General Formulation of Optical Paths for Large Zenith Angles in the Earth's Curved Atmosphere

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ABSTRACT

Formulas that can be used to determine the optical path between two points along an atmospheric ray path are derived for the case when the local zenith angle of the ray path is larger than 70°. For angles less than 70°, these formulas reduce to the airmass function; *viz*., the secant of the zenith angle. The formulation presented in this paper is general enough to be applicable to a wide variety of atmospheric conditions, such as spherical and nonspherical atmospheres, and vertically and horizontally homogeneous as well as inhomogeneous atmospheres. Formulation for the case when atmospheric refraction is important also is presented here.

1. Introduction

Calculation of the solar radiation transmitted through the atmosphere between an arbitrary point and a reference point along a ray path requires knowledge of the two main quantities; *viz*., the number density of the absorbing gases and scattering media, and the distance between them.

The distance of an arbitrary point along the ray path with respect to the reference point, is, in general, calculated in an approximate manner by using the air mass function; namely, the secant of the local zenith angle that the ray path makes at the reference point. This distance, together with a vertical profile of the distribution of the species, will allow us to obtain the column density and the optical thickness for the transmission calculation. However, the airmass function approximation is a direct consequence of modeling the atmosphere as a plane-parallel layer. In general, the plane-parallel approximation is considered valid for a small local zenith angle Z at any given point in the atmosphere; when $Z < 70°$. When $Z > 70°$, this approximation is no longer valid (Rozenberg, 1966; McCartney, 1976). In many satellite remote sensing techniques, one encounters cases in which $Z \approx 90°$ is of particular importance; for example, in the satellite solar occultation measurements, the twilight measurements, or in the simulation of the diurnal variation of minor atmospheric constituents. Several treatments for the computation of optical paths or depth for $Z > 70°$ have appeared in literature; e.g., Swider, 1964; Wilkes, 1954; Chapman, 1931; Goody, 1963; Green and Martin, 1966; Murphy and Kim, 1975. All these papers deal with one-dimensional cases for a spherically symmetric atmosphere. In some cases, that particular situation can be solved by introducing the so-called Chapman function or generalized Chapman function (GCF) (see, e.g., Green and Martin, 1966). This approach, like the others mentioned, is appropriate for the particular case when atmospheric conditions vary with altitude only. In other words, the state of the atmosphere is assumed to have spherical symmetry. The Chapman function is the special case of the GCF and is used when atmospheric constituents follow an exponential vertical distribution. In addition to the restriction of spherically symmetric atmosphere, the abovementioned approaches become unfeasible for problems where the time variation of the atmospheric state is important.

The number density, the second quantity required for the transmissivity calculation, is obtained conventionally by employing a model atmosphere of one-dimensional distribution of the optically active species. With increased knowledge of the earth's atmosphere and the demand for better results of transmissivity calculations, the distribution of atmospheric species based on observations in two or three dimensions should be used. For example, in the computation of thermal heating due to solar absorption by atmospheric constituents, one needs to consider two or three dimensional treatment of this problem. Thus, determination of the position includ-

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ing altitude, latitude and longitude of an arbitrary point along the ray is important for this purpose.

The goal of this paper is to derive formulas for the general problem. These formulas give the latitude, longitude and altitude of any point on the ray passing through a given point in the atmosphere, if the distance between the two points is given. An example is given of the application of these general formulas to determine the ratio of the diurnal average solar flux to the flux at noon for different wavelength bands, altitudes and latitudes. The results thus obtained are compared with those of Rundel (1977). Formulation for the case when refraction is important is also presented here.

2. Theoretical formulation

We shall derive the formulas for two cases, namely, without and with atmospheric refraction being present in the calculations.

a. Without atmospheric refraction

Consider first the path length, i.e., the distance between a given point and the edge of the earth's atmosphere in the direction toward the sun. The relative position of the ray path from an arbitrary point A with respect to the earth's center (O) is shown schematically in Fig. 1. The definition of notation used is given below:

- A' normal projection of A on the equatorial plane
- B point along the ray path characterized by showing the shortest distance between the path and the center of earth
- C point at which the ray path intersects the y, z plane
- L phase angle (local time) of A measured from noon
- P denotes top of the atmosphere; the absorbing gases between this point and the sun are assumed to have negligible effect
- X an arbitrary point along the ray path between P and A
- X' normal projection of X on the equatorial plane
- Z zenith angle of A
- β solar declination, positive or negative depending on whether the sun is above or below the xy plane
- ϕ latitude of A.

Four normal views of the plane OPA are presented in Figs. 2a–2d, which correspond to four different situations of A, viz., (a) $\beta > 0°, Z \leqslant 90°$; (b) $\beta > 0°$, $Z > 90°$; (c) $\beta < 0°, Z \leqslant 90°$; and (d) $\beta < 0°, Z > 90°$, respectively.

