

Convergence Rate and Stability of Ocean-Atmosphere Coupling Schemes with a Zero-Dimensional Climate Model

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ABSTRACT

A zero-dimensional climate model is considered with three thermal reservoirs, i.e., the atmosphere, the surface mixed layer and the intermediate water of the ocean. Realistic values are adopted for the rates of heat transfer between those reservoirs. If heat is added suddenly to the atmosphere, the atmospheric temperature increases a small amount in a few days. Thereafter, the atmosphere and mixed layer increase in temperature. About one-half the mixed-layer response occurs on a time scale of two years and the rest on a time scale of about 100 years. Numerical solution of atmosphere-ocean general circulation models may require asynchronous coupling strategies to link the two models. A semi-implicit approach is considered which generalizes previously suggested schemes. Its convergence and stability are examined by application to the zero-dimensional climate model. It may be unstable if it is made too explicit. On the other hand, the fully implicit approach greatly slows down the response of the mixed layer unless a very short coupling interval is used. With some fraction about equal to 0.05 of the atmosphere-ocean heat transfer made implicit, the asynchronous coupling solutions are close to the correct solution even if a large coupling interval is used.

1. Introduction

One of the fundamental problems of climate modeling is the determination of the response of the climate system to changes in global external forcing. Of particular interest is the change of the global surface temperature in response to changes in the global heat balance as might be the result of a changing solar constant or changes in the concentrations of radiatively active gases such as CO₂. This problem is being studied using a hierarchy of modeling approaches (as reviewed, e.g., by Schneider and Dickinson, 1974; Ramanathan and Coakley, 1978; North *et al.*, 1981).

The present note is intended to discuss the properties of the simplest climate model that distinguishes separately between the atmosphere, the ocean surface mixed layer, and the deep ocean. There is currently much effort being devoted to the utilization of the most elaborate of such models (i.e., atmosphere-ocean general circulation models or AOGCM's) for climate studies. These efforts have been reviewed recently by Manabe *et al.* (1979). One of the key issues in integrating such models is the optimum coupling strategy to be used for energy exchanges between the atmosphere and oceans. This issue is primarily one of computational efficiency and arises because the atmospheric models require much more computer time per year of integration than do the

ocean models, whereas the ocean models require a much longer time to reach equilibrium than do the atmospheric models. Schlesinger (1979) has briefly reviewed the coupling strategies that have been applied up to now and has called for further research on this issue. Thus the present note investigates the elementary numerical consequences of various coupling strategies that might be adopted for AOGCM's.

To do this, we introduce in Section 2 a global-average two-layer ocean model, coupled to a "zero-dimensional" atmospheric model. Model parameters are chosen to be as realistic as possible. Section 3 describes the behavior of this system through an approximate analytic solution. Next in Section 4, the atmosphere and ocean mixed layer components are solved separately and linked by means of an asynchronous coupling procedure. This analysis is intended to illustrate the nature of the numerical errors inherent in such a procedure and how these might be minimized. Two separate issues are 1) the distortion of the model time evolution, and 2) changes in the model time required to achieve steady state. Conclusions inferred from the analysis are given in Section 5.

2. Formulation

Our current understanding of the steady-state global temperature change ΔT due to a change in the external heating $\Delta \hat{Q}$ can be summarized in terms of

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the extremely simple “zero-dimensional” climate model

$$\Delta T = \Delta \hat{Q} / \hat{\lambda}. \tag{1}$$

The feedback parameter $\hat{\lambda}$ describes the change of outgoing thermal and reflected solar radiation at the top of the atmosphere (or more precisely at the tropopause), and has the likely value $\hat{\lambda} = 1.7 \pm 0.8 \text{ W m}^{-2} \text{ K}^{-1}$ (NAS, 1979).

The time-dependent generalization of (1) is

$$C \frac{\partial T}{\partial t} + \hat{\lambda} T = \Delta \hat{Q}, \tag{2}$$

where C is the heat capacity of the atmosphere-ocean climate system, and since all variables represent perturbations, we have dropped the Δ 's. One conceptual difficulty in applying (2) is that the climate system has a number of thermal reservoirs with differing heat capacities. Consequently, the most appropriate value of C depends on the time scale of the problem. On time scales of years to decades, we must consider the heat capacity of the mixed layer and intermediate layer waters of the oceans. For a model which distinguishes between atmospheric and oceanic temperatures, the T multiplying $\hat{\lambda}$ in Eq. (2) should be some weighted average of the atmospheric and ocean surface temperatures. Since, as we shall see, both these two temperatures closely follow each other, it is a reasonable approximation to simply use atmospheric temperature in this term. The simplest appropriate model can then be written

$$C_a \frac{\partial T_a}{\partial t} + \lambda_{am}(T_a - T_m) + \lambda T_a = \Delta Q, \tag{3}$$

$$C_m \frac{\partial T_m}{\partial t} + \lambda_{am}(T_m - T_a) + \lambda_{mo}(T_m - T_o) = 0, \tag{4}$$

