On the Amplitudes Reached by Baroclinically Unstable Disturbances

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ABSTRACT

The maximum eddy energies reached by baroclinically unstable disturbances in some simplified numerical models (Gall, 1976b; Simmons and Hoskins, 1978) are compared to a scale-dependent measure of the energy available to the disturbances. It is found that, except for the longest wavelengths, the ratio of the maximum eddy energies to this available energy (the eddy efficiency) tends to remain constant relative to the large variations in the maximum energies. The roles of the growth rate of the initial disturbance and the β-effect are discussed with reference to results from a maximally simplified model. It is shown that dimensional arguments paralleling Charney (1971) lead to the maximum energy versus scale relationships found in the numerical results.

1. Introduction

The simplest climate models require parameterization of the meridional heat flux due to transient large-scale disturbances. Closure theories assume the disturbance energies as a function of the zonal mean state. The theory of Green (1970) assumes that the eddy kinetic energy is proportional to the total available potential energy in the baroclinically unstable region, assumed to be the order of the radius of the earth. Stone (1972) scales the eddy kinetic energy by the available potential energy (APE) contained in a latitude band with a width L of the Rossby radius corresponding to a scale height H, $L = NH/f$. Held (1978) postulates that the eddy kinetic and potential energies of disturbances with vertical scale $h < H$ should be proportional to the available potential energy contained in a volume of height h and a latitude band of width $l = Nh/f$ (i.e., a generalization of Stone's parameterization). Recent numerical models which examined the evolution of a single adiabatic, initially small-amplitude normal-mode disturbance on a baroclinically unstable, zonally symmetric initial state, where the disturbance was allowed to interact with the mean flow (Gall, 1976b, referred to as G; Simmons and Hoskins, 1978, hereafter referred to as SH) produce results which suggest that the maximum energies that the disturbance achieves is limited by the initial available potential energy in the restricted region where the disturbance amplitude is significant. It therefore would seem that the above scaling considerations (particularly that of Held) are applicable to the non-equilibrium situation studied in the numerical calculations. However, it is not so obvious in what manner the numerical results reflect on scaling arguments for the problem of the eddy heat flux resulting from the balance between forcing and dissipation.

In extending the results of linearized calculations of the growth of instabilities to the finite-amplitude regime, it is generally assumed that the most rapidly growing disturbance will grow to the largest amplitude. The above numerical calculations did not show this to be the case. A rapidly growing disturbance of small spatial scale may attain a much smaller maximum energy than a more slowly growing disturbance of large spatial scale. An estimate of the maximum energy that an initially small-amplitude, baroclinically unstable disturbance of limited vertical and meridional extent can achieve is derived by assuming that the maximum disturbance energy is equal to the available potential energy of the initial state in the region where disturbance amplitudes are significant. The estimate is applied to the results of the numerical models. This scale-dependent measure of the maximum amplitude that the various disturbances can reach, although estimated crudely, provides a better predictor of the maximum disturbance energies than does the growth rate.

Additionally, the limited data suggest that slower growing waves achieve a smaller fraction of their maximum possible energy than the more rapidly growing waves. In order to further investigate the dependence of the maximum eddy energies on the growth rate, some experiments analogous to those of G and SH were performed using a highly truncated two-level, quasi-geostrophic model [the model of Charney and Strauss (1980) without topography]. The behavior of initially exponentially growing

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waves with the same vertical and meridional scales is determined as a function of zonal wavenumber m, β and initial zonal mean temperature gradient θ_{A_I}. For the same θ_{A_I} a more rapidly growing wave usually, but not always, achieves a larger maximum eddy energy. The model exhibits simple behavior, however, in that the maximum eddy energy is found to be a function of θ_{A_I}, m and the critical temperature gradient $\theta_{A_c}(m, \beta)$. It is $\theta_{A_c}(m, \beta)$ rather than the initial growth rate which turns out to be the relevant predictor of the maximum wave energy in the two-level model. This model also produces wavenumber-dependent upper level eddy kinetic energy enhancement relative to lower level eddy kinetic energy in the maturing wave, a feature qualitatively similar to the results of SH.

Finally, the reason for the similarity between the energy versus scale relationship for the non-equilibrium initial value problems and the equilibrium medium-scale quasi-geostrophic turbulence regime (Charney, 1971) is explored. It is found that the similarity is not accidental as the same dimensional argument applies in either case. The k^{-3} wavenumber dependence of the maximum energy spectrum expected by dimensional analysis in the non-equilibrium initial value problems for the medium-scale waves is shown to be that of the spectrum of eddy energy production in statistical equilibrium.