A normal view of plane AA′X′X is also shown in Fig. 2e. By referring to Fig. 2a, and by using a trigonometric identity relation, we have

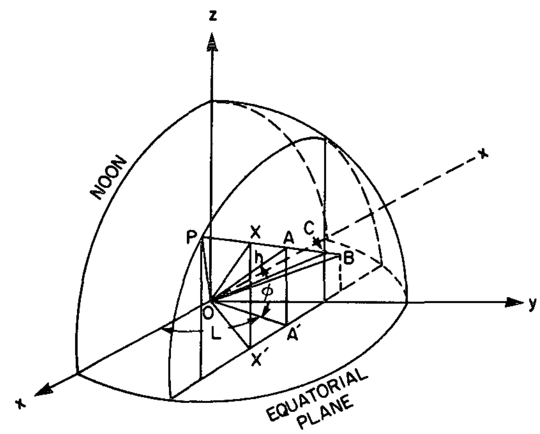

FIG. 1. The position of the ray path $\overline{PB}$ of point $A(h, \phi, L)$ relative to the center of the earth (O). Point P is the location at which the ray path merges into the atmosphere. Point B is a point along the ray path $\overline{PB}$ that exhibits the shortest distance to O.

$$\overline{PB}^2 = \overline{PO}^2 - \overline{BO}^2$$

or

$$\overline{PB} = (\overline{PO}^2 - \overline{BO}^2)^{1/2}. \quad (1)$$

Since $\overline{BO} = \overline{AO} \sin Z$, and $\overline{PB} = \overline{PA} + \overline{AB}$, we have

$$\overline{PA} + \overline{AB} = (\overline{PO}^2 - \overline{AO}^2 \sin^2 Z)^{1/2}$$

or

$$\overline{PA} = (\overline{PO}^2 - \overline{AO}^2 \sin^2 Z)^{1/2} - \overline{AB}.$$

From triangle $\triangle OAB$, we get $\overline{AB} = \overline{AO} \cos Z$. Therefore, $\overline{PA} = (\overline{PO}^2 - \overline{AO}^2 \sin^2 Z)^{1/2} - \overline{AO} \cos Z$. Let $\overline{PO} = h_t$, and $\overline{AO} = h$, finally, we obtain

$$S = (h_t^2 - h^2 \sin^2 Z)^{1/2} - h \cos Z, \quad (2)$$

where $\overline{PA} = S$, the ray pathlength. In Eq. (2), h_t and h are the distances from points P and A to the center of the earth, respectively. For convenience, we will denote $h_t(h)$ as the absolute altitude of point P(A).

Neglecting the effect of refraction, zenith angle Z of a given point in the atmosphere is a function of latitude ϕ, the phase angle L and solar declination β, and is given by

$$\cos Z = \sin\beta \sin\phi + \cos\beta \cos\phi \cos L. \quad (3)$$

As a result, for a given solar declination, the path length can be determined for a given point $A(h, \phi, L)$ by using Eqs. (2) and (3). It is easy to show that the latitude and the phase angle of $P(h_t, \phi_t, L_t)$ are given by

$$\sin\phi_t = h_t^{-1}[S \sin\beta + h \sin\phi], \quad (4)$$

$$\sin L_t = \frac{h}{h_t} \frac{\cos\phi}{\cos\phi_t} \sin L, \quad \phi_t \neq 90°, \quad (5)$$

respectively, and that $L_t = 0°$ for $\phi_t = 90°$.

Although the above formulation is referred to the

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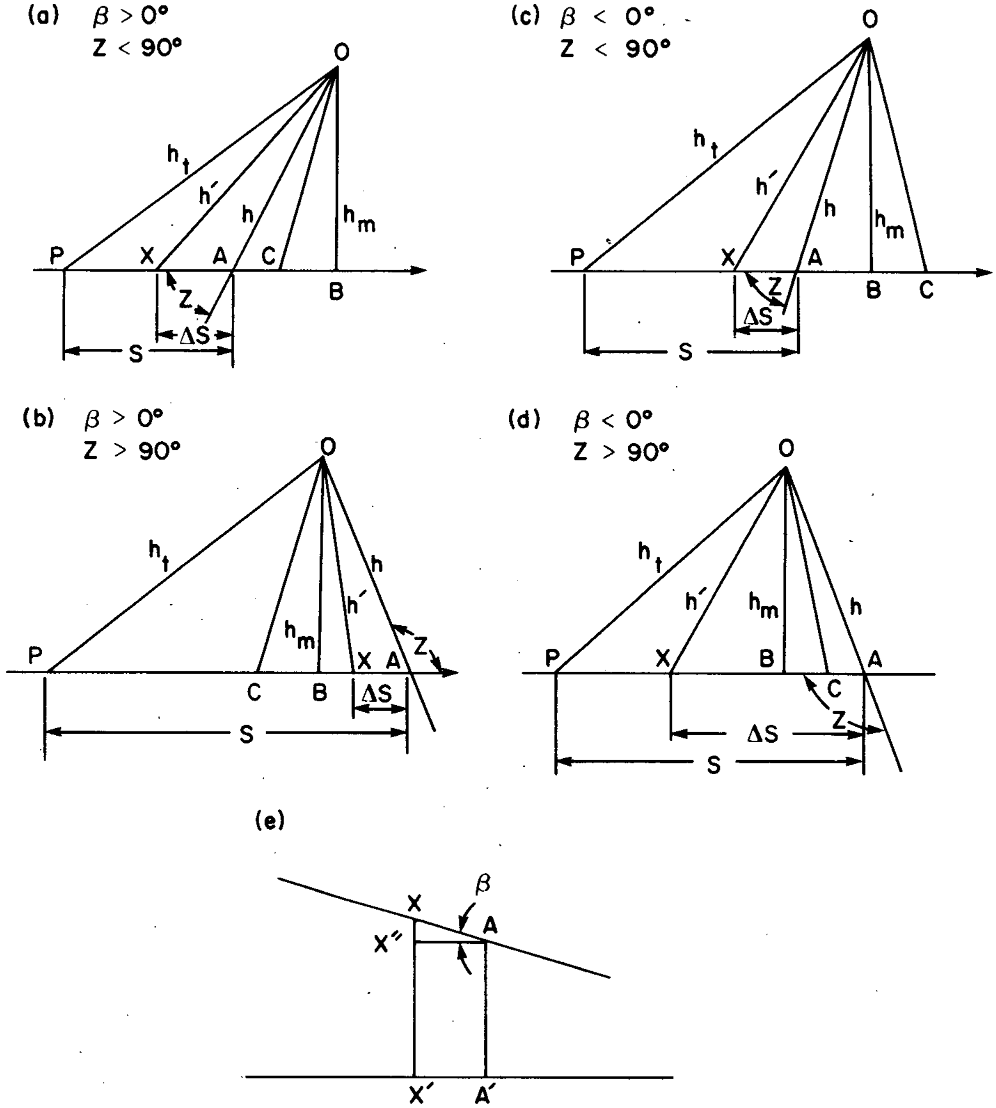