$$C_o \frac{\partial T_o}{\partial t} + \lambda_{mo}(T_o - T_m) = 0, \tag{5}$$

where the subscripts a, m, o refer to atmosphere, mixed layer of the ocean, and ocean intermediate water, respectively. Since we are concerned with the ocean which covers 0.7 of the earth's surface, the atmospheric heat capacity, feedback parameter $\hat{\lambda}$ and heat input $\Delta \hat{Q}$ have been multiplied by 1.4; i.e., $\lambda = 1.4\hat{\lambda} = 2.4 \text{ W m}^{-2} \text{ K}^{-1}$ and $\Delta Q = 1.4\Delta \hat{Q}$ to represent parameters per unit ocean area. The λ 's with subscripts represent exchange terms. For the present purposes, it is convenient to measure time t in years, and heat capacities in units of $\text{W m}^{-2} \text{ year}^{-1} \text{ K}^{-1}$. Unit heat capacity corresponds to 7.8 m water using a density of 1.035 that of fresh water and a heat capacity for sea water of 0.933 that of fresh water (Ekman, 1914). It is generally believed that only the first 500–1000 m of ocean exchange a significant amount of heat with the atmosphere on the decadal

TABLE 1. Model layers and assumed physical properties. (Since the model is formulated for unit ocean area, the atmospheric mass has been multiplied by the ratio of total global area to ocean area.)

Layer	Mass ($\times 10^3 \text{ kg m}^{-2}$)	Heat capacity ($\text{W m}^{-2} \text{ year K}^{-1}$)
Atmosphere	14	$C_a = 0.45$
Ocean mixed layer	81	$C_m = 10$
Ocean intermediate water	807	$C_o = 100$

time scale. Hence we adopt the values for model layer masses given in Table 1.

The ΔQ has been assumed to be entirely in the atmospheric equation for simplicity. More generally some fraction would go directly into the mixed layer. However for realistic parameters where the atmosphere responds to the mixed layer rapidly compared to the mixed layer response time this generalization would not significantly change any of the results of the following analysis.

Energy is exchanged across the atmosphere-ocean interface through solar and thermal radiative fluxes and fluxes of sensible and latent heat. The latter two terms are often parameterized in terms of the air-sea temperature difference and represent the bulk of the temperature-dependent energy exchange at this interface. Solar fluxes are insensitive to temperature, upward thermal radiation F_u depends on T_m and downward F_d on T_a . For small changes $\Delta F_u = (\partial F_u / \partial T_m) \Delta T_m$ and $\Delta F_d = (\partial F_d / \partial T_a) \Delta T_a$ so for the form (3) and (4) to be valid, it is necessary that

$$\frac{\partial F_u}{\partial T_m} = \frac{\partial F_d}{\partial T_a}. \tag{6}$$

(a) (b)

Wetherald and Manabe (1975) give changes in $F_u - F_d$ for their sector model with solar constant change from which we infer that term (b) in (6) is $\sim 15\%$ larger than term (a). Hence, equality is a reasonable first approximation.

If we did not impose (6), we would have to add to (3) and subtract from (4) a term of the form

$$\left(\frac{\partial F_u}{\partial T_m} - \frac{\partial F_d}{\partial T_a} \right) \Delta T,$$

where the T in ΔT is T_a or T_m or some combination, depending on the precise definition of λ_{am} . We have confirmed by numerical integration that this leads only to small changes in the second decimal place. As formulated, Eqs. (3)–(5) have equilibrium solutions $T_a = T_m = T_o$ for the perturbation temperatures. More general formulations with additional terms such as just mentioned would no longer retain this convenient property.

TABLE 2. Energy exchange coefficients (per unit ocean area).

Parameter	Definition	Assumed value (W m ⁻² K ⁻¹)
λ	Climate feedback parameter due to atmospheric radiative damping	1.7/0.7 = 2.4
λ_{am}	Atmosphere-mixed layer exchange	45
λ_{mo}	Mixed layer-intermediate water exchange	2(0.5, 8)

From some limited sensitivity calculations with an NCAR GCM, we infer that for a global-average area weighted over oceans

$$\frac{\partial F_u}{\partial T_m} \approx 6 \text{ W m}^{-2} \text{ K}^{-1},$$

and for the same conditions,

$$\frac{\partial F_{\text{SEN}}}{\partial T_m} = 11 \text{ W m}^{-2} \text{ K}^{-1},$$

$$\frac{\partial F_{\text{LAT}}}{\partial T_m} \approx 28 \text{ W m}^{-2} \text{ K}^{-1},$$

where subscripts SEN and LAT refer to sensible and latent fluxes, respectively. Hence we infer $\lambda_{am} \approx 45 \text{ W m}^{-2} \text{ K}^{-1}$.

