NOTES AND CORRESPONDENCE

On the (Nearly) Symmetric Instability

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ABSTRACT

It is shown in the limit of small Ekman number that the preferred mode of the symmetric instability exhibits a slight angle of inclination with the direction of the mean flow. The sign of the angle depends on the sign of P − 1, where P is the Prandtl number. It is likely that owing to this effect the range of Richardson numbers for which the instability occurs is increased significantly beyond the limits derived by Kuo (1956) and by McIntyre (1970). Numerical computations are needed to establish this property quantitatively.

1. Introduction

The theory of the instability of a plane parallel baroclinic flow owing to disturbances that are independent of the coordinate in the direction of the flow was first analyzed in detail by Solberg (1936) and has been extended by many authors since that time. In the early work, the instability was considered in the form of the axisymmetric instability of a baroclinic vortex and became known for this reason as the symmetric instability in contrast to the nonaxisymmetric baroclinic instability which is much better known in the meteorological literature. Because the range of Richardson number for which the instability occurs is small, at least for fluids with a Prandtl number of the order unity, there appear to be few meteorological applications. It has been suggested, though, that rain bands or squall lines are caused by this instability (Bennets and Hoskins, 1979; Emanuel, 1979), and Stone (1967) has proposed that the band structure of Jupiter's atmosphere is caused by the symmetric instability.

The importance of doubly diffusive effects for the symmetric instability was discovered by McIntyre (1970), who demonstrated their strongly destabilizing influence on baroclinic shear flows. Thus, the stability of a baroclinic shear flow depends on the Prandtl number, $P \equiv \nu/k$, even in the limit when the kinematic viscosity ν and the thermal diffusivity κ approach zero. This contrasts with the stability criterion derived earlier for an inviscid fluid. Only for $P = 1$ do the two stability criteria agree. The Prandtl number dependence of the critical Richardson number was originally derived by Kuo (1956), but at that time applications to systems with a Prandtl number much different from unity appeared to be remote.

The main purpose of this note is to point out that the symmetric instability is not quite symmetric, in general, and that an inclination of the realized roll-like disturbance with respect to the direction of the mean flow must be expected. Only in a special case with $P \approx 1$ does the angle of inclination vanish. The following analysis will be based on a perturbation approach using the wavenumber α in the direction of the mean flow as small parameter. This approach determines the dependence of the direction of inclination of the preferred mode on the Prandtl number, but does not allow computation of the physically realized angle. The dependence of the sign of the angle on P alone provides potentially useful information in discriminating between different interpretations of observed instability phenomena. But a more detailed understanding of the preferred instability requires a numerical analysis similar to that carried out by Emanuel (1979) in the symmetric case. Work on such a numerical analysis is currently in progress and its results will be reported in a future paper.

2. Mathematical formulation of the problem

We consider a horizontal fluid layer of depth d which is rotating about a vertical axis with the angular velocity Ω. A constant temperature gradient is prescribed

$$\nabla T = \frac{\Delta T}{d}(\mathbf{k} - \epsilon \mathbf{j}), \tag{1}$$

such that the density is stably stratified, $\mathbf{k}$ is the vertical unit vector opposite to the direction of gravity g and $\mathbf{j}$ is a horizontal unit vector. The hori-

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zontal component of the temperature gradient (1) gives rise to a buoyancy torque which is balanced by the Coriolis torque of the thermal wind shear u_0,

$$u_0\mathbf{j} \times \mathbf{k} = \mathbf{j} \times \mathbf{k} g\hat{\gamma}\epsilon\Delta T/2\Omega d, \qquad (2)$$

where $\hat{\gamma}$ is the coefficient of thermal expansion. By arranging for a motion of the upper and lower rigid boundaries with the speeds $u_0 d/2$ and $-u_0 d/2$ in the direction of the shear, the generation of spiraling motion in the Ekman layers at the boundaries can be avoided. Using d, $(2\Omega)^{-1}$, and the temperature difference ΔT between upper and lower boundary as scales for length, time and temperature, respectively, the equations of motion for an infinitesimal disturbance velocity field $\mathbf{v}$ and the heat equation for the deviation θ from the basic temperature distribution can be written in dimensionless form,