FIG. 2. Four normal views of the plane *OXPB* corresponding to four different cases: (a) $\beta > 0°$; $Z \leqslant 90°$; (b) $\beta > 0°$, $Z > 90°$; (c) $\beta < 0°$, $Z \leqslant 90°$; (d) $\beta < 0°$, $Z > 90°$. (e) Normal view of plane AA′XX′ in Fig. 1 for the case (a).

case $\beta > 0$ and $Z < 90°$ (Fig. 2a), it can be shown that it applies equally well to the other three cases.

Calculation of the solar radiation transmitted in the atmosphere involves the determination of the column number density for the path S. Taking ozone as an example (since it absorbs significant amounts of solar UV radiation in the upper atmosphere), the column number density may be written as

$$u(S) = \int_A^P \rho_3(X)ds', \tag{6}$$

where $\rho_3(X)$ is the number density of ozone at X (Fig. 2a). To perform the integration numerically, one must know ozone number density at discrete points along the ray path. Therefore, it is necessary to locate the position of each of those points. This can be done by using Eqs. (2), (4) and (5) with simple modification. In doing this, we replace S by ΔS, the increment of distance used in numerical calculation, and replace h by h', the absolute height of X associated with ΔS (Fig. 2). Then, we have

$$h' = (\Delta S^2 + h^2 + 2\Delta Sh \cos Z)^{1/2}, \tag{7}$$

and also the equations governing the latitude and phase angle of $X(h', \phi', L')$

$$\sin\phi' = \frac{1}{h'}[\Delta S \sin\beta + h \sin\phi] \tag{8}$$

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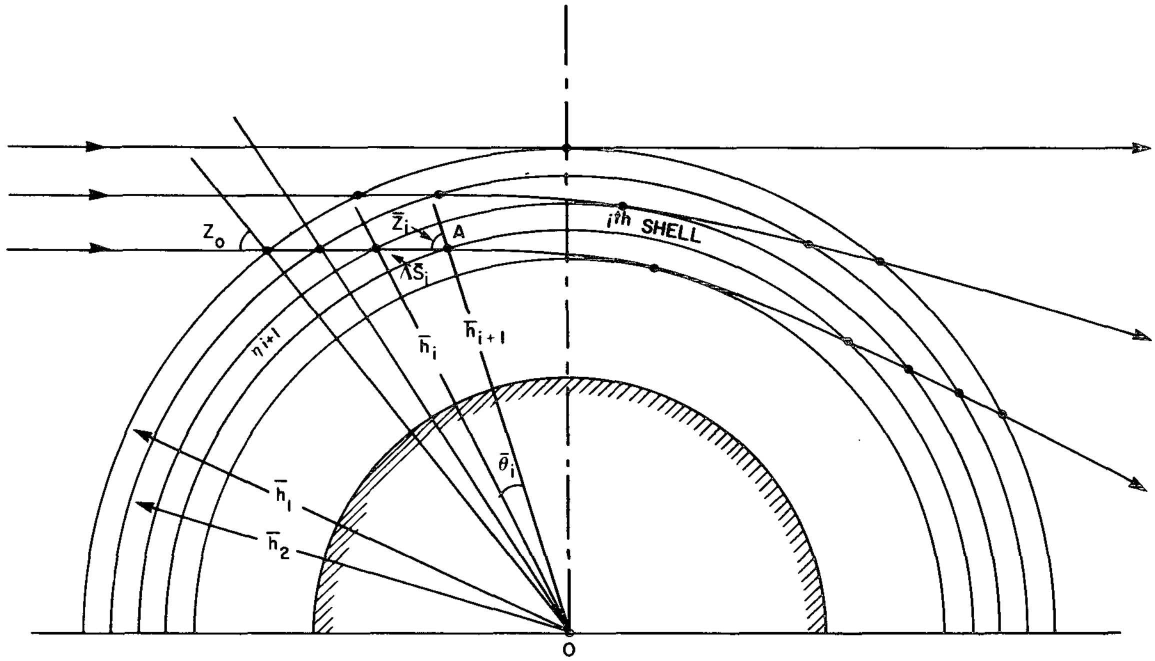

FIG. 3. Schematic diagram shows the effect of atmospheric refraction on ray path.

and

$$\sin L' = \frac{h}{h'} \frac{\cos\phi}{\cos\phi'} \sin L, \tag{9}$$

respectively.

