The Boundary-Layer Growth Equation with Reynolds Averaging

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ABSTRACT

A local interfacial boundary condition between a turbulent and non-turbulent layer is constructed in a manner that allows local entrainment. The Reynolds average of this equation produces a mean boundary-layer growth rate that contains additional terms not heretofore included in boundary-layer growth models. One of these terms represents the net effect of fluctuating horizontal entrainment and can be absorbed into the usual net entrainment term. The other two terms may become important in horizontally inhomogeneous boundary layers: one represents mean horizontal entrainment across an interface which slopes in the mean, and the other is a divergence of a small-scale horizontal flux of boundary-layer height which presumably acts to smooth variations in the mean boundary-layer height.

1. Introduction

At the interface between two miscible fluids, the lower fluid (I) generally being denser than the upper fluid (II), the well-known condition which links the mean growth rate of fluid I to the mean entrainment rate \( W_e \) is (e.g., see Carson, 1973):

\[
\partial h/\partial t + \tilde{v}(\tilde{h}) \cdot \nabla \tilde{h} = \tilde{w}(\tilde{h}) + W_e, \tag{1}
\]

where \( \tilde{v}(\tilde{h}) \) is the average fluid velocity at the mean interface height \( \tilde{h}(x,y,t) \), and \( \tilde{w}(\tilde{h}) \) is the mean vertical velocity of the fluid at the mean interfacial height. If the lower fluid is turbulent and the upper fluid non-turbulent, \( W_e \) is taken to be positive when fluid II entrains into fluid I. In (1), the overbar average is considered to be either an ensemble average or a running horizontal average over an area which may be rather limited in extent, as in a mesoscale model. Hence, \( \tilde{h} \) can be a function of horizontal coordinates \( x, y \) as well as time \( t \) and the horizontal advection of \( h \) therefore appears on the left side of (1). For simplicity, no detrainment of fluid I into fluid II is assumed to take place.

The question which this note addresses is how (1) can be derived from a more fundamental relationship before Reynolds averaging has been applied. Without such a derivation, Eq. (1) may seem to be ad hoc and without sufficient physical basis.

2. The kinematic interfacial condition without entrainment

In the absence of entrainment, the kinematic interfacial condition between two immiscible fluids is well known (Haurwitz, 1951) and is given by

\[
\partial h/\partial t + v(h) \cdot \nabla h = w(h), \tag{2}
\]

where \( v(h) \) is the fluid velocity at \( z = h(x,y,t) \). This condition applies locally, provided that \( h \) is single-valued in \( z \) and continuous in \( (x,y,t) \). Condition (2) expresses the assumption that in the absence of entrainment, a fluid particle initially at the local interface between the two fluids remains there, i.e., that the interface is a material surface. If free slip is allowed to occur between fluids I and II, Eq. (2) can be written twice, once in which \( v(h) \) is labeled \( v_I(h) \), and once in which it is labeled \( v_II(h) \). Since in almost all geophysical applications fluid viscosity is present, however, and prevents any abrupt slippage, this distinction will not be treated here.

Eq. (2) also states that the interface is locally carried along by the fluid velocity. Thus, Eq. (2) does not permit any of fluid II to cross the interface into fluid I, so it does not permit entrainment. Note that only the horizontal components of \( v(h) \) participate in the dot product in (2) since \( \partial h/\partial z = 0 \).

3. The interfacial condition with entrainment

In order that entrainment occur, the interface must be such that fluid II particles can be considered to have penetrated it at certain times and places and, subsequently, to have become transformed (largely through molecular diffusion) into fluid I. A schematic picture of this process is shown in Fig. 1, where regions of entrainment through the interface are indicated by dotted interfacial lines. These redefined interfacial regions must be inserted in order that the local interface not be required to follow thin sheets or convoluted ribbons of fluid II far down into the body of fluid I. This redefinition is also necessary if \( h(x,y,t) \) is to remain a single-valued, continuous function, and if the interfacial area is to remain finite. In these local regions, particles of fluid II move downward across the redefined interface and do not carry the interface with them. Thus (2) does not apply.

For purposes of the present analysis, it is not important precisely how the artificially imposed re-
4. The Reynolds average of the interfacial condition

The Reynolds average of (5) is

\[
\frac{\partial h}{\partial t} + \overline{v(h)} \cdot \nabla h = -\overline{v_e} \cdot \nabla h
+ \overline{w(h) + \overline{w_e}},
\]

(6)

where, following Reynolds' averaging procedures, we have expanded

\[
\overline{v_e} = \tilde{v}_e + \tilde{v}_e' \bigg| \quad h = \tilde{h} + h',
\]

and assumed

\[
\overline{\tilde{v}_e} = 0 = \overline{h'}.
\]