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{V}\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{V} + \mathbf{k}\times\mathbf{v} = -\nabla\pi + \mathbf{k}\theta\,\mathrm{R}(u_0/2\Omega)^2 + \mathrm{E}\nabla^2\mathbf{v}, \qquad (3a)$$

$$\nabla\cdot\mathbf{v} = 0, \qquad (3b)$$

$$\frac{\partial}{\partial t}\theta + \mathbf{V}\cdot\nabla\theta = -\mathbf{v}\cdot\mathbf{k} + \epsilon\mathbf{v}\cdot\mathbf{j} + \mathrm{E\,P}^{-1}\nabla^2\theta, \qquad (3c)$$

where $\mathbf{V} \equiv u_0\mathbf{j} \times \mathbf{kk}\cdot\mathbf{r}/2\Omega d$ is the dimensionless thermal wind. The Boussinesq approximation has been assumed and all terms that can be written as a gradient have been combined into $\nabla\pi$. The Richardson number R, the Ekman number E, and the Prandtl number P are defined by

$$\mathrm{R} \equiv \hat{\gamma}g\Delta T/u_0^2 d, \quad \mathrm{E} = \nu/2\Omega d^2, \quad \mathrm{P} = \nu/\kappa,$$

where ν is the kinematic viscosity and κ the thermal diffusivity. A fourth dimensionless parameter is given by $u_0/2\Omega$ which is sometimes called the baroclinicity.

It is convenient to introduce a Cartesian system of coordinates with the z coordinate in the $\mathbf{k}$ direction, the x coordinate in the direction of the thermal wind $\mathbf{V}$, and with the origin on the middle plane of the layer. Eq. (3b) can be eliminated by using the general representation for the solenoidal velocity field $\mathbf{v}$,

$$\mathbf{v} = \nabla \times (\nabla \times \mathbf{k}\hat{\phi}) + \nabla \times \mathbf{k}\hat{\psi}.$$

In the following, the attention will be restricted to steady solutions $\mathbf{v}$ which correspond to the surface of neutral stability in the parameter space of the problem, since McIntyre (1970) has shown that oscillatory disturbances are decaying at the neutral surface of monotonic disturbances. Accordingly, functions ϕ and ψ of the form

$$[\hat{\phi}, \hat{\psi}] = [\phi(z), \psi(z)]\exp(-i\hat{\beta}y + i\hat{\alpha}x)$$

will be assumed. By taking the curl and the curl curl of (3a) and eliminating θ by using (3c), the following equations for ψ and ϕ are obtained:

$$(i\alpha z - \mathrm{E}\nabla^2)\psi - D\phi + i\beta\phi = 0, \qquad (4a)$$

$$(i\alpha z - \mathrm{E\,P}^{-1}\nabla^2)[(i\alpha z - \mathrm{E}\nabla^2)\nabla^2\phi + D\psi] - \mathrm{R}(\alpha^2 + \beta^2)\phi - i\beta D\phi - i\alpha\psi = 0, \qquad (4b)$$

where D denotes the differentiation with respect to z and

$$\alpha \equiv \hat{\alpha}u_0/2\Omega, \quad \beta \equiv \hat{\beta}u_0/2\Omega,$$

has been introduced to simplify the notation. Assuming rigid boundaries at $z = \pm\frac{1}{2}$ we obtain as boundary conditions for ϕ and ψ

$$\phi = D\phi = \psi = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \qquad (4c)$$

When the additional assumption is made that $u_0 \gg 2\Omega$, the horizontal part of the Laplacian ∇^2 can be neglected with respect to the vertical differentiation, i.e.,

$$\nabla^2 \approx D^2, \qquad (5)$$

thereby reducing the number of external parameters of the problem. An alternative and equivalent way of arriving at approximation (5) is based on the hydrostatic assumption (see Walton, 1975). Eqs. (4) can be solved for sufficiently small values of $|\alpha|$ by expanding the variables in power of α,

$$\phi = \phi_0 + \alpha\phi_1 + \cdots, \qquad (6a)$$

$$\mathrm{R} = \mathrm{R}_0 + \alpha\,\mathrm{R}_1 + \cdots. \qquad (6b)$$

In zeroth order, the well-known equation for the symmetric instability is obtained,

$$\mathrm{E}^2\,\mathrm{P}^{-1}\nabla^6\phi_0 + D^2\phi_0\,\mathrm{P}^{-1} - \mathrm{R}_0\beta^2\phi_0 - i\beta D\phi_0(1 + \mathrm{P}^{-1}) = 0. \qquad (7)$$