For a given $A(h, \phi, L)$ and an increment ΔS, Eqs. (7), (8) and (9) provide us with the exact position of $X(h', \phi', L')$ along the ray path. The number density at X can then be obtained by using an interpolation method based on a given distribution of ozone. Now, with this new information one may locate the next point at a distance ΔS from X along the path S. This procedure can be repeated until the entire path S is covered. Finally, the integration in Eq. (6) can be carried out. It should be mentioned, when $Z \geqslant \pi/2$, the sunlight cannot reach A if

$$h_m = h \sin(\pi - Z) < h_b, \tag{10}$$

where h_b is the distance from O to either the earth's surface or to the highest altitude level below which the atmosphere is totally opaque.

Obviously, there is no difficulty in applying the column number density calculation described above to atmospheric conditions in more than one dimension. Furthermore, since the derivation is based on a spherical atmosphere, the case of large zenith angle is automatically included.

b. With atmospheric refraction

Finally, we shall discuss the effect of refraction on the ray path. Below ~35 km in the atmosphere, this effect becomes increasingly important as the altitude decreases. Extensive discussion on the refraction phenomena in the curved atmosphere has been reported numerous times in the literature including Weisbrod and Anderson (1959), Tverskoi (1962), Rozenberg (1966), Selby and McClatchey (1972) and Selby *et al.* (1976). Unlike the nonrefracted case, zenith angle now depends not only on the latitude, phase angle and solar declination but also on the atmospheric refraction. The determination of the exact position of an arbitrary point along the ray path of a given point is very difficult. One way to simplify the problem is by assuming that the atmospheric refractive index is only a function of altitude. As a result, the ray path can be described by the well known Snell's Law for spherical stratification which states

$$\bar{h}\eta \sin\bar{z} = \text{constant}, \tag{11}$$

where $\eta = \eta(h)$ is the refractive index, and the overbar indicates a quantity which is directly related to a ray path subjected to the effect of refraction in the atmosphere. As case (a), the column density of interested species can be obtained numerically. In doing this, the atmosphere is divided into many thin concentric shells whose thickness is much smaller than the earth's radius (Fig. 3). By applying Eq. (11) to the lower boundary of the ith shell (Figs. 3 and 5), we have

$$\bar{h}_{i+1}\eta_{i+1} \sin\bar{Z}_i = \text{constant}. \tag{12}$$

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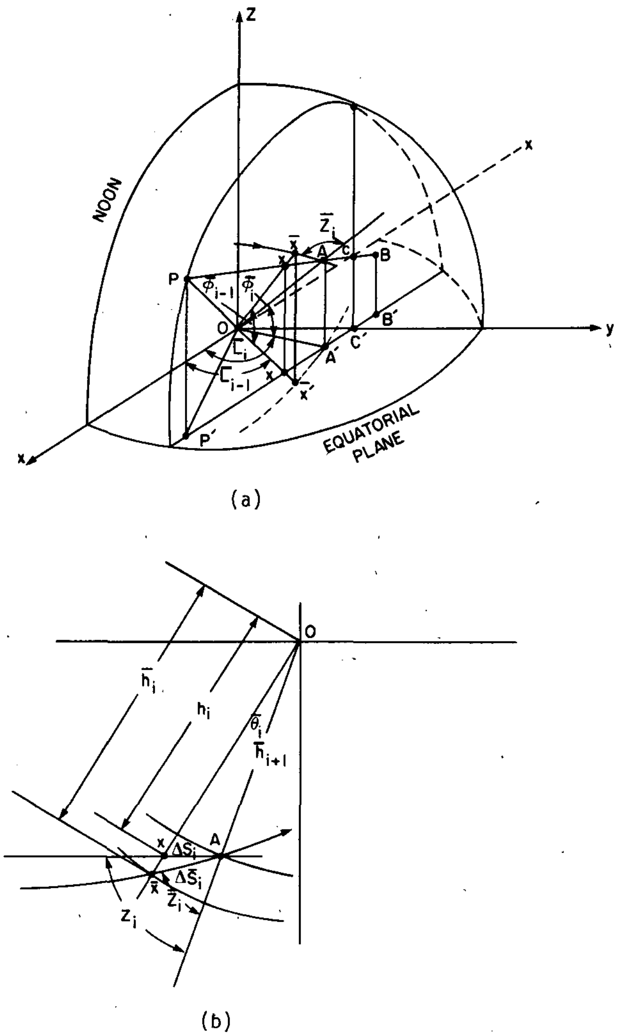

FIG. 4. (a) As in Fig. 1 except the ray path is subject to the effect of atmospheric refraction. (b) Normal view of the plane AOX in (a).

