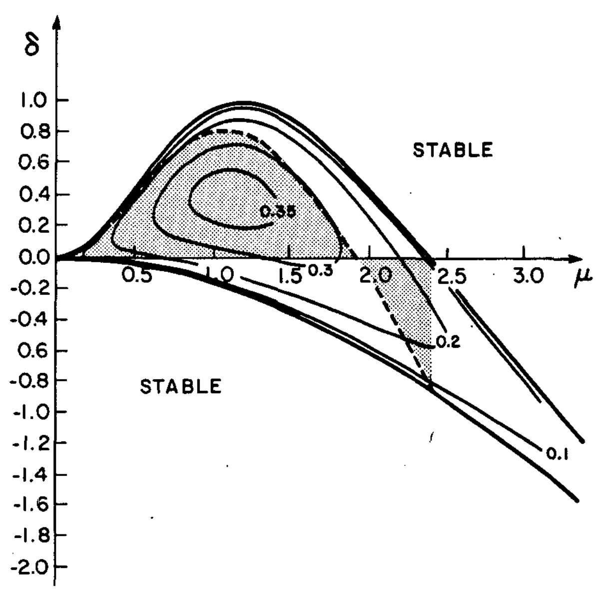

FIG. 2. As in Fig. 1 except for the Eady model with sloping bottom.

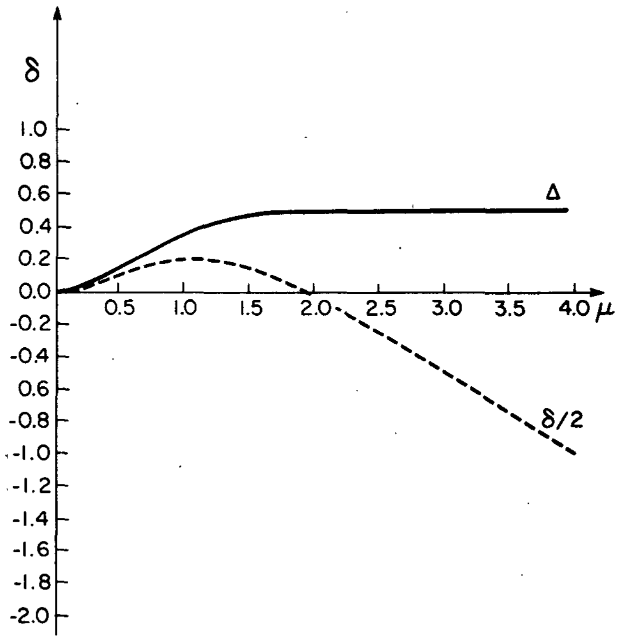

FIG. 3. The value of $\delta/2$ that gives the maximum growth rate for a fixed μ (dashed line) and the corresponding value of Δ (solid line).

For a given μ, the value of $\delta/2$ that gives the maximum growth rate can be computed by using (3.13). These values are indicated by the dashed line in Fig. 3. The corresponding values of Δ are given by the solid line in the same figure. Inspection of Fig. 3 reveals that, for the most unstable waves, Δ is very close to 0.5 when μ is greater than 1.25. Then we can anticipate that, when looking for the most unstable waves, we will find longer wavelengths when the bottom slope is positive and shorter ones when it is negative. These considerations apply to the interpretation of the different regions in Figs. 1 and 2.

Contours of the ratio between the slope of the projection of the trajectories on the meridional plane (w/v) and the slope of the isentropes of the basic state are plotted as a function of the phase of the wave in Fig. 4 for the most unstable case corresponding to (a) $\delta = -0.4$, (b) $\delta = 0.0$ and (c) $\delta = 0.4$. The shaded areas indicate regions where this ratio is negative or greater than 1. For the typical Eady case ($\delta = 0$), we see that the vertical and meridional velocities are in phase at $z = 0.5$. Therefore, $\overline{\phi w}$