2. Energetic considerations

The available potential energy (APE) of a volume of atmosphere v is defined (after Holton, 1972) by

$$\mathrm{APE}(V) = \iiint_V (g^2/N^2)\rho \frac{\theta'^2}{2[\bar{\theta}]^2} dv, \qquad (1)$$

where g is the acceleration of gravity; N the Brunt-Väisälä frequency; $[\leqslant\theta(z)]$ the mean potential temperature at height z; $\theta'(x, y, z)$ the deviation of the potential temperature from $[\bar{\theta}]$; and ρ the mass density. Formally, APE(V) is relevant to discussing energy transformation only when there is no motion across the imaginary internal boundaries of v. For a zonally symmetric field of potential temperature with constant horizontal gradient,

$$\left.\begin{aligned} \bar{\theta}(y, z) &= [\bar{\theta}(z)] + y \frac{\partial\bar{\theta}}{\partial y} \\ \theta' &= y \frac{\partial\bar{\theta}}{\partial y} \end{aligned}\right\}, \qquad (2)$$

where y is the meridional distance from the mean latitude of V. When V is an annular region of width L_y and constant height h, centered at latitude ϕ_0, and N^2 and $[\bar{\theta}]^2$ are approximately constant,

$$\begin{aligned} \mathrm{APE}(V) &= D(\phi_0) \frac{g}{2[\bar{\theta}]^2 N^2} \int_{-L_y/2}^{L_y/2} \\ &\times dy \int_{p_T}^{p_0} dp \left(y \frac{\partial\bar{\theta}}{\partial y}\right)^2 \\ &= D(\phi_0) \frac{g}{N^2[\bar{\theta}]^2} \left(\frac{\partial\bar{\theta}}{\partial y}\right)^2 \frac{L_y^3(\Delta p)}{24}. \end{aligned} \qquad (3)$$

Here $D(\phi_0) = 2\pi a \cos\phi_0$ is the circumference of the sphere of radius a at latitude ϕ_0, and $\Delta p = p_0 - p_T$ is the pressure difference between the top of V and the ground. Averaged over a hemisphere the energy density due to APE(V) is then

$$\begin{aligned} \overline{\mathrm{APE}}(V) &= \frac{\mathrm{APE}(V)}{2\pi a^2} \\ &= \frac{g \cos\phi_0}{N^2[\bar{\theta}]^2} \left(\frac{\partial\bar{\theta}}{\partial y}\right)^2 \left(\frac{L_y}{a}\right) \frac{L_y^2 \Delta p}{24}, \end{aligned} \qquad (4)$$

where a is the radius of the sphere.

A disturbance confined to V may achieve a maximum energy equal to the total (available plus kinetic) basic-state energy contained in V. The zonal mean kinetic energy in V, ZKE(V) will be neglected in the following discussion. ZKE(V)/APE(V) turns out to be close to 1, using the scales estimated in the next section, but the energy transformations found by G and SH during the growing stage of the disturbance show that the dominant effect is ZAPE $\rightarrow$ wave energy. Thus an approximate upper bound on the energy density reached by an initially small-amplitude disturbance restricted to an annular region V will be taken to be $\overline{\mathrm{APE}}(V)$.

A "square" wave of wavelength L_x, $L_y \sim L_x$, and vertical scale $L_z \sim f/N$, L_x (Held's scaling) can achieve a maximum eddy energy averaged over the surface area of V which varies as k^{-3} when $L_z/H \ll 1$ ($\Delta p \alpha L_z$) and k^{-2} when $L_z \gg H$ ($\Delta p \alpha H$ = constant), where k is the dimensional zonal wavenumber $k = 2\pi/L_x$ and H is the atmospheric scale height.

3. Model energies

The studies of G and SH examined the effects of wave/mean flow interaction on the evolution of a flow that consisted initially of a small-amplitude, exponentially growing normal mode of specified horizontal wavelength superimposed on a geostrophically balanced zonal mean flow. Both sets of calculations were performed on a spherical earth, assumed an adiabatic thermodynamic equation, and included viscous dissipation of kinetic energy.

In G, the structures of the initial disturbances were taken as those of the most rapidly growing modes of zonal wavenumbers 5, 7, 9 and 15, while the initial zonal mean flow, derived from a GFDL

TABLE 1. Application of Eq. (4) to Gall's calculation.

	Zonal wavenumber			
	5	7	9	15
L_y (deg latitude)	45	40	27	16
Δp (mb)	960	940	800	600
ϕ_0 (°N)	52	45	43	40
Maximum model EKE ($J\ m^{-2} \times 10^4$)	10.5	13	9	1
$\overline{APE}(V)$ ($J\ m^{-2} \times 10^4$)	118	93	25	4.1
EKE efficiency	0.089	0.14	0.36	0.24

general circulation experiment, was the same for each case. Zonal wavenumbers different from those of the initial disturbance were excluded. The wave evolved until it reached a maximum kinetic energy. The maximum kinetic energies of waves $m = 7$, 9 and 15 were strongly decreasing for m increasing. The energy for $m = 5$ was ~75% of the energy for $m = 7$. The $m = 15$ wave was initially the most rapidly growing, but extracted the least energy from the initial zone mean flow. The initial disturbances had significant amplitude over latitude bands and depths that increased as the zonal wavenumber increased (Gall, 1976a). The finite-amplitude disturbances appear to have had roughly similar scales.