At this point, the problem is still quite difficult to solve for a general distribution of $\eta(h)$. In this paper we will solve the problem based on the approach of Weisbrod and Anderson (1959) by assuming the gradient of $\eta(h)$ is a constant in each shell. With this assumption together with the condition that the latitude ($\bar{\phi}_i$), the phase angle ($\bar{L}_i$), and the zenith angle ($\bar{Z}_i$) at the altitude $\bar{h}_{i+1}$ are known, it can be shown (the Appendix) that the ray path length $\Delta\bar{S}_i$ in the ith shell is given by

$$\Delta\bar{S}_i = \left\{\left[\frac{1}{r_i} + (\bar{h}_i - \bar{h}_{i+1})\right]^2 - \frac{1}{r_i^2}\sin^2\bar{Z}_i\right\}^{1/2} - \frac{1}{r_i}\cos\bar{Z}_i, \quad (13)$$

where

$$r_i = \frac{1}{\bar{h}_{i+1}} - \frac{\eta_{i+1} - \eta_i}{(\bar{h}_i - \bar{h}_{i+1})\eta_{i+1}},$$

and the associated angle $\bar{\theta}_i$ (Figs. 3, 4 and 5) is

$$\bar{\theta}_i = \bar{\gamma}_i - (\bar{Z}_i - \bar{Z}_{i-1}), \quad (14)$$

(the Appendix), where $\bar{\gamma}_i$ is the bending of the ray path (Fig. 5) when it passes through the ith shell and is given by

$$\bar{\gamma}_i = \frac{2(\eta_{i+1} - \eta_i)}{\cot\bar{Z}_i + \cot\bar{Z}_{i-1}}. \quad (15)$$

It should be mentioned that this equation is equivalent to Eq. (16) of Weisbrod and Anderson (1959). It is easy to show that Eq. (13) will reduce to Eq. (2) when the refractive index of the atmosphere is a constant (the Appendix). Since the atmospheric species of interest may exhibit two- or three-dimensional distribution, one needs to know the locations of the points along the ray path at each level in order to calculate the column density. The equations governing the latitude ($\bar{\phi}_{i-1}$) and the phase angle ($\bar{L}_{i-1}$) of the point at an absolute altitude $\bar{h}_i$ along the ray path, i.e. $\bar{X}$, are found to be (Fig. 4)

$$\sin\bar{\phi}_{i-1} = \frac{1}{h_i}(\Delta S_i \sin\beta + \bar{h}_{i+1}\sin\bar{\phi}_i), \quad (16)$$

$$\sin\bar{L}_{i-1} = \frac{\bar{h}_{i+1}}{h_i}\frac{\cos\bar{\phi}_i}{\cos\bar{\phi}_{i-1}}\sin\bar{L}_i, \quad \bar{\phi}_{i-1} \neq 90°, \quad (17)$$

where

$$h_i = \bar{h}_{i+1}\sin Z_i \csc(Z_i - \bar{\theta}_i),$$

$$\Delta S_i = [h_i^2 - \bar{h}_{i+1}^2\sin^2(Z_i)]^{1/2} - \bar{h}_{i+1}\cos Z_i.$$

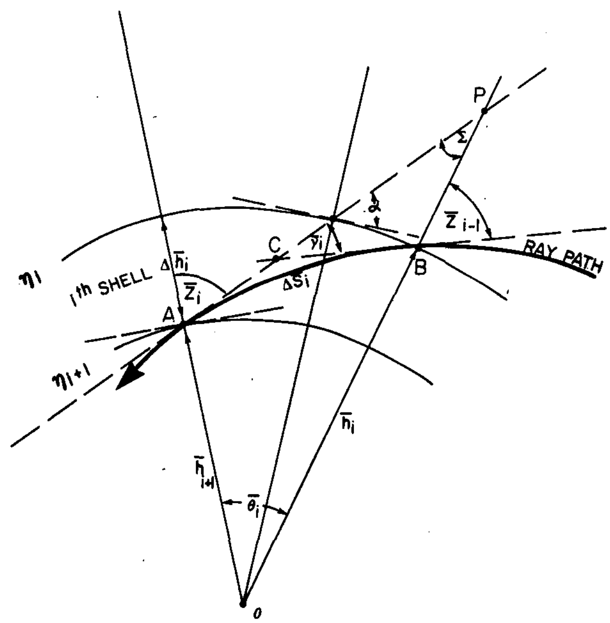

FIG. 5. Schematic diagram shows the refracted ray path when it passes the ith shell.

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One should bear in mind that when refraction is important, the apparent zenith angle $\bar{Z}$ of the incident radiation at the volume element of interest is different from the one when radiation is not subjected to the effect of refraction. Thus, zenith angle $\bar{Z}$ at the reference point A cannot be determined by using Eq. (3) and has to be obtained by other means. Briefly, this can be done with a first guess of the zenith angle at A. Let it be $\bar{Z}_g(A)$. Then, the location as well as zenith angle $\bar{Z}_i$ ($i = 1, 2 \cdots N_{35\,\mathrm{km}}$) of points along the ray path at levels associated with the shells can be determined by applying Eqs. (13)–(17). Since, above 35 km, the effect of refraction is negligible, we may use Eq. (3) to calculate the true zenith angle Z (35 km) based on the information of ray path location i.e., coordinates, at 35 km obtained according to the guessed zenith angle $\bar{Z}_g(A)$. If the first guess of $\bar{Z}_g(A)$ is very close to the apparent one, $\bar{Z}_i$ ($i = N_{35\,\mathrm{km}}$) should be also very close to Z (35 km). Now, if the difference between Z (35 km) and $\bar{Z}_i$ ($i = N_{35\,\mathrm{km}}$) is not acceptable, one may take another guess of $\bar{Z}_g(A)$, and repeat the procedure until the result is of the required accuracy. Once this apparent zenith angle at the reference point is obtained, one can calculate numerically the column density of the interested species along the refracted path. In doing this, it is necessary to apply Eqs. (13)–(17) to each shell again and the procedure is similar to the one described in the nonrefracted case in Section 2a.