Before similarly expanding \(\overline{v(h)} \cdot \nabla h\) in (6), however, let us examine \(\overline{w(h)}\), which differs from \(\overline{w(h)}\) of (1). To see this, we consider the identity

\[
w(h) = \overline{w(h)} + \int_{\tilde{h}^h}^h (\partial w/\partial z) dz.
\]

(7a)

Upon utilizing the incompressible approximation to the equation of continuity

\[
\frac{\partial w}{\partial z} = -\nabla h \cdot \nu,
\]

(7b)

where \(\nabla h \cdot \nu\) is the horizontal divergence operator, and using Leibniz's rule for interchanging the order of the derivative and integral when the latter has variable limits, we find that (7a) becomes

\[
w(h) = \overline{w(h)} + \int_{\tilde{h}}^h \nu dz
+ \overline{v(h)} \cdot \nabla h - \overline{v(h)} \cdot \nabla \tilde{h}.
\]

(7c)

The Reynolds average of (7c) is

\[
\overline{\overline{w(h)}} = \overline{\tilde{w}(h)} + \overline{\overline{v(h)}} \cdot \nabla \tilde{h} - \overline{\overline{v(h)}} \cdot \nabla \tilde{h}.
\]

(7d)

where

\[
J = \int_{\tilde{h}}^h \nu dz.
\]

(7e)

The last term in (7d), which vanishes under conditions of horizontal homogeneity, will be discussed later.

We may most simply examine \(\overline{\overline{v(h)}} \cdot \nabla \tilde{h} - \overline{\overline{v(h)}} \cdot \nabla \tilde{h}\) on the right of (7d) for a case in which the averaging domain is so large that \(\nabla h^\infty \approx 0\). Then \(\overline{\overline{v(h)}} \cdot \nabla \tilde{h} \approx -\tilde{h} \nabla \tilde{h} \cdot \overline{v(h)}\). It is further deduced that \(\tilde{h} \nabla \tilde{h} \cdot \overline{v(h)} > 0\) and is of significant magnitude relative to \(\overline{\tilde{v}_e}\), since occurrence of domes or hummocks at the top of a boundary layer, where \(\tilde{h}\) is greater than average, is positively correlated with horizontal divergence or spreading of the domes. Similarly, near cusps or wedges where entrainment occurs, \(\tilde{h}\) tends to be smaller and horizontal convergence tends to
FIG. 2. Depiction of the local entrainment velocity \( v_e \). In the regions of entrainment the flow crossing from fluid II into fluid I has a velocity directed generally opposite from \( v_e \).

In this example the overbar averaging scale would be assigned a length greater than that of the region shown and \( \bar{u}_e \) is considered to be zero; thus \( u_e = u_e' \).

occur. These correlations also follow from Phillips' (1972) observation that mixed-layer hummocks or domes are regions where interfacial area is created, and the cusps or wedges are regions where it is destroyed. Thus, \( \bar{w}(\bar{h}) \) in (7d) is significantly smaller or more negative than \( \bar{w}(\bar{h}) \).

Substitution of (7d) into (6) produces

\[
\frac{\partial \bar{h}}{\partial t} + \bar{v}(\bar{h}) \cdot \nabla \bar{h} = \bar{w}(\bar{h}) + \bar{w}_e - \bar{v}_e \cdot \nabla \bar{h} - \nabla \cdot \bar{J}.
\]  

Before comparing (8) with (1), we note that the term \( -\bar{v}_e \cdot \nabla \bar{h}' \) always gives a positive contribution to \( \partial \bar{h}/\partial t \); when \( \partial \bar{h}/\partial x < 0, u_e' > 0 \) (see Fig. 2), and vice versa, by virtue of the entrainment velocity being directed normal to the local interface and outward from I to II. One interpretation of (8), then, is that the boundary-layer growth rate is enhanced by fluctuating horizontal entrainment associated with increased interfacial slopes and increased interfacial surface area. This interpretation has been made by Townsend (1976, p. 232) and by Mahrt (1979). It is certainly valid if it is possible to distinguish \( \bar{w}_e \) from the horizontal entrainment.

Another interpretation is that only the net entrainment of fluid II into fluid I matters, where this net entrainment \( \bar{W}_e \) is

\[
\bar{W}_e = \bar{w}_e - \bar{v}_e \cdot \nabla \bar{h}'.
\]  

Only this net value can be estimated experimentally or observationally. Moreover, physical processes that serve to increase \( \bar{w}_e \) also serve to increase interfacial slopes and \( |v_e'| \) concurrently. Thus, Eq. (9) will be used here, for the additional reason that \( \bar{W}_e \) must be unaffected by the details of how the local interface is defined in regions of entrainment and near-infinite slopes, whereas \( \bar{w}_e \) and \( -\bar{v}_e \cdot \nabla \bar{h}' \) individually depend on such details. Thus (8) may be rewritten

\[
\bar{h}/\partial t + [\bar{v}(\bar{h}) + \bar{v}_e] \cdot \nabla \bar{h} - \bar{v}_e \cdot \nabla \bar{h} - \nabla \cdot \bar{J}.
\]  

The mean entrainment vector \( \bar{v}_e \) in (10) usually has a horizontal component much less in magnitude than the mean velocity at \( \bar{h} \). It is thus understandable that the term is neglected in the meteorological literature, as in (1). An instance in which it may not be entirely negligible, however, is along a sloping internal boundary layer, with air above a shallow, overwater boundary layer passing into the boundary layer above heated ground. Then \( |\nabla \bar{h}| \) is significant, as may also be \( |\bar{v}_e| \) if the mean wind is weak. This situation is depicted in Fig. 3.