In SH, various initial zonal mean flows (basic states) were considered. Wave disturbances with the structure of the most rapidly growing normal mode appropriate to the initial zonal mean flow and $m = 3$, 6 and 9 were considered. For experiments with the same basic state, $m = 6$ disturbances achieved maximum energies (potential plus kinetic) 1.4–14 times the energies achieved by the $m = 9$ disturbances. $m = 3$ achieved the same energy as $m = 6$ in the comparable broad 30° jet experiments. The growth rate of $m = 6$ was greater than the growth rate of $m = 9$ for the 45° jet, while the opposite was true for the 30° jets. Disturbance amplitudes were significant only over limited latitudinal bands. For the same basic state, disturbances of larger zonal wavelengths produced heat fluxes occupying larger bands of latitude. m = 3 and 6 heat fluxes occupied a greater depth of the model atmosphere than $m = 9$.

The maximum energies achieved in the experiments of G and SH are compared to estimates of the maximum energies available to the disturbances, from (4). The results of these estimates are presented in Tables 1 and 2, for the experiments of G and SH, respectively. In Table 1 for the results of G, the quantity of interest is the eddy kinetic energy (EKE) efficiency, defined as the ratio of maximum EKE to $\overline{APE}(V)$. The EKE efficiency is larger for the shorter waves than for the longer waves. In Table 2 for the results of SH, the total eddy efficiency, the ratio of maximum EKE + EAPE to $\overline{APE}(V)$ is found. This efficiency again varies much less strongly than the energies themselves and is order 1. The least efficient wavenumber is $m = 3$, which has an efficiency of 0.31.

The estimates are, of course, extremely crude. In the first place, (4) does not take into account the latitudinal variations in $\partial\bar{\theta}/\partial y$ or the vertical variations of N^2 and $[\bar{\theta}]^2$ that occur in the basic states of the numerical calculations. Also, the specification of L_y and Δp are largely arbitrary, but it is hoped that a reasonable level of internal consistency for each set of calculations is achieved by applying the same objective criteria where possible. These criteria are described below.

For G, $\partial\bar{\theta}/\partial y = -6.1 \times 10^{-6}\ km^{-1}$ was found using the thermal wind equation with tropospheric maximum $\bar{u}$ of 30 m s^{-1} at 40° latitude and 200 mb (Fig. 2 of Gall, 1976a). The parameters L_y and Δp were found using the latitudes and heights of 0.1 × maximum contour for the initial geopotential perturbations (Fig. 12 of Gall, 1976a). ϕ_0 is the central latitude of the initial disturbance; N is taken as $1.2 \times 10^{-2}\ s^{-1}$ and $[\bar{\theta}]$ as 317 K.

For SH, $\overline{\partial\theta/\partial y}$ was estimated from the basic state using the thermal wind equation at the latitude of the appropriate jet maxima. Data for these estimates were taken from Simmons and Hoskins (1977). The basic states considered were jets at 55 and 45° and a narrow jet at 30° (30°N) with maximum $\bar{u}$ of 40 m s^{-1}, and a broad jet at 30° (30B) with maximum $\bar{u}$ of 37 m s^{-1}. The vertical and horizontal

TABLE 2. Application of Eq. (4) to Simmons and Hoskins calculations.

	Jet latitude, zonal wavenumber							
	(55, 6)	(45, 6)	(45, 9)	(30B, 3)	(30B, 6)	(30B, 9)	(30N, 6)	(30N, 9)
L_y (deg latitude)	30	30	17	40	30	20	20	20
Δp (mb)	820	820	640	870	820	640	820	640
ϕ_0 (°N)	57	47	47	45	40	40	37	37
Maximum model eddy energy ($J\ m^{-2} \times 10^5$)	7.5	7.0	0.6	3.0	3.5	1.2	0.7	0.5
$\overline{APE}(V)$ ($J\ m^{-2} \times 10^5$)	7.9	7.2	1.1	9.7	3.8	1.0	1.3	1.0
Eddy energy efficiency	0.95	0.97	0.55	0.31	0.92	1.2	0.54	0.5