3. Sample application

As an example, the formulas derived in Section 2a are used to determine the ratio of diurnal average solar flux to the flux at noon at different altitudes, latitudes and wavelength intervals for the case of an absorbing gas such as ozone. The results will be compared with those obtained with the empirical formula of Rundel (1977). In these calculations, ozone is assumed to be the only absorber and ozone vertical profile of the standard atmosphere of 1976 is used for the computation of the ozone column number density. Ozone absorption cross sections at

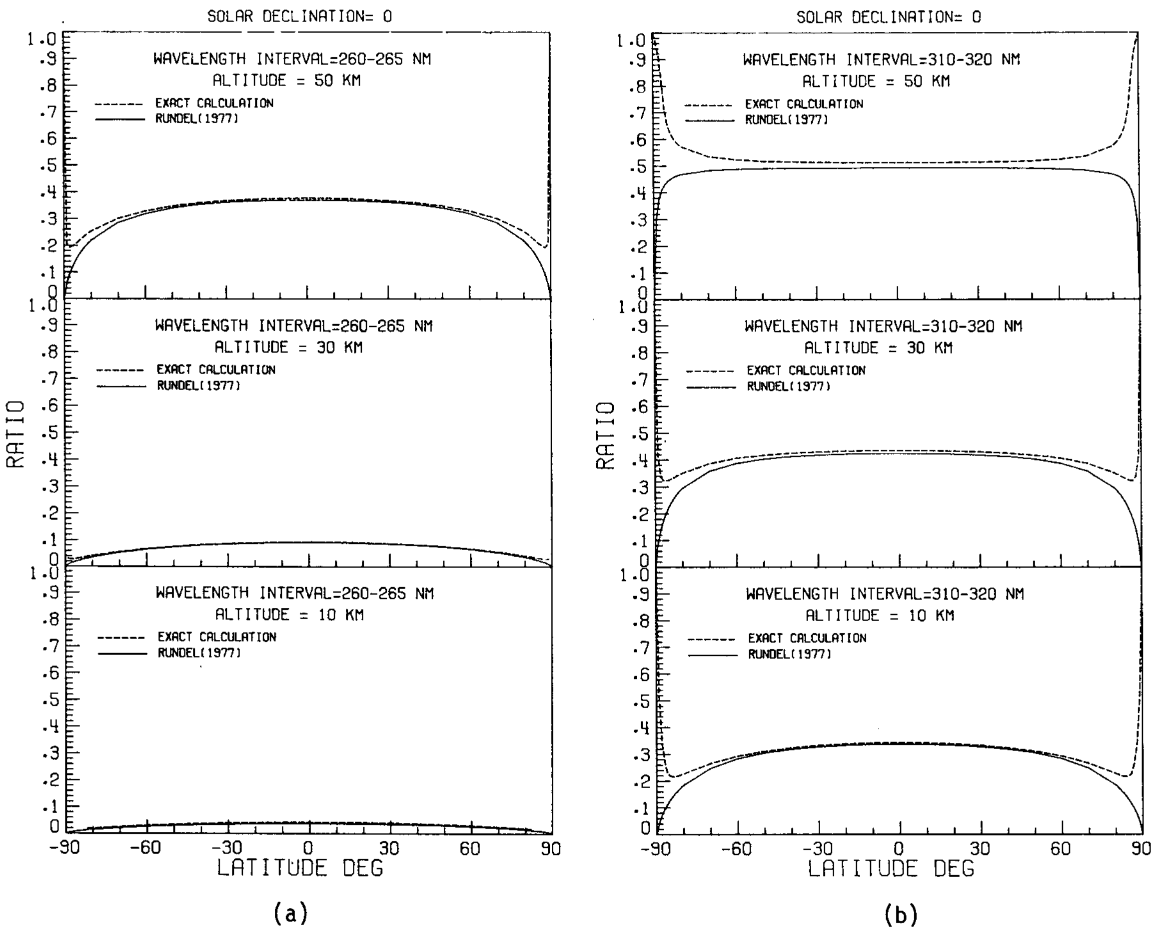

FIG. 6. A comparison of the ratio (see text) between exact calculation and the empirical formula of Rundel (1977) for the case solar declination $\beta = 0$: (a) wavelength interval 260–265 nm and (b) wavelength interval 310–320 nm.