The energy exchange between the mixed layer and deeper ocean waters occurs through the exchange of water, e.g., winter convection forming intermediate waters and Ekman pumping (NAS, 1979). Hence, the parameterization in terms of the exchange coefficient λ_{mo} is not physical and at best can provide the effect of time scale. The time scale for the exchange of intermediate waters with the surface can be inferred from radioactive tracer data (e.g., Broecker *et al.*, 1979) which can be modeled with a λ_{mo} in the range 1–4 W m⁻² K⁻¹. The value 4 comes directly from the observed tritium penetration of 284 m in 10 years. The value 1 is inferred on the assumption of diffusive transfer, i.e., for a diffusion equation with constant diffusion coefficient, time scale \approx depth of penetration squared divided by the diffusion coefficient, so for a reservoir twice as large (i.e., 600 m), the time scale and hence λ^{-1} is 4 \times as large. We shall take $\lambda_{mo} = 2 \text{ W m}^{-2} \text{ K}^{-1}$ as a standard value and also consider 0.5 W m⁻² K⁻¹ as a slow value and $\lambda_{mo} = 8 \text{ W m}^{-2} \text{ K}^{-1}$ as a fast value. The values adopted for energy exchange coefficients are shown in Table 2.

The system Eqs. (3)–(5) is sufficiently obvious that no claim for originality can be made. In particular, ocean geochemists are quite familiar with essentially the same model applied to, e.g., CO₂ (cf. Broecker *et al.*, 1979). Thompson and Schneider

(1979) have considered an analogous, but somewhat more complicated, climate model. Nevertheless, it is helpful to examine some aspects of the solution of such a simple system as preparation for considering a possible coupling approach for AOGCM's. This exercise can also help to clarify the nature of the transient response of the global-average ocean to climate perturbations.

3. Analytic solution

In this section we develop analytic solutions to the atmosphere-ocean climate model defined in the previous section. In doing so, we provide a framework for consideration of possible iterative approaches to their solution.

Eqs. (3)–(5) as a linear system of equations can in principle be solved exactly. However, the nature of these solutions is much more clearly seen through an asymptotic approach that takes advantage of the smallness of the parameters $\delta = C_a/C_m$ and $\epsilon = C_m/C_o$, i.e., the heat capacity of the atmosphere is much less than that of the mixed layer which in turn is much less than that of the intermediate ocean waters (see Table 1).

For a specific example, assume a perturbation heating ΔQ is switched on at $t = 0$ or T_m is made nonzero. Then Eq. (3) has the approximate solution

$$T_a = \frac{(\Delta Q + \lambda_{am} T_m)}{\lambda + \lambda_{am}} [1 - \exp(-\alpha t)], \quad (7)$$

where $\alpha = (\lambda + \lambda_{am})/C_a \approx 100 \text{ year}^{-1}$ (or $\alpha^{-1} = 4$ days). Thus, T_a responds in order of a few days to heat provided either directly (ΔQ) or from the mixed layer ($\lambda_{am} T_m$). Note that for $T_m = 0$ and $\Delta Q = 1 \text{ W m}^{-2}$, T_a is small; $T_a = 0.02 \text{ K}$ as $t \rightarrow \infty$ in (7), or $\sim 4\%$ of its ultimate response given by (1). For time scales of one week or longer, we take the term in brackets in (7) equal to unity, eliminate T_a in Eq. (4) and write Eqs. (4)–(5) as

$$\left. \begin{aligned} \frac{\partial T_m}{\partial t} + \gamma(T_m - T_o) + \sigma T_m &= \frac{\sigma \Delta Q}{\lambda} \\ \frac{\partial T_o}{\partial t} + \epsilon \gamma(T_o - T_m) &= 0 \end{aligned} \right\}, \quad (8)$$

where

$$\gamma = \lambda_{mo}/C_m,$$

$$\epsilon = C_m/C_o,$$

$$\sigma = \frac{\lambda \lambda_{am}}{C_m(\lambda + \lambda_{am})}.$$

Note that $\gamma \approx \sigma \approx 0.2$ with inverse time being measured in units of per year and $\epsilon \approx 0.1$. We assume ϵ is a small parameter, and introduce the slow time scale, $\tau = \epsilon \sigma t$. We infer that on the time scale σt

$$\frac{\partial T_m}{\partial t} + (\gamma + \sigma)T_m \approx \gamma T_o(\tau) + \frac{\sigma}{\lambda} \Delta Q, \quad (9)$$

and that on the slow time scale

$$T_m(\tau) = (1 + \gamma/\sigma)^{-1}[\gamma/\sigma T_o(\tau) + \Delta Q/\lambda], \quad (10)$$

and that

$$\tilde{t}_2 \frac{\partial T_o}{\partial \tau} + T_o = \frac{\Delta Q}{\lambda}, \quad (11)$$

with $\tilde{t}_2 = (1 + \sigma/\gamma)$. Hence on the time scale $t_1 = 1/(\gamma + \sigma) \approx 2.3$ (in years), T_m reaches the balance with T_o given by (10), and on longer time scales (11) governs the joint decay of T_m and T_o . Thus for $t = O(t_1)$

$$\left. \begin{aligned} T_m &= \frac{\Delta Q}{\lambda(1 + \gamma/\sigma)} (1 - e^{-t/t_1}) \\ T_o &= 0 \end{aligned} \right\}, \quad (12)$$