scales and ϕ_0 were found from the 0.2 × maximum heat flux contours presented for the life-cycle averaged disturbances. The scales of the life-cycle averaged eddy momentum fluxes were not used, since the momentum fluxes gain their largest amplitudes in the decaying phase of the disturbance. When the Reynolds stresses contribute significantly to the energetics of the growing disturbance, the eddy energy efficiency can turn out to be much smaller (Simmons and Hoskins, 1980). As the heat flux is a quadratic quantity of the disturbance amplitude, it is expected that these scales are smaller than would be found if the scale of the disturbance amplitude was used as in application to G's results (0.2 × maximum heat flux should correspond roughly to $\sqrt{0.2}$ × maximum disturbance amplitude). The efficiencies are, therefore, larger than those in Table 1 because of the comparison of total eddy energy rather than EKE to $\overline{\text{APE}}(V)$ and the bias toward larger $\overline{\text{APE}}(V)$ for the method of estimation applied in producing Table 1. The scales of the $m = 9$ disturbances on the two 30° jets were not available. The pressure scale was taken in both of these cases to be the same as the pressure scale for $m = 9$ on the 45° jet, and the meridional scale was taken in both cases to be 20° latitude (approximately ½ wavelength, or in the case of the narrow jet, the width of the region of large meridional temperature gradients). The values of N and $[\bar{\theta}]$ are the same as those used in calculating Table 1.

The results presented in Tables 1 and 2 illustrate that for the shorter wavelengths, for which the β effect is relatively unimportant, the maximum disturbance energies tend to be proportional to $\overline{\text{APE}}(V)$.

4. Effect of growth rate

SH noted a tendency for larger maximum eddy energies for a particular zonal wavenumber to be associated with larger growth rates in the initial linear phase. In order to isolate the role of the initial growth rate on the maximum eddy energies achieved, integrations are performed using a maximally simplified, wave-mean flow interaction model from which the scale effects described in the previous sections have been eliminated.

A quasi-geostrophic two-level, adiabatic, inviscid, constant N^2, highly truncated spectral model on a β-plane channel is used for the integrations. The nondimensionalized equations for this model are presented in Charney and Strauss [1980, Eqs. (2.28)–(2.35)] and topographic effects are neglected. The behaviors of various zonal wavenumbers m disturbances with the gravest meridional structure are calculated for a channel of width 5000 km and a depth of 1 atm. There are no Reynold stresses due to the truncation, and the equations conserve energy. The two-step time-differencing scheme of Lorenz (1963) is employed. Programming was verified against results published in Charney and Strauss. For each m there is a critical shear of the initial zonal mean wind and, consequently, a critical meridional temperature gradient, represented by $\theta_{A_c}(m, \beta)$ (the dimensionless temperature structure at the mid-level is $\theta_A\sqrt{2}\cos y$, $0 \leq y \leq \pi$, so that $\theta_A > 0$ represents $\partial T/\partial y < 0$). Exponentially growing normal-mode perturbations of zonal wavenumber m exist only when $\theta_{A_I} > \theta_{A_c}(m, \beta)$, for $\theta_{A_I} > 0$, where θ_{A_I} represents the basic-state temperature gradient. The integrations are initialized with a small-amplitude disturbance of wavenumber m (the wavenumber for the earth at 45° latitude), with the structure appropriate for initial normal-mode exponential growth, superimposed on the basic state represented by θ_{A_I}. The total eddy energy grows to some maximum value, which occurs when the absolute value of the zonal mean temperature gradient reaches it minimum value. Denoting the minimum value of θ_A reached in the integration as $\theta_{A_{\min}}$, the maximum possible eddy energy, which equals the available potential energy of the basic state and is independent of m or β, is achieved if $\theta_{A_{\min}} \leq 0$.

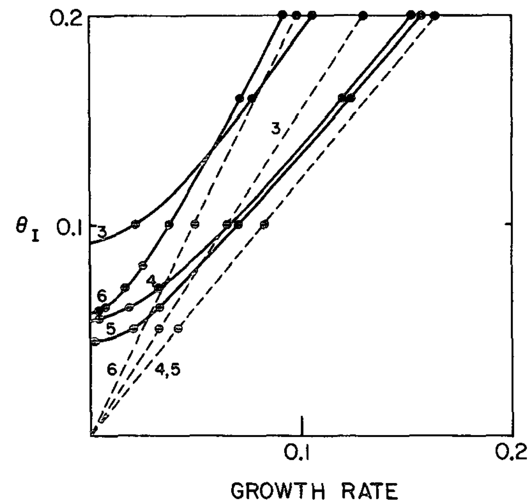

FIG. 1. Nondimensional growth rate σ_i as a function of the zonal mean temperature gradient θ_{A_I} for $\beta = 0$ (dashed lines) and $\beta = \beta_{45}$ (solid lines) for various zonal wavenumbers in the two-level model. Numbers on the curves refer to zonal wavenumber.