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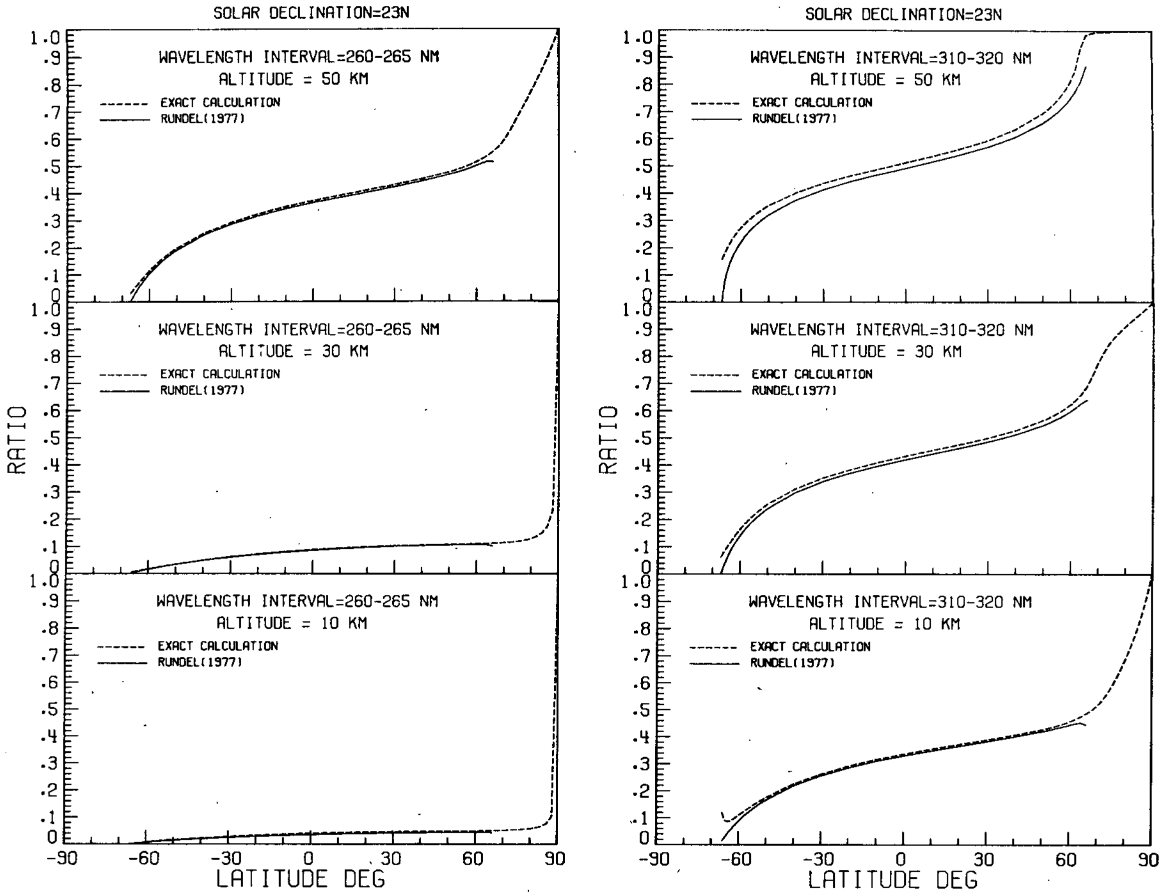

FIG. 7. As in Fig. 5 except solar declination β = 23°N.

different wavelengths are adopted from Rundel *et al.* (1978). Since the distribution of ozone employed is in one dimensional, this example shows only part of the applicability of the derived formulas.

The results for the case when $\beta = 0$, are given in Fig. 6. In general, the empirical expression of Rundel (1977) yields reasonable results at almost all latitudes, except when the latitude is $\geqslant 70°$. It should be mentioned that his empirical formula is valid only when the sum of the absolute values of the latitude and solar declination is less than 90°. In our formulation, this limitation is nonexistent as shown in Figs. 6 and 7. At both poles, our calculations based on formulas for the nonrefracting case (Section 2a) lead to a value of unity at all different altitudes. This is expected, since the poles are constantly exposed to the sun at all altitudes when $\beta = 0$. The difference between Figs. 6a and 6b is due to the difference in the value of ozone absorption cross section. The exact calculation for the case when β = 23°N, is presented in Fig. 7. Differences can be found again in the higher latitude. It is interesting to point out that, in the region between 60 and 90°N at 50 km, the exact calculation is nearly unity. This reflects the fact that at the altitude and wavelength of interest the optical depth does not change very much around a constant latitude circle.

4. Concluding remarks

When refraction is negligible, the location of an arbitrary point along an atmospheric ray path of a given point as well as the distance between them can be determined exactly. This result can be applied to the calculation of atmospheric transmissivity of solar radiation for atmospheric conditions in more than one dimension, and can be applied to the case of solar zenith angle > 90°. When refraction is important, the information along the ray path can be determined approximately. This difference stems primarily from the fact that in the former case we have a closed system of equations. But this is not the case when refraction becomes important (i.e., we must iterate).

The example described in the paper shows the applicability of the formulas to the computation of atmospheric optical path for any local zenith angle for an absorbing gas in one dimension. An applica-

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tion to a two-dimensional atmosphere can be found in the paper by Hong and Wang (1980), in which the atmospheric thermal heating is calculated based on ozone and water vapor distributions in two dimensions, the altitude and the latitude. The refracted path treatment given in this paper is only a theoretical development. Preliminary numerical evaluations for the refractive case, in which the vertical profile of refractive index is assumed to decrease upward monotonically, do not present any computational problem. However, it is quite likely that singularities and roundoff error may cause problems if nonstandard (i.e., realistic) refractivity profiles are used. Work is in progress in checking our iterative formulation for such refractive profiles, and results will be published in the near future. It should be mentioned that these formulas are not restricted to the computation for absorbing gases only; they can be used equally well for the atmospheric scattering species. In addition, work is in progress for applying these formulas to the scattering case, and will be reported subsequently.