and for $t \gg t_1$

$$\left. \begin{aligned} T_m &= \frac{\Delta Q}{\lambda(1 + \gamma/\sigma)} \left[\frac{\gamma}{\sigma} (1 - e^{-t/t_2}) + 1 \right] \\ T_o &= \frac{\Delta Q}{\lambda} (1 - e^{-t/t_2}) \end{aligned} \right\}, \quad (13)$$

where $t_2 = (\epsilon\sigma)^{-1}\tilde{t}_2 = \epsilon^{-1}(\sigma^{-1} + \gamma^{-1})$. The above approximate solutions require $t_1 \ll t_2$. Since $(\sigma + \gamma)^{-1} < \sigma^{-1} + \gamma^{-1}$, it is sufficient that $\epsilon \ll 1$ or equivalently that the mixed layer be in thermal contact with a reservoir of sufficiently greater heat capacity.

It is of interest to note the relationships between the mixed layer and the deeper layer time scales. We can write the time scale for the deeper layer as

$$t_2 = t_o + \frac{C_o}{C_m} t_{mr},$$

where

$$t_o = C_o/\lambda_{m0}$$

is the intrinsic deeper layer time scale,

$$t_{mr} = \sigma^{-1} = t_m + t_r$$

combines the mixed-layer and radiative times scales,

$$t_m = C_m/\lambda_{am}$$

is the time scale for heat exchange between the mixed layer and the atmosphere to act on the mixed layer, and

$$t_r = C_m/\lambda$$

is the time scale for atmospheric radiative damping to act on the mixed layer. The mixed layer damps on the time scale

$$t_1 = \left(t_{mr}^{-1} + \frac{C_o}{C_m} t_o^{-1} \right)^{-1}.$$

For the standard parameters, $t_m = 0.2$, $t_r = 4.2$, $t_o = 50$, $t_{mr} = 4.4$, $t_1 = 2.3$, $t_2 = 94$ (all measured in years). Note from Eqs. (12)–(13) that the ratio of steady state T_m on the t_1 time scale to steady state T_m on the t_2 time scale $= (1 + \gamma/\sigma)^{-1} = t_1/t_{mr} = 0.53$.

In summary, for the standard parameters of Table 2, on a time scale of $t_1 \approx 2$ years, about one-half the steady-state response of T_m occurs. The remaining response occurs on a time scale $t_2 \approx 100$ years according to the asymptotic approximation. The magnitudes of t_1 and t_2 receive about equal contributions from t_o and t_{mr} which are controlled, respectively, by the rate of water exchange between the mixed and intermediate ocean water, and by the atmospheric radiative damping rate. In other words, *the mixed layer responds twice as fast as it would without coupling to the deeper layer, and the deeper layer adjusts to the mixed layer only half as fast as it would without the atmospheric radiative damping.*

If there is sufficiently fast coupling between the mixed layer and intermediate water, the deeper layer time scale t_o primarily controls the t_1 time scale, whereas the t_2 time scale is controlled largely by the radiative damping time t_r ; e.g., if $t_o = 12$ years, then $t_2 = 56$ years, $t_1 \approx 1$ year and about 0.2 of T_m responds on the first (t_1) time scale. The converse holds for slow coupling between the mixed layer and intermediate water, e.g., if $t_o = 200$ years then $t_2 \approx 250$ years and $t_1 \approx 4$ years. Also, about 0.8 the response of T_m occurs on the t_1 time scale. In general, only for slow coupling to the deeper layer is the mixed-layer response (magnitude and time scale) controlled primarily by radiative damping. In this limit, the deeper layer subsequently slowly warms itself to the radiative equilibrium temperature by leakage of heat from the mixed layer. In the other limit of fast coupling to the deeper layer, after an initial rapid but small adjustment, the mixed layer becomes transparent to the heat added from the atmosphere, and the whole atmosphere-ocean system warms together until radiative damping balances the added heating.

Fig. 1 shows the numerical solution for T_m divided by its equilibrium value for the three values of λ_{m0} considered and for an uncoupled mixed layer. For $\lambda_{m0} = 0$, we see about 90% and for $\lambda_{m0} = 0.5$ about 80% of the mixed-layer response occurs in 10 years. For $\lambda_{m0} = 8$ about 35% of the response occurs in 10 years, whereas for the standard case of $\lambda_{m0} = 2$ about 55% occurs in 10 years. As a generality, it appears that for approximately the first 100 years, that slower transfer of heat to intermediate waters increases the magnitude of the response.