Fig. 1 shows the dimensionless growth rate of the initial perturbation σ_i as a function of θ_{A_I}, for $m = 3$, 4, 5, and 6 and two values of β, $\beta = 0$ and $\beta = \beta_{45}$ (β_{45} represents the β effect on the earth at 45° latitude). For $\beta = \beta_{45}$, $m = 6$ grows faster than $m = 3$ when $\theta_{A_I} \leq 0.14$, while $m = 3$ grows faster than $m = 6$ for $\theta_{A_I} > 0.14$. $\sigma_i(m, \beta = \beta_{45})$ approaches $\sigma_i(m, \beta = 0)$ as $\theta_{A_I} \to \infty$. $\theta_{A_c}(m, \beta)$ is given

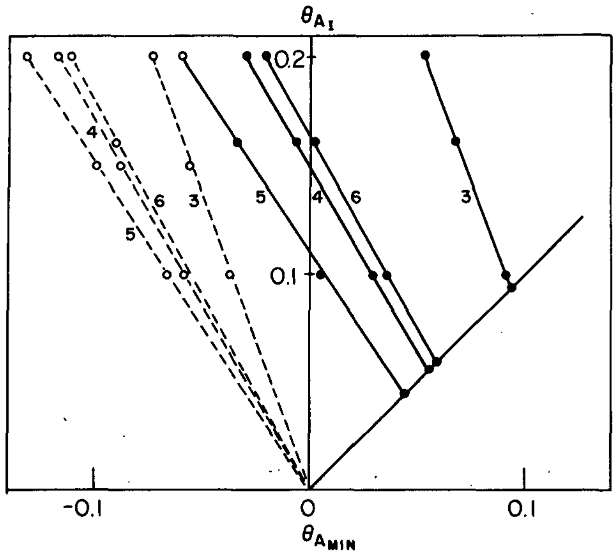

FIG. 2. Results of integrations with the two-level model. The minimum zonal mean temperature gradient $\theta_{A_{min}}$ versus the initial zonal mean temperature gradient θ_{A_I} for various zonal wavenumbers, $\beta = 0$ (dashed lines) and $\beta = \beta_{45}$ (solid lines). Numbers on the curves refer to zonal wavenumber. The line $\theta_{A_{min}} = \theta_{A_I}$ also is plotted.

by the value of θ_{A_I} on the appropriate curve as $\sigma_i(m, \beta) \to 0$.

Fig. 2 summarizes the results of integrations for $m = 3$, 4, 5 and 6, and for $\beta = 0$ and $\beta = \beta_{45}$. $\theta_{A_{min}}(m, \beta)$ is plotted as a function of θ_{A_I}. The response is linear for each m and is described by

$$\theta_{A_{min}}(m, \beta) = \theta_{A_c}(m, \beta) - A(m)[\theta_{A_I} - \theta_{A_c}(m, \beta)]. \quad (5)$$

A is a function of m only; the slopes of the response curves do not depend on β. Calculations done with other values of β confirm this feature. A is positive for $m > m_c \sim 2.25$ and negative for $m < m_c$. $A < 0$ also very close to the short-wave cutoff ($m_{sw} \sim 6.3$, for $\beta = 0$). That is, for $m > m_c$, $\theta_{A_{min}} < \theta_{A_c}$, so that these wavenumbers continue growing (for $\beta > 0$) even when the zonal mean state is stable. For $\beta = 0$ and $m > m_c$ (except near m_{sw}), $\theta_{A_{min}} < 0$ and the waves release all of the initial ZAPE. For $\beta > 0$, $m > m_c$, and large enough supercriticality [defined as $\theta_{A_I} - \theta_{A_c}(m, \beta)$], the waves also reverse the zonal mean temperature gradient and achieve the maximum possible eddy energy. For $m < m_c$, $A < 0$ and the waves stop growing while the zonal mean temperature gradient is still supercritical.

The stability properties of this model differ from those of the two-level model discussed in Holton (1972, Section 9.2) in which the basic state vertical wind shear and temperature gradients are independent of latitude. Instability is always possible for $m \to 0$ and large enough vertical shear in the model which Holton discusses, while there is an absolute long wave cutoff at $m \approx 1.1$ for all $\beta \geq 0$ in the Charney-Strauss model.

The relationship between the growth rate of the initial wave and the maximum eddy energy $E_{max}(m)$ may be seen by comparing Figs. 1 and 2. The ratio of the maximum eddy energy to the basic-state ZAPE (the eddy efficiency) is $1 - [(\theta_{A_{min}})^2/(\theta_{A_I})^2)$ when $\theta_{A_{min}} > 0$ or 1 when $\theta_{A_{min}} < 0$. For $\beta = 0$ and $m_{sw} > m > m_c$, the efficiency is 1 and the growth rate is irrelevant. For $\beta = \beta_{45}$ and $\theta_{A_I} < 0.14$ the faster growing initial waves are more efficient, but for $\theta_{A_I} > 0.14$, the slower growing $m = 6$ wave is more efficient than the more rapidly growing $m = 3$ wave. Thus even when the scale effects on the maximum energies have been eliminated, the more rapidly growing wave does not always achieve the larger maximum eddy energy. When β is important and the supercriticality becomes small, the maximum waves energies are much less than the ZAPE of the initial state.