In the last term in (10), \( \bar{J} \) may be rewritten, from (7e), as

\[
\bar{J} = v(h_{\ast})h'
\]  

by the mean value theorem, where \( h_{\ast} \) lies between \( \bar{h} \) and \( h \). Therefore

\[
\bar{J} = v'(h_{\ast})h',
\]  

where

\[
v'(h_{\ast}) = v(h_{\ast}) - \bar{v}(h_{\ast}).
\]  

We now notice that the term \( \nabla \cdot \bar{J} \) in (7d) is the divergence of a small-scale horizontal flux of \( h \). Upon invoking the classical downgradient diffusion hypothesis, \( \bar{J} \) would be expressed as

\[
\bar{J} = -K \nabla \bar{h},
\]  

where \( K \) is a positive eddy coefficient. A reason for suggesting that \( K \) may be positive lies in sus-
5. The shallow-water height equation

Instead of (10), the shallow-water height equation is often used to predict $\bar{h}$. To see the connection between the two approaches, we integrate the mean incompressibility condition [the Reynolds average of (7b)] vertically from a variable lower surface at $z = \bar{z}_s(x,y)$ to $\bar{h}(x,y,t)$:

$$\int_{\bar{z}_s}^{\bar{h}} \nabla_h \cdot \bar{v} dz = -\bar{w}(\bar{h}) + \bar{w}(\bar{z}_s).$$  \hspace{1cm} (12a)

Upon interchange of the derivative and integral through use of Leibniz's rule, Eq. (12a) becomes

$$\nabla_h \cdot \int_{\bar{z}_s}^{\bar{h}} \bar{v} dz = -\bar{w}(\bar{h}) + \bar{w}(\bar{z}_s).$$  \hspace{1cm} (12b)

If the vertically integrated mean wind is designated by

$$\bar{v}_m = (\bar{h} - \bar{z}_s)^{-1} \int_{\bar{z}_s}^{\bar{h}} \bar{v} dz$$

and if the lower surface is a material surface in the mean

$$\bar{w}(\bar{z}_s) = \bar{v}(\bar{z}_s) \cdot \nabla \bar{z}_s,$$

then (12b) becomes

$$\nabla \cdot [(\bar{h} - \bar{z}_s)\bar{v}_m] - \bar{v}(\bar{h}) \cdot \nabla \bar{h} = -\bar{w}(\bar{h}).$$  \hspace{1cm} (12c)

Addition of (12c) and (10) produces

$$\partial \bar{h}/\partial t + \nabla \cdot [(\bar{h} - \bar{z}_s)\bar{v}_m] = \bar{W}_e - \bar{v}_e \cdot \nabla \bar{h} - \nabla_h \cdot \bar{J}.$$  \hspace{1cm} (13)

Except for the last two terms on the right, which may often be negligible, Eq. (13) is the usual shallow-water fluid-depth equation allowing for mean entrainment.

6. Discussion and summary

The derivations of (10) and (13) indicate how the local interface condition (5) can form the starting point from which the Reynolds-averaged expressions containing net entrainment emerge. One extra term on the right of the resulting equations represents the effect of mean horizontal entrainment when the mean interface slopes. An additional term, $-\nabla_h \cdot \bar{J}$, is also obtained; it is the divergence of a small-scale horizontal flux of $h$.

In the derivation of (10) the net entrainment [Eq. (9)] is found to be represented by two terms, one the usual (vertical component of) entrainment and the second the contribution from local horizontal entrainment where the interface slopes locally. For convenience, the two contributing terms have been lumped into one entrainment term.

It is recognized that in (5) the local entrainment velocity $v_e$ cannot be defined precisely and unequivocally at each point on the interface, since the positioning of the latter is somewhat arbitrary in regions of entrainment. However, this difficulty does not carry over to the Reynolds-averaged results (10) and (13).

It is suggested that the additional term, $-\nabla_h \cdot \bar{J}$ in (10) and (13), may be represented by $\nabla_h \cdot K \nabla h$, where $K$ is an eddy coefficient. Such a smoothing term in computations of $h(x,y,t)$ has been postulated by Deardorff (1972) and others, but has not previously been justified from first principles.

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