Walton (1975) has shown that the solution of this equation has a characteristic vertical scale of the order $\mathrm{E}^{1/3}$ if E is small, and that it can be written in the form

$$\phi_0(z) = \cos m\pi z\,\exp(i\gamma z), \qquad (8)$$

with

$$\gamma = \beta(1 + \mathrm{P})/2, \qquad (9a)$$

$$\mathrm{R}_0 = \frac{(1 + \mathrm{P})^2}{4\,\mathrm{P}}[1 - \mathrm{E}^2\gamma^4 - (m\pi/\gamma)^2], \qquad (9b)$$

if terms of the order $\mathrm{E}^{1/3}$ are neglected in (7). The Richardson number R_0 reaches its maximum value

$$\mathrm{R}_{0c} = \frac{(1 + \mathrm{P})^2}{4\,\mathrm{P}}[1 - 3(\mathrm{E}\pi^2/2)^{2/3}] \qquad (10a)$$

for $m = 1$ and

$$\gamma = \gamma_c \equiv (\pi/\sqrt{2}\,\mathrm{E})^{1/3}. \qquad (10b)$$

In the following analysis, these properties are not needed, since the expansion in α is independent of the expansion in powers of $E^{1/3}$ of the symmetric solution.

3. First-order perturbation

The terms proportional to α in (4b) yield

$$L\phi_1 = R_1\beta^2\phi_0 + i(1 + P^{-1})\psi_0 - i(1 - P^{-1})zD\psi_0 + i(1 + P^{-1})zED^4\phi_0, \quad (11)$$

where L denotes the operator on the left-hand side of (7) and terms of higher order in E have been neglected. The solvability condition for (11) requires that the right-hand side vanishes after multiplication by ϕ_0^+ and integration over the interval $-\frac{1}{2} \leqslant z \leqslant \frac{1}{2}$. This follows from the fact that the complex conjugate ϕ_0^+ of ϕ_0 satisfies the adjoint problem of (7). Before the solvability condition can be applied for the computation of R_1, Eq. (4a) must be solved for ψ_0. Neglecting the term proportional to α in that equation, the solution satisfying the boundary conditions (5) for ψ_0 can be obtained in the form

$$E\psi_0 = \{i[(\gamma - \beta)(\gamma^2 + \pi^2) - 2\gamma\pi^2]\phi_0 + \pi[2\gamma(\gamma - \beta) - \gamma^2 - \pi^2] \times (\phi_0 \tan\pi z - 2z \cos\gamma/2 - i \sin\gamma/2)\} \times [(\gamma^2 + \pi^2)^2 - 4\gamma^2\pi^2]^{-1}. \quad (12)$$

Using this expression, the solvability condition can be evaluated,

$$R_1\beta^2 = \beta(P - 1)/2\ E(\gamma^2 + \pi^2) + \cdots, \quad (13)$$

where terms of higher order in $E^{1/3}$ have not been given explicitly.

The result (13) demonstrates that the range of unstable Richardson numbers is extended by disturbances with a small but finite value of α. The critical value R_c of the Richardson number exceeds R_{0c} when α assumes the same sign as $(P - 1)$, i.e.,

$$R_c > R_{0c} \quad \text{for} \quad \begin{cases} \alpha < 0, & \text{if} \quad P < 1 \\ \alpha > 0, & \text{if} \quad P > 1. \end{cases} \quad (14)$$

The perturbation approach used here is not suitable for more than a qualitative analysis of the problem. An inspection of the equations shows that R_2 is a quantity of the order unity just as has been found for R_1 according to expression (13). A maximum of R as a function of α must thus be expected for a value of α of the order unity at which point higher order terms are no longer negligible. While a value of α of the order unity would translate into only a small angle with direction of the thermal wind, since β is of the order $E^{-1/3}$ in this analysis, it would significantly change the critical value of the Richardson number. Numerical calculations are in progress presently to establish that a maximum of R is indeed reached for values of α of the order unity as suggested by this analysis.