The initial growth rate, however, does play an important role in the rate of energy generation in the two-level experiments. The zonal mean temperature gradient θ_A behaves periodically in these experiments, varying between extreme values of $\sim\theta_{A_I}$ and $\theta_{A_{min}}$ with period P. The eddy energies have single or one double maximum with value E_{max} per period depending on whether $\theta_{A_{min}} >$ or < 0. The rate of eddy energy generation is estimated to be $E_{max}/(P/2)$. $1/P\alpha\sigma^{0.8}$ is found for $\beta = \beta_{45}$, $\theta_{A_I} = 0.1$ and $\theta_{A_I} = 0.06$, and $\beta = 0$, $\theta_{A_I} = 0.1$. The rate of energy generation thus varies approximately as $\sigma_i E_{max}$.

An interesting feature of the integrations that lends credence to the results of this highly simplified two-level model is that the preferential upper level eddy kinetic energy growth found by SH in the maturing disturbance is qualitatively simulated. Longer waves show more preferential upper level growth than shorter waves, and the effect does not depend strongly on β. The time behavior of $m = 5$, the most rapidly growing wave, for $\theta_{A_I} = 0.1$ and $\beta = \beta_{45}$ is shown in Fig. 3. The ratio of the upper level to lower level eddy kinetic energy, EKE_u/EKE_L is shown as well as the zonal mean temperature gradient, θ_A, and the ratio of the total eddy energy to the initial ZAPE. EKE_u/EKE_L is 2.47 in the small amplitude stage of development, 3.8 at the time when the zonal mean temperature gradient equals the critical gradient, and 10 when the maximum eddy energy is reached. The preferential upper level growth is most significant when θ_A is subcritical, i.e., when exponential structure preserving growth is no longer possible. In SH, the preferential upper level growth for $m = 6$ becomes significant when the surface temperature gradient is eliminated, which is a necessary condition for the

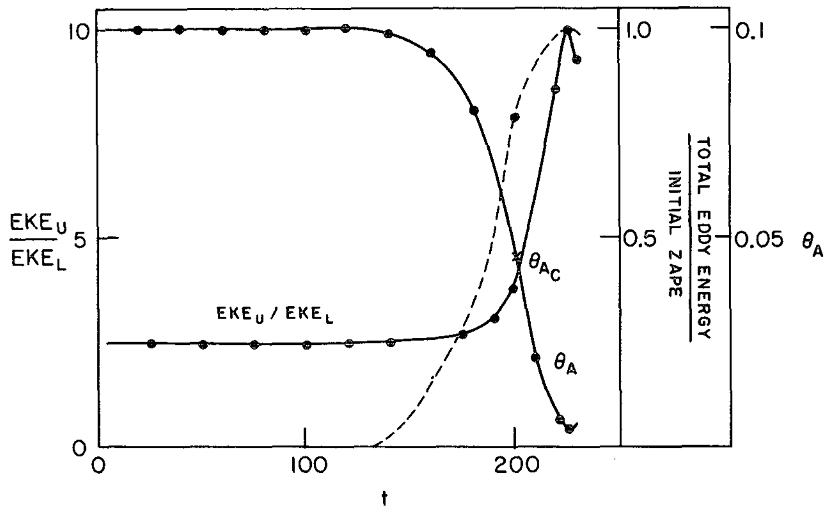

FIG. 3. Time behavior of zonal wavenumber 5, $\theta_{A_I} = 0.1$ integration in two-level model. Zonal mean temperature gradient, θ_A (upper solid curve), the ratio of the upper level to lower level eddy kinetic energy, EKE_u/EKE_L (lower solid curve), and the ratio of the total energy to the initial ZAPE (dashed curve). The point where $\theta_A = \theta_{A_c}$ is marked with an ×.

elimination of exponentially growing normal modes in the continuous model (Charney and Stern, 1962). For the case of $m = 6.11$ in the simple model with the same basic state ($\theta_{A_I} = 0.1$, $\beta = \beta_{45}$), a wavenumber close to the short-wave cutoff, little preferential upper level growth is found: $EKE_u/EKE_L = 1.85$ in the linear regime and $EKE_u/EKE_L = 2.2$ when the wave achieves its maximum energy. SH find little tendency for upper level amplification in the shorter wave $m = 9$ case. For $m = 3.5$ in the simple model, $EKE_u/EKE_L = 7.54$ in the linear phase and $EKE_u/EKE_L = 230$ at maximum amplitude. Thus, there is qualitative similarity between the structural changes found in the finite-amplitude regime in the simple model and the SH model. Quantitative differences are apparent, however, as most of the EKE in the initial $m = 6$ wave of SH is concentrated near the ground, while the two-level model gives $EKE_u/EKE_L > 1$ for the initial waves.