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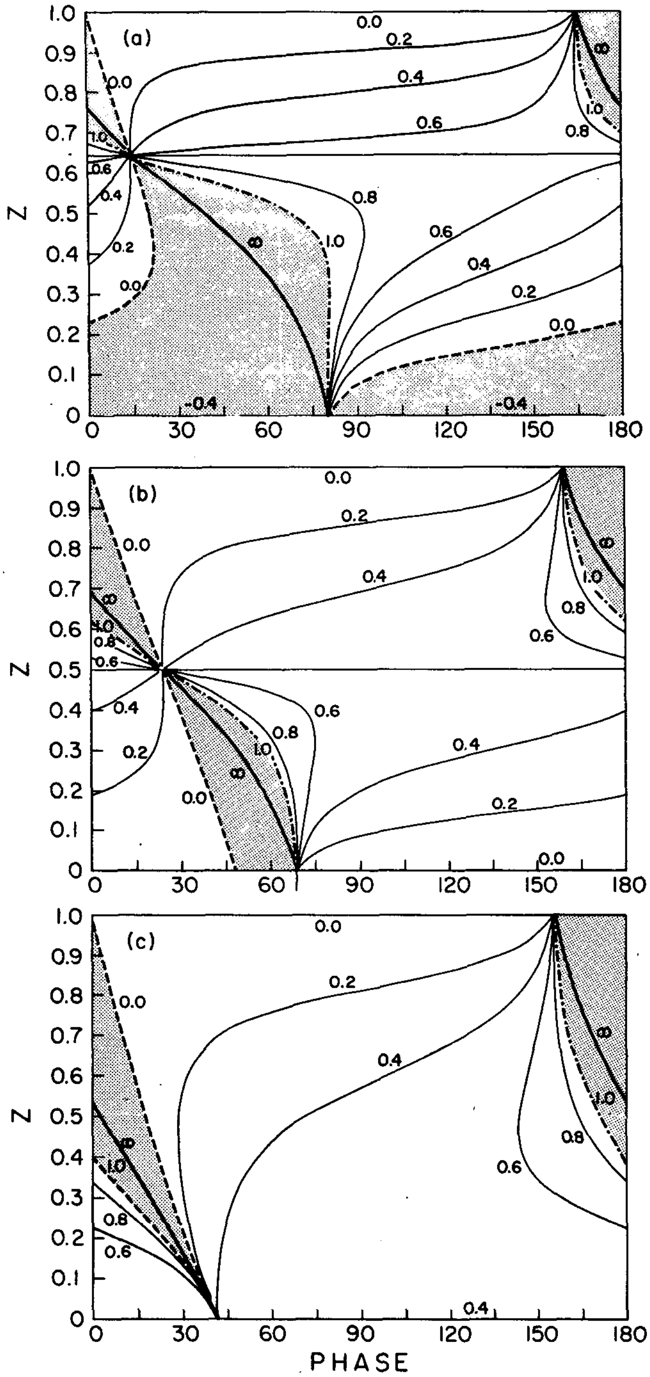

FIG. 4. Contours of the ratio between the slopes of the projection of the trajectories on the meridional plane (w/v) and the slope of the isentropes of the basic state for the most unstable wave when (a) $\delta = -0.4$, (b) $\delta = 0.0$ and (c) $\delta = 0.4$.

$= 0$ at that level and hence the vertical transport of energy vanishes there. The values of $\overline{\phi w}$ for the three cases mentioned above are plotted as a function of height in Fig. 5. The sign of the vertical energy flux convergence $\partial(\overline{\phi w})/\partial z$ indicates that the region of the fluid next to the lower boundary is a sink of energy for the perturbation when $\delta = -0.4$ and $\delta = 0$. Such energy flux convergence is almost zero at the lower boundary when $\delta = 0.4$. To analyze such behavior of the vertical energy flux at the lower boundary we consider the way in which this boundary constrains the trajectories over it. For $0 < \delta < 1$ the sloping bottom forces the parcels to move at the angle appropriate for the conversion of available potential energy of the mean flow to kinetic energy of the perturbation to take place. The conversion increases with δ until a certain slope is reached. Except for very long waves, this slope is close to the bisector of the angle that the isentropes of the basic state form with the horizontal. A further increase of δ decreases the energy conversion until it vanishes for $\delta = 1$. For $\delta < 0$, however, no release of energy from the mean flow to the perturbation takes place at the lower boundary. From the arguments given above, we may expect that the most unstable wave in this generalization of the Eady problem in which the bottom slope is allowed to vary, will be found for a positive value of δ and its wavenumber will be smaller than the one obtained in the horizontal bottom case.

The most unstable wave is found by solving the system

$$\frac{\partial \eta_i}{\partial \mu} = \frac{\partial \eta_i}{\partial \delta} = 0. \qquad (4.2)$$

The condition for the wave to have zero vertical energy flux convergence at $z = 0$ is

$$\delta = \frac{\mu}{\tanh \mu} - 1. \qquad (4.3)$$

It is simple to show that the particular values of δ and μ that satisfy (4.2) also satisfy (4.3). Fig. 6 illustrates this property. Hence, for the most unstable wave

$$\Delta = \delta, \qquad (4.4)$$

according to (4.1) and (4.3).

5. The general case when both δ_B and δ_T are allowed to vary

In this case, the solution of the system (4.2) is

$$\delta_B = \delta_T = 0.5.$$

Therefore, the maximum growth rate is reached when the bottom and top boundaries are along the bisector of the angle that the isentropes of the basic state form with the horizontal. However, in this case

$$\mu = 0$$

and the wave becomes infinitely long. The parcels are forced by the boundaries to follow the optimum trajectories for extracting potential energy from the mean flow.