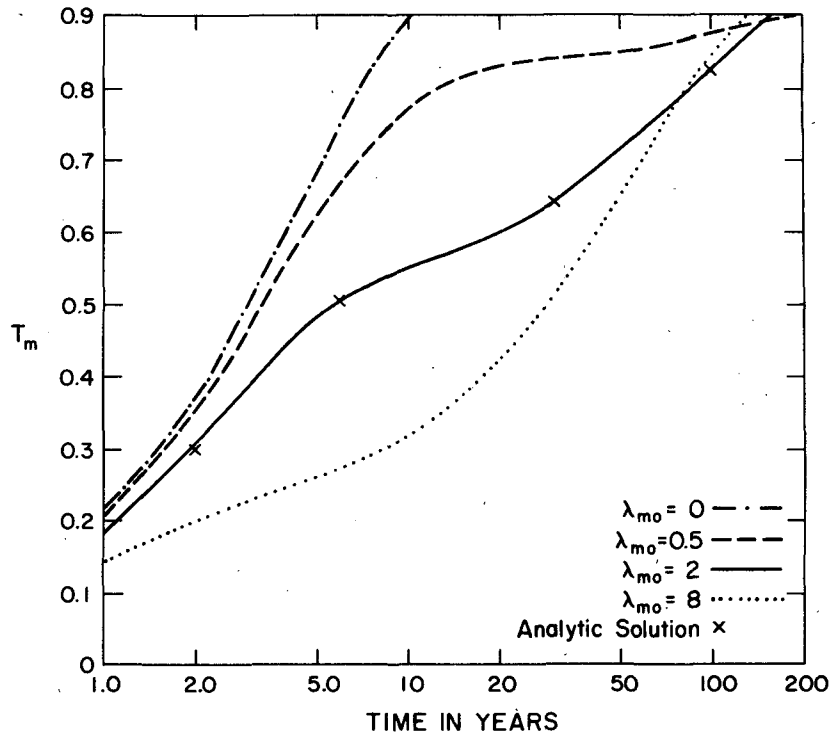


FIG. 1. Temperature of the mixed layer (divided by steady-state value) in response to a sudden shift in atmospheric heating. The value $\lambda_{mo} = 2 \text{ W m}^{-2} \text{ K}^{-1}$ represents the preferred transfer rate between the mixed layer and ocean intermediate waters. Values of 0.5 and 8 represent extreme lower and upper limits for very slow or very fast coupling. The \times 's indicate the analytic solution described in the text.

We also show by \times 's on the $\lambda_{mo} = 2$ curve values for the analytic solution $= 0.52 \times [1 - \exp(-0.43t)] + 0.48 \times [1 - \exp(-0.01t)]$. The agreement is seen to be quite satisfactory.

Finally, we note that the air-mixed-layer energy transfer rate λ_{am} essentially controls only the very short time scale response of the atmosphere as given by Eq. (7). On the longer time scales, the air-ocean temperature difference $T_a - T_m$ adjusts itself to translate the atmospheric radiative damping directly to the mixed-layer temperature anomaly.

4. Asynchronous coupling

For AOGCM's, it may be economical of computer time to run the atmospheric model to a statistical steady state with the ocean prescribed, then use the atmospheric model to provide boundary conditions to integrate the ocean model over some time interval Δt , and then iterate. This strategy (first used by Manabe and Bryan, 1969) is referred to as asynchronous coupling.

Denote by T_a^N and T_m^N the atmospheric and mixed layer temperatures after this procedure has been carried out N times. From Eq. (7), we have

$$T_a^N = \frac{\Delta Q + \lambda_{am} T_m^N}{\lambda + \lambda_{am}} \tag{14}$$

Eqs. (8) are then integrated forward in time to get T_m^{N+1} and T_o^{N+1} . One basic issue is the manner in which the air-mixed layer energy transfer H_{am} is evaluated. In the present context,

$$H_{am} = \lambda_{am}(T_a - T_m). \tag{15}$$

Manabe *et al.* (1979) suggest that for seasonal AOGCMs', H_{AM} be evaluated in the ocean part of the model since its evaluation in the atmospheric part, as done earlier by Manabe and Bryan (1969) in a nonseasonal model, leads to instability. We refer to these two possible approaches as fully implicit and explicit coupling, respectively, by analogue to the terminology use in developing finite-difference solutions to differential equations. In the context of (15) we generalize these possibilities through a partially implicit scheme,

$$H_{am} = \lambda_{am}[T_a^N - rT_m - (1 - r)T_m^N]. \tag{16}$$

Here $r = 1$ for the fully implicit scheme, i.e., H_{am} is evaluated in the ocean model; and $r = 0$ for the

explicit scheme, i.e., H_{am} is evaluated in the atmosphere model.