5. Relation to theory of quasi-geostrophic turbulence

There is an interesting resemblance in G (Fig. 2) between the vertically integrated maximum kinetic energy in the wave-mean flow interaction model plotted as a function of wavenumber and the vertically integrated kinetic energy spectrum at 45.6°N in a GCM simulation. A curve given by $KE(m) = KE(7) \times (7/m)^3$ fits both sets of data reasonably well for $7 \leq m < 15$. [Actually m^{-4} better describes the KE(9)/KE(15) ratio for the wave-mean model.] In terms of the dimensional wavenumber $k = (2\pi/\text{wavelength})$, both sets of data behave approximately as k^{-3} for the synoptic-scale waves. This is the scale behavior of the kinetic and available potential energy predicted by the theory of Charney [1971, Eq. (31)] by use of dimensional arguments for the statistical equilibrium energy spectrum of the medium-scale wavelengths. It will be shown below that the similarity between the shape of the maximum energy curve and the shape of the quasi-geostrophic energy spectrum is not likely to be accidental, because essentially the same dimensional argument can be applied in either case.

Charney's (1971, p. 1091) argument goes as follows: Assume that the horizontal scales are small enough so that the β effect and Reynolds stresses are unimportant. Also, assume that diabatic and frictional effects are unimportant for these scales. Assume isotropy in x, y and $\zeta = (N/f)z$. "Then the only parameters on which the eddy energies can depend at statistical equilibrium are $|\mathbf{V}_\zeta|$ and the wavenumber k_H, suggesting energy spectra of the form

$$\left.\begin{aligned} K(k_H) &= C_1 |\mathbf{V}_\zeta|^2 k_H^{-3} \\ P(k_H) &= C_2 |\mathbf{V}_\zeta|^2 k_H^{-3} \end{aligned}\right\} \tag{31}$$

for the kinetic and potential energies, respectively." $\mathbf{V}_\zeta$ is the vertical shear of the mean flow in the stretched coordinates, and the energies have dimensions energy/(unit wavenumber).

A restatement of the above argument for the scale variation of the maximum energies achieved by an initially baroclinically unstable, small-amplitude normal mode on a zonally symmetric basic state is offered below.

Assume that the horizontal scales are small enough so that the β effect and Reynolds stresses are unimportant for the growing phase of the disturbance. Also, assume that diabatic and frictional effects are unimportant for these scales, and that the variation of growth rate with scale can be neglected. Assume that the disturbance is bottom trapped and isotropic in x, y and $\zeta = (N/f)\zeta$ (i.e., that $L_y \propto L_x$, where L_x is zonal wavelength and that the vertical scale $L_z \propto fL_x/N$), and that the variation of density with height may be neglected. Then the only parameters on which the maximum eddy energies, which are taken as the eddy energies per unit volume for convenience of interpretation, can depend are $\bar{u}_\zeta$, the vertical shear of the basic state in the stretched vertical coordinate, L ($=L_x$), and $\tilde{\rho}$, the density in the stretched coordinate $[\tilde{\rho} = (f/N)\rho]$. The only quantity that can be constructed from $\bar{u}_\zeta$, L and $\tilde{\rho}$ with the dimensions of energy/(unit volume) is

$$E = \tilde{\rho}(\bar{u}_\zeta)^2 L^2. \tag{6}$$

Then the energy/(unit area) in the disturbed region should scale as $E \times L$, and the energy/(unit area) for an annular disturbed region (surface area $\propto La$) averaged over the total surface area (a^2) should scale as $\bar{E} = E \times (L^2/a)$. Use of the approximate thermal wind equation

$$f\frac{\partial \bar{u}}{\partial z} = -\frac{g}{\bar{\theta}}\frac{\partial \bar{\theta}}{\partial y} \tag{7}$$

and $\Delta p = \tilde{\rho} g L$ then produces

$$\bar{E} = C_1 \overline{\mathrm{APE}}(V). \tag{8}$$

$\bar{E}$ may be interpreted as kinetic, potential or total maximum eddy energy with C_1 representing a different constant for each case. $\bar{E} \sim k^{-4}$ for the scales under consideration.

The extent to which the maximum eddy energies for different wavenumbers scale as (8) (essentially k^{-4}) for numerical calculations should be indicative as to the extent that the assumptions used in deriving (8) are satisfied. For example, the long-wave calculations of G, $m = 5$, and SH, $m = 3$, are situations in which β becomes important and in which the relationship between the scales changes, and these waves achieve significantly smaller energies than would be expected by extrapolating (8) from the short-wave results. The results of Simmons and Hoskins (1980) show that waves of the same scales growing on basic states characterized by the same $\bar{u}_z$ can achieve widely varying maximum energies when the basic states are constructed so that Reynolds stresses are important during the growing phase of the disturbance. On the other hand, as the shorter wave energies in G do tend to scale as (8), it is not necessary, as supposed in G, to appeal to enhanced frictional effects for shorter wavelengths to explain the results.