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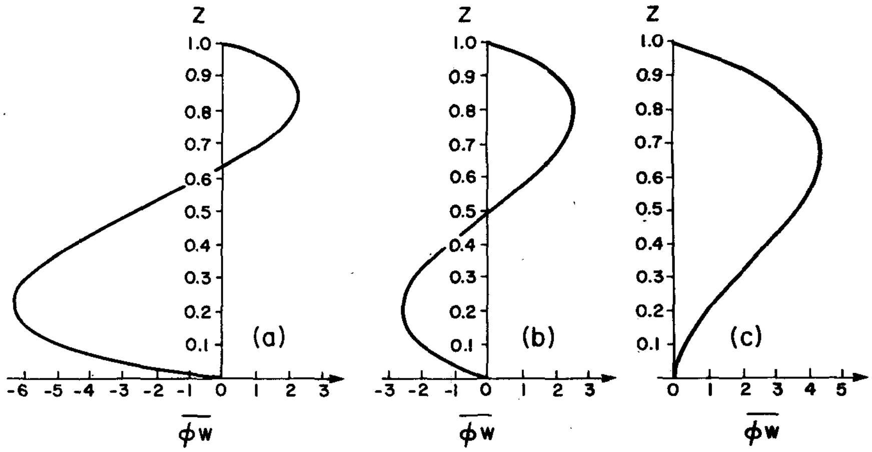

FIG. 5. The vertical transport of energy for the most unstable wave when (a) $\delta = -0.4$, (b) $\delta = 0.0$ and (c) $\delta = 0.4$.

6. Concluding remarks

The ratios of the slopes of the boundaries to that of the interface between the two fluid layers in the two-layer model or that of the isentropes in Eady's model appear to be fundamental to the stability of these systems. No instabilities are possible when either one of those ratios is greater than 1. We have analyzed the energetics for the Eady model with sloping bottom and top boundaries and found that no instabilities are possible when Δ, the ratio between the horizontal and vertical heat transports by the wave divided by the slope of the isentropes of the basic state, is greater than one or smaller than zero. We have described the relation between Δ and $\delta = \delta_B$ for the case with $\delta_T = 0$. According to this relation, Δ is a function of the wavenumber plus $\delta/2$. Therefore, the bottom slope directly affects the slope of the heat transport. This explains why the system is stabilized when the absolute value of δ is sufficiently large. It also explains why waves shorter than the short-wave cutoff are destabilized by a negative sloping bottom. Finally, the relation between Δ and δ describes the variation of the wavelength of the most unstable wave with the bottom slope.

We feel that this interpretation of the influence of the topography on the linear stability of baroclinic waves is clearer than others which make reference to the slope of the Eulerian parcel trajectories. This field has a complicated pattern, as shown in Fig. 4. The consideration of the geometric forcing at the boundaries is important because of its influence on the vertical energy flux convergence in adjacent regions. When the bottom slope is positive (and smaller than the slope of the isentropes of the basic state), it forces overlying parcels to move in such a way that available potential energy from the mean flow is released to the perturbation. Thus, the most unstable wave is found for a positive value of the bottom slope. However, the energy conversions in other regions of the fluid also change when compared to the horizontal bottom case. The result is that the most unstable wave has zero vertical energy flux convergence at the sloping bottom.

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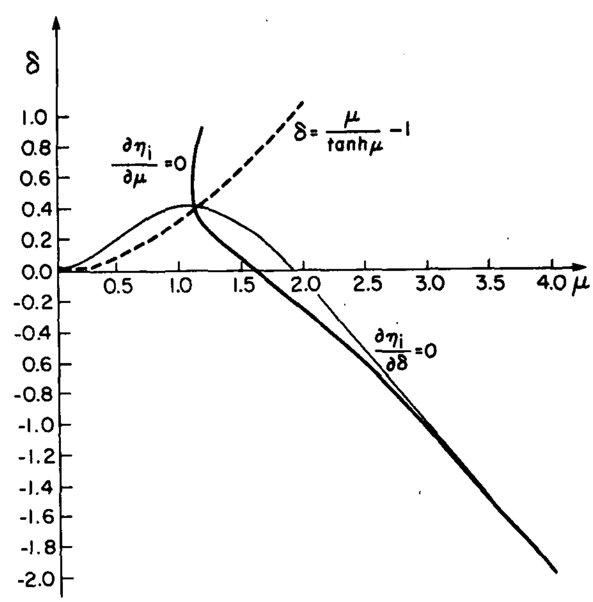

FIG. 6. Determination of the most unstable wave for the Eady model with sloping bottom.

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