The problem is seen in simpler fashion if we take $\gamma \ll \sigma$, so that the intermediate waters decouple from the mixed-layer calculation. Then Eq. (4) is written

$$C_m \frac{\partial T_m}{\partial t} + \lambda_{am}(\hat{T}_m - T_a^N) = 0, \quad (17)$$

where $\hat{T}_m = rT_m + (1 - r)T_m^N$. Then Eq. (17) integrated over one ocean cycle becomes

$$T_m^{N+1} = T_m^N \exp(-\beta t^*) + r^{-1}[T_a^N + (r - 1)T_m^N] \times [1 - \exp(-\beta t^*)], \quad (18)$$

where we have introduced the parameters

$$\beta = \frac{r(\lambda + \lambda_{am})}{\lambda}, \quad (19a)$$

$$t^* = [\lambda\lambda_{am}/(\lambda + \lambda_{am})]\Delta t/C_m = \sigma\Delta t. \quad (19b)$$

After using (14) to eliminate T_a , we rearrange (18) to

ASYNCHRONOUS:

$$T_m^{N+1} = [1 - \beta^{-1}E(\beta t^*)]T_m^N + \frac{1}{\beta\lambda}E(\beta t^*)\Delta Q, \quad (20)$$

where $E(x) = 1 - \exp(-x)$. For comparison, the synchronous coupling solution to Eq. (3)–(4) for the same assumptions, i.e., $C_a \approx 0$, $\lambda_{mo} \ll \lambda$, can be written for the interval Δt as

SYNCHRONOUS:

$$T_m^{N+1} = [1 - E(t^*)]T_m^N + \lambda^{-1}E(t^*)\Delta Q. \quad (21)$$

The rate at which the solutions (20) or (21) converge to a steady state is examined in terms of the ratio

$$R = (T_m^{N+1} - T_m^N)/(T_m^N - T_m^{N-1}),$$

i.e.,

$$R_{\text{asyn}} = 1 - \beta^{-1}E(\beta t^*), \quad (22)$$

$$R_{\text{syn}} = 1 - E(t^*) = \exp(-t^*). \quad (23)$$

These convergence ratios are obviously identical if $\beta = 1$.

Note also the limiting cases: 1) $t^* \ll 1$, $\beta = O(1)$ whence

$$\left. \begin{aligned} R_{\text{asyn}} &= 1 - t^* + 1/2\beta t^{*2} + O(t^{*3}) \\ R_{\text{syn}} &= 1 - t^* + 1/2t^{*2} + O(t^{*3}) \end{aligned} \right\}; \quad (24)$$

and 2) $t^* \gg 1$

$$R_{\text{asyn}} = 1 - \beta^{-1} + o(t^{*-n}), \quad (25)$$

where the last term in (25) is an asymptotic remainder smaller than any polynomial in t^{*-1} .

We infer from (24) and (25) that for $\beta > 1$ and either small or large $\sigma\Delta t$, the asynchronous coupling

converges more slowly than the synchronous coupling. However, (25) also indicates that for large enough $\sigma\Delta t$ and $\beta < 1$, the transient part of the solution will oscillate from one iteration to the next. Indeed, if

$$\beta < 0.5E(\beta\sigma\Delta t) < 1/2, \quad (26)$$

the magnitude of R_{asyn} given by (22) exceeds unity, which means the iteration is unstable and will diverge.

To better compare the asynchronous coupling solution with the correct solution, it is useful to look at the ratio of the logarithms of the expressions given by (22) and (23), i.e., the ratio of actual to calculated decay time scale

$$q = \frac{-\ln R_{\text{asyn}}}{t^*}. \quad (27)$$

This quantity is shown in Fig. 2 versus β for several values of t^* . A value of $\beta = 20$ corresponds to the fully implicit coupling scheme. Values of q near unity are necessary for the calculated model transient behavior to conform to solutions of the original equations. Since the mixed layer has a decay time of several years, values of $q \ll 1$ imply the asynchronous coupling scheme would require many decades to reach a steady state. Note $t^* = 0.25$ corresponds to a coupling time interval of $\Delta t = 1.1$ years for the standard parameters so Fig. 2 suggests a fully implicit coupling scheme would require $\Delta t \ll 1$ year to satisfactorily model the mixed layer thermal response to an insertion of heat in the atmosphere. Conversely, by selecting β sufficiently close to unity, it should be possible to use coupling time intervals of up to several years without sacrificing realistic mixed layer temperature variations. Note that none of this discussion refers to time increments used to numerically integrate the ocean model by itself. The ocean model integration was done analytically here due to the simplicity of the model.

5. Discussion and conclusions

In the context of our simple ocean model, the asynchronous partially implicit coupling procedure, Eqs. (14)–(17), forms a simple difference-differential equation. This system has solutions which may be compared just as finite difference solutions are compared with solutions to the original differential equations (e.g., Gear, 1971). The three practical questions to be answered are (i) Does the discrete solution converge to the solutions of the differential equation as $\Delta t \rightarrow 0$? (ii) Is the discretization stable? and (iii) how small does Δt have to be so that the discrete solution adequately approximates the exact