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APPENDIX

Ray Path Length Equation for Refracted Case

Since we assume that the index of refraction is a function of altitude, i.e., $\eta = \eta(h)$, the ray path will follow Snell's law for a spherical stratified medium, *viz.*,

$$\eta h \sin Z = \text{constant}. \qquad \text{(A1)}$$

Applying this equation to the boundaries of an arbitrary shell, say the ith shell (Fig. 5), we have

$$\eta\bar{h} \sin\bar{Z} = \eta_{i+1}\bar{h}_{i+1} \sin\bar{Z}_i = \eta_i\bar{h}_i \sin\bar{Z}_{i-1} = \text{constant}. \qquad \text{(A2)}$$

From (A2), we get

$$\sin\bar{Z} = \left(\frac{\eta_{i+1}\bar{h}_{i+1}}{\eta\bar{h}}\right)\sin\bar{Z}_i, \qquad \text{(A3)}$$

$$\cos\bar{Z} = \left(\frac{\eta_{i+1}\bar{h}_{i+1}}{\eta\bar{h}}\right)\left[\left(\frac{\eta\bar{h}}{\eta_{i+1}\bar{h}_{i+1}}\right)^2 - \sin^2\bar{Z}_i\right]^{1/2}, \qquad \text{(A4)}$$

The ray pathlength equation in the differential form is

$$ds = \sec\bar{Z}d\bar{h} \qquad \text{(A5)}$$

or

$$ds = d\bar{h}/\cos\bar{Z}.$$

Substituting Eq. (A4) into (A5), we obtain the equation for the pathlength in the integrated form

$$\Delta\bar{S}_i = \int_{\bar{h}_{i+1}}^{\bar{h}_i} \frac{\left(\dfrac{\eta\bar{h}}{\eta_{i+1}\bar{h}_{i+1}}\right)}{\left[\left(\dfrac{\eta}{\eta_{i+1}\bar{h}_{i+1}}\right)^2 - \sin^2\bar{Z}_i\right]^{1/2}} d\bar{h}. \qquad \text{(A6)}$$

To carry out the integration of Eq. (A6), we introduce the following assumptions:

(a) $\left.\dfrac{d\eta(\bar{h})}{d\bar{h}}\right|_i = -k_i = \text{constant},$

(b) $\bar{h}_i - \bar{h}_{i+1} \ll \bar{h}_{i+1}.$

Assumption (a) implies

$$\frac{\eta_{i+1} - \eta_i}{\bar{h}_i - \bar{h}_{i+1}} = \frac{\eta_{i+1} - \eta}{\bar{h} - \bar{h}_{i+1}} = k_i, \qquad \text{(A7)}$$

and assumption (b) leads to

$$\bar{h} - \bar{h}_{i+1} \ll \bar{h}_{i+1}, \qquad \text{(A8)}$$

in the ith shell.

Employing Eqs. (A7) and (A8), we find

$$\left(\frac{\eta\bar{h}}{\eta_{i+1}\bar{h}_{i+1}}\right) \simeq 1 + r_i(\bar{h} - \bar{h}_{i+1}), \qquad \text{(A9)}$$

where

$$r_i = \left(\frac{1}{\bar{h}_{i+1}} - \frac{k_i}{\eta_{i+1}}\right). \qquad \text{(A10)}$$

Substituting Eq. (A9) into Eq. (A6) yields

$$\Delta\bar{S}_i = \int_{\bar{h}_{i+1}}^{\bar{h}_i} \frac{[1 + r_i(\bar{h} - \bar{h}_{i+1})]d\bar{h}}{\{[1 + r_i(\bar{h} - \bar{h}_{i+1})]^2 - \sin^2\bar{Z}_i\}^{1/2}}. \qquad \text{(A11)}$$

By integrating Eq. (A11), we obtain

$$\Delta\bar{S}_i = \{[r_i^{-1} + (\bar{h}_i - \bar{h}_{i+1})]^2 - r_i^{-2}\sin^2\bar{Z}_i\}^{1/2} - r_i^{-1}\cos\bar{Z}_i. \qquad \text{(A12)}$$

This equation gives the path length of the ray passing through the ith shell.

Now, we shall examine the case when $k_i \to 0$, i.e., an atmosphere with a constant refractive index.

From Eq. (A10), we find that

$$r_i = \frac{1}{\bar{h}_{i+1}}, \quad \text{when} \quad k_i = 0.$$

As a result, Eq. (A12) becomes

$$\Delta\bar{S}_i = \{\bar{h}_i^2 - \bar{h}_{i+1}^2\sin^2\bar{Z}_i\}^{1/2} - \bar{h}_{i+1}\cos(\bar{Z}_i).$$

This equation is identical to Eq. (2) in Section 2a if we write $h_i = h$ and this is what one would expect.

To find the associated angle $\bar{\theta}_i$, let us refer to Fig. 5, again. From $\triangle$APO, we find

$$\bar{\theta}_i = \pi - \Sigma - (\pi - \bar{Z}_i)$$

or

$$\bar{\theta}_i = \bar{Z}_i - \Sigma. \qquad \text{(A13)}$$

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Using triangle $\triangle$BCP, we get

$$\Sigma = \bar{Z}_{i-1} - \bar{\gamma}_i.$$

Therefore, we can write

$$\bar{\theta}_i = \bar{Z}_i - (\bar{Z}_{i-1} - \bar{\gamma}_i)$$

or

$$\bar{\theta}_i = \bar{\gamma}_i - (\bar{Z}_{i-1} - \bar{Z}_i). \tag{A14}$$

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