The single wave mean flow experiments may be used to define a spectrum of eddy energy generation, as discussed in Section 4. That is, for a particular basic state, the maximum eddy energies $E_{\max}(k, L_y)$ of initially small-amplitude unstable modes of zonal wavenumber k and meridional scale L_y are found, and the eddy energy generation spectrum $G(k, L_y)$ is then approximately

$$G(k, L_y) \propto \sigma_i(k, L_y) E_{\max}(k, L_y). \tag{9}$$

This generation spectrum may be useful for estimating the generation spectrum in calculations with many interacting waves which have reached statistical equilibrium.

In application to Charney's (1971) geostrophic equilibrium calculation described above, the energy releasing eddies are assumed to be bottom trapped, isotropic and horizontally homogeneous. $E_{\max}(k)$ is taken as the mean state $\underline{\mathrm{ZAPE}}$ per unit disturbance area. That is, $E_{\max}(k) \propto \overline{\mathrm{APE}}(v)/L_y \propto k^{-3}$. As there is only one time scale in this formulation, $(\partial\bar{u}/\partial z)^{-1}$, $\sigma_i(k)$ = constant independent of k is taken. Then $G(k) \propto k^{-3} \propto \bar{E}(k)$.

To apply (9) to the "equilibrium" spectra of experiments 1 and 2 of Gall *et al.* (1979) for the energy releasing medium scales (wavenumber 10–20), the same estimate of $E_{\max}(k) \propto k^{-3}$ is chosen. In their experiment 2, with fixed zonal flow the calculated growth rates are approximately k independent, leading to G $(k) \propto k^{-3}$, and they find EKE$(k) \propto k^{-3}$. In their experiment 1, where the zonal mean flow is not forced, the calculated initial growth rates decrease from wavenumbers 10 to 20, with the growth rate of wavenumber 10 approximately 2.3 times the growth rate of wavenumber 20. G $(k) \propto \sigma_i(k)k^{-3}$. The EKE$(k)$ spectrum in their experiment behaves roughly as this estimate of G (k).

6. Conclusion

Initially small-amplitude baroclinically unstable normal-mode disturbances in some simplified numerical models grow to amplitudes that are consistent with release of the locally available potential energy. Results from calculations with a simplified model suggest that more supercritical disturbances should be more efficient in transforming available zonal mean energy into eddy energy, while the rate of eddy energy generation is closely related to the growth rate. The rapid decrease in maximum eddy energy with decreasing wavelength for the medium scales in the numerical calculations is to be expected from dimensional considerations.

The wavelengths responsible for the largest equilibrium heat fluxes (the largest scales) are wavelengths for which it is expected that the assumptions required by the dimensional arguments do not apply. Additionally, as the longest waves are not expected to lose their energy by an energy cascade to shorter waves (Charney, 1971), the roles of frictional and radiative dissipation in determining the equilibrium heat flux cannot be neglected.

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REFERENCES

Charney, J. G., 1971: Geostrophic turbulence. *J. Atmos. Sci.*, **28**, 1087–1095.

——, and M. E. Stern, 1962: On the stability of internal baroclinic jets in a rotating atmosphere. *J. Atmos. Sci.*, **19**, 159–172.

——, and D. M. Strauss, 1980: Form-drag instability, multiple equilibria and propagating planetary waves in baroclinic, orographically forced, planetary wave systems. *J. Atmos. Sci.*, **37**, 1157–1176.

Gall, R., 1976a: A comparison of linear baroclinic instability theory with the eddy statistics of a general circulation model. *J. Atmos. Sci.*, **33**, 349–373.

——, 1976b: Structural changes of growing baroclinic waves. *J. Atmos. Sci.*, **33**, 374–390.

——, R. Blakeslee and R. C. J. Sommerville, 1979: Baroclinic instability and the selection of the zonal scale of the transient eddies of middle latitudes. *J. Atmos. Sci.*, **36**, 767–784.

Green, J. S. A., 1970: Transfer properties of the large-scale eddies and the general circulation of the atmosphere. *Quart. J. Roy. Meteor. Soc.*, **96**, 157–185.

Held, I. M., 1978: The vertical scale of an unstable baroclinic wave and its importance for eddy heat flux parameterization. *J. Atmos. Sci.*, **35**, 572–576.

Holton, J. R., 1972: *An Introduction to Dynamic Meteorology*. Academic Press, 319 pp.

Lorenz, E. N., 1963: Deterministic nonperiodic flow. *J. Atmos. Sci.*, **20**, 130–141.

Simmons, A. J., and B. J. Hoskins, 1977: Baroclinic instability on the sphere: Solutions with a more realistic tropopause. *J. Atmos. Sci.*, **34**, 581–588.

——, and ——, 1978: The life cycles of some nonlinear baroclinic waves. *J. Atmos. Sci.*, **35**, 414–432.

——, and ——, 1980: Barotropic influences on the growth and decay of nonlinear baroclinic waves. *J. Atmos. Sci.*, **37**, 1679–1684.

Stone, P. H., 1972: A simplified radiative-dynamical model for the static stability of rotating atmospheres. *J. Atmos. Sci.*, **29**, 405–418.