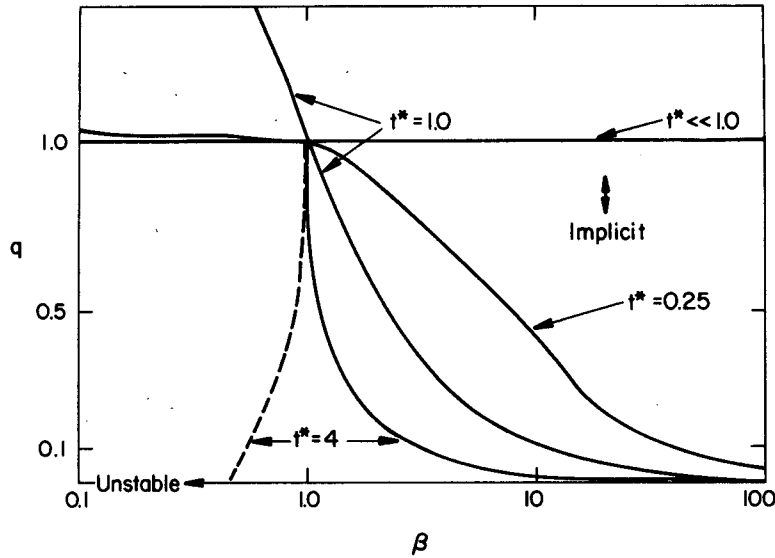


FIG. 2. The ratio of asynchronous to correct decay rate of mixed layer temperature, i.e., the quantity q given by Eq. (27), versus β defined by Eq. (19a) which essentially gives the ratio of the implicit part of the atmosphere-mixed layer coupling rate to the climate feedback parameter. The parameter t^* given by (19b) represents the ocean model coupling time in units of mixed layer time scale. The dashed line shows the presence of an oscillating solution. The value of β referred to as "implicit" is the value for the fully implicit asynchronous coupling procedure, i.e., for $r = 1$.

solution? The answer to (i) is yes as seen from Eqs. (24). The answers to (ii) and (iii) depend crucially on the coupling parameter r as used in Eq. (16) and the radiative and air-ocean energy transfer rates, λ and λ_{am} as combined in the parameter β given by (19a). For a given Δt , if r is too small, β will satisfy Eq. (26) and the iterative procedure will be unstable. This instability occurs in the form of growing oscillations. It can be avoided provided $\beta \geq 0.5$ but unphysical oscillations can occur for values of β up to twice as large as those instituting instability. The absence of oscillations can be insured by taking $\beta \geq 1$. According to Eq. (24), we can always adequately approximate the exact solution by the discrete solution provided

$$\sigma\Delta t \ll 1, \quad \beta\sigma\Delta t \ll 1. \quad (28)$$

For the fully implicit scheme, $r = 1$, and for the parameters of Table 2, $\beta \approx 20$, and $\sigma = 0.23$, requiring $\Delta t \ll 0.2$ years. A value of $\beta \leq 1$ according to Eq. (24) relaxes this constraint to $\Delta t \ll 5$ years. However, note from Eqs. (22)–(23) that for $\beta \approx 1$, the approximate solution should follow the exact solution well beyond the small $\sigma\Delta t$ range. Fig. 2 suggests how close β should be to 1 for the exact solution to be adequately approximated. The value $\beta = 1$ occurs essentially when the implicit part of the atmosphere-ocean energy transfer matches the climate feedback rate λ . Note that with large coupling intervals and large β , the limit Eq. (25) is likely to

be applicable. In this limit, the ocean warms by $\beta^{-1} = r^{-1}\lambda/(\lambda + \lambda_{am})$ each coupling to the atmosphere, independent of how long the ocean model is run by itself.

So far we have only been able to treat analytically the effect of asynchronous coupling on the system (8) for $\gamma = 0$. Some numerical solutions for an asynchronous coupling solution of the system Eq. (8) are shown in Fig. 3 for T_m and (divided by its steady state value) compared with the exact solution. The slowest convergence shown occurs for a large time step, $t^* = 4$, and fully implicit coupling ($\beta = 20$). As either β or t^* decreases we see that the solution compares more favorably with the exact solution which is approached either by letting $\beta \rightarrow 1$ or $t^* \rightarrow 0$.

We infer from these and other numerical calculations that: provided the mixed-layer time scale remains short compared to the previously inferred "exact" expression for the deeper layer time scale, the expressions (12) and (13) are still approximately valid except the short time scale time history of the mixed-layer temperature T_m is that inferred from Eq. (22) for the mixed layer alone. In other words, the partitioning of T_m between fast and slow time scales is essentially unchanged but the fast time scale is lengthened by the factor q given by (27). The slow time-scale evolution is essentially unchanged. This description applies to the $t^* = 0.5$ implicit and $\beta = 10, t^* = 4$ curves of

Fig. 3. For the $t^* = 4$, both implicit ($\beta = 20$) and $\beta = 10$ curves in Fig. 3, the limit (25) applies. For $\beta = 20$ and $\beta = 10$, the mixed layer temperature initially grows, respectively, by 0.05 and 0.1 over each coupling interval, nearly independent of Δt . From these values, we infer $t_1 = 100$ and $t_1 = 50$, respectively, (in years). For the latter case, the solution grows to about half its steady state in 50 years and further equilibrates on the $t_2 = 100$ year time scale. In the former case, $t_1 = t_2$ and the solution increases with a 100 year e -folding time for both small and large times. For much larger t^* , the mixed-layer T_m growth is slow compared to the deeper layer long time scale. In this case, numerical calculations show that the analysis of asynchronous coupling for the mixed layer alone approximately describes the time history of the mixed layer coupled to a deeper layer even after several hundred years.

It should be noted that our assumption of no coupling to the ocean below the intermediate waters is likely to give solutions that are too close to equilibrium on time scales of hundreds of years when the yet deeper (abyssal) waters also are significant heat reservoirs. One possible simple approach to improving the range of time scales adequately described by such a model is to replace the lower

layer by an eddy diffusion process as first recommended by Oeschger *et al.* (1975) for oceanic tracer data. Some such one-dimensional models for oceanic thermal response have been published while this paper was being revised [i.e., Hoffert *et al.* (1980), Cess and Goldenberg (1981)]. The ocean response of the present standard model is in approximate agreement with theirs on time scales less than 100 years. With regard to asynchronous coupling, we would expect the general principles inferred from the above discussion to apply, e.g., if asynchronous coupling reduces the heat uptake of T_m to a 100 year time scale, the correct solution would be approximately followed after several hundred years.

In summary, we have considered a simple zero-dimensional climate model with an atmosphere linked to a two-layer ocean. With parameters selected to mimic the surface mixed layers and intermediate waters, we have discussed the time scales determining the response of the mixed-layer temperature. We have discussed the stability and convergence of a partially implicit coupling scheme for connecting the atmospheric to the oceanic model and showed that it can satisfactorily approximate the correct solutions provided either that the time interval is small enough or that the

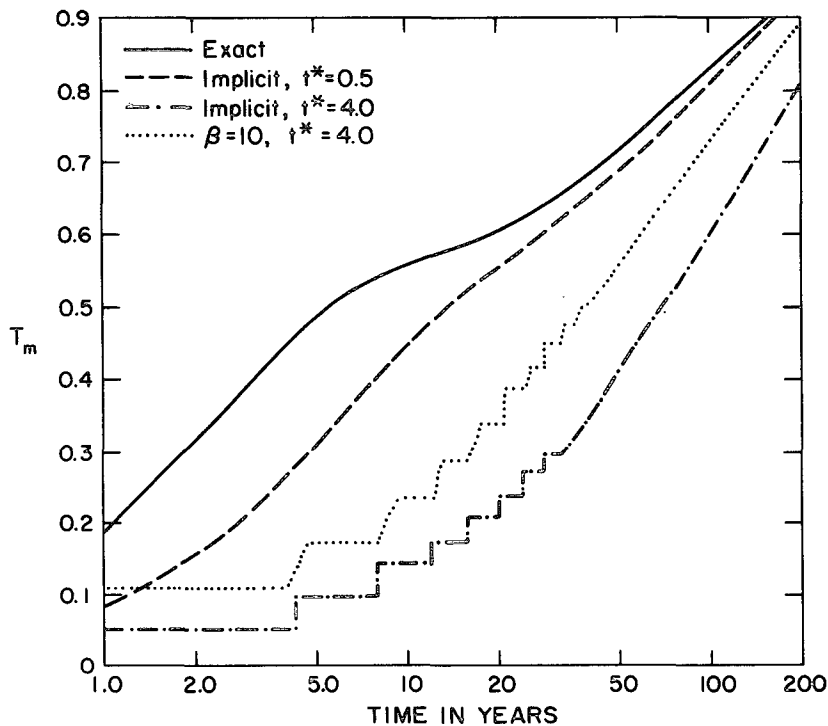


FIG. 3. The solid line is the same as Fig. 1 for $\lambda_{m0} = 2 \text{ W m}^{-2} \text{ K}^{-1}$. It is compared with various possibilities for asynchronous coupling. It is seen that as $t^* = \sigma \Delta t$ is reduced in the fully implicit scheme or β is reduced from its fully implicit value of 20, the asynchronous coupling solution moves closer to the correct solution.

factor r giving the fraction of the air-sea heat exchange to be evaluated implicitly in the ocean model is properly selected.

A strategy for locating empirically the optimum r for an AOGCM with asynchronous coupling to most rapidly achieve a near steady state for the mixed layer would be to find the smallest value of r that does not lead to oscillations in the global average temperature of the top layer of the ocean model from one iteration to the next. Our analysis suggests this optimum value of r is likely to be ~ 0.05 . However, our model is too oversimplified to indicate how much improvement could be achieved in an AOGCM by finding a single "optimum" r . The decrease of rates of atmosphere-ocean thermal coupling from low to high latitudes suggests that further improvement might be sought by taking r to decrease with increasing ocean temperature. Finally, it should be noted that the errors of implicit coupling such as illustrated in Figs. 2 and 3 are likely to be much more pronounced on the shorter (mixed-layer) ocean time scales. Provided the asynchronous coupling does not slow the mixed-layer response to rates slower than the slow time scales of the deeper waters, the asynchronous coupling may not significantly increase the time required to reach an ultimate statistical steady state. Consequently, the fully implicit system may be computationally efficient for some AOGCM calculations where a statistical steady state is the primary objective.

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