4. Discussion

In order to understand the physical effects that favor inclined disturbances relative to the symmetric ones, the mechanism of the symmetric instability must be recalled briefly. The growing symmetric disturbance gains kinetic energy from the mean flow by exchanging fluid parcels at different levels of the layer while conserving the total momentum. Insight into the Prandtl number dependence of this mechanism can be gained by considering the orientation of the wave vector, $\gamma\mathbf{k} - \hat{\beta}\mathbf{j}$. The angle η of this vector with the vertical is given by

$$\eta \approx \hat{\beta}/\gamma = 2\epsilon\ R/(1 + P) \approx \epsilon(1 + P)/2P. \quad (15)$$

For $P = 1$, we find $\eta \approx \epsilon$, i.e., the disturbing motion is primarily parallel to the isotherms of the basic state in order to minimize the inhibiting effect of the stable stratification. An angle $\eta > \epsilon$ corresponds to an increased vertical component of motion which, in turn, provides a more efficient release of the kinetic energy of the mean flow. But this is possible only in the case of Prandtl numbers $P < 1$, where thermal diffusion moderates the inhibiting effect of thermal stratification. At a finite negative value of α, however, the motion can stay more closely parallel to the isotherms for $\eta > \epsilon$ because the second part

$$-i\alpha\psi_0 \approx \alpha(\gamma - \beta)\phi_0/\gamma^2 = \alpha\beta(P - 1)\phi_0/2\gamma^2 \quad (16)$$

of the y component of the velocity field adds to the part, $\beta\gamma\phi_0$, present in the symmetric case.

For $P > 1$, additional energy is released by tapping the potential energy stored in the fluid layer owing to the horizontal component of the basic temperature gradient. This requires $\eta < \epsilon$ since fluid parcels of higher temperature must be exchanged with those of lower temperature, but at a higher level. Again, the effectiveness of the energy release is increased by the increase of the y component of the velocity owing to the contribution (16). Generally speaking, a finite value of α allows the disturbance motion to be more perpendicular to the mean flow, which is advantageous, since the parallel component does not contribute to the release of energy. While direct observational evidence for finite values of α may be difficult to obtain because of its smallness in comparison with β, the increase of the critical Richardson number for effective Prandtl numbers different from unity is likely to be a significant effect.

The fact that the symmetric instability is not quite symmetric may suggest that the mechanism of instability is more closely related to the nonaxisymmetric barolinic instability than is commonly assumed. But the analysis of this paper does not indicate a *shift* toward geostrophy associated with a finite value of α. The fact that the same inclination with respect to the direction of the mean flow has also been found in a numerical analysis of shear flow instabilities in a fluid of vanishing Prandtl number (Busse and Chen, 1981) where the non-axisymmetric barolinic instability does not exist, indicates that the nearly symmetric instability represents a distinct mechanism of instability.

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Energy Dissipation Rates of Turbulence in the Stable Free Atmosphere

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ABSTRACT

For stable stratification, it is pointed out that there exists a strong correlation between the intensity of atmospheric turbulence and the energy dissipation rate ϵ. It is given in terms of the variance of vertical velocity σ_w^2 and the Brunt-Väisälä frequency ω_B by $\epsilon = 0.4\,\sigma_w^2\omega_B$. This relation is argued to have a wider range of applicability in the stratosphere than the previous relation $\epsilon = \beta\sigma_w^3$, where β is taken to be constant.

1. Introduction

It was demonstrated by Chen (1974) and Heck and Panofsky (1975)[1] that there often exists a strong correlation between the intensity of atmospheric turbulence and the energy dissipation rate ϵ. This correlation is given in terms of σ_w^2, the variance of vertical velocity, by the relation

$$\epsilon = \beta\sigma_w^3, \tag{1}$$

where β is treated as a "constant" inverse length adjusted to fit observations. However, it was pointed out by Chen that ". . . a single curve can hardly represent the dissipation state in the atmosphere." Evidence of this limitation was given by Heck and Panofsky (1975)[1] who found that (1) is quite good for larger σ_w but not as good for smaller σ_w; i.e., it is seen in Fig. B1 of Heck and Panofsky[1] that (1) predicts values of ϵ that are often too small by a factor of 8 when $\sigma_w < 50$ cm s^{-1}. Indeed, that Figure suggests that ϵ is proportional to σ_w^2, at least as well as it suggests ϵ is proportional to σ_w^2, over the entire range of $\sigma_w(0 \leqslant \sigma_w \leqslant 200$ m s^{-1}). An inadequacy of (1) is also suggested by Fig. 1 of Chen (1974) which shows that extrapolation of (1) down to $\sigma_w = 2$ cm s^{-1} underestimates ϵ by two orders of magnitude—when β is adjusted to agree with the high-altitude clear air turbulence data of project HICAT (Crooks *et al.*, 1967).[2] The fact that (1) underestimates ϵ at small σ_w was pointed out by Heck *et al.* (1977).

It is clear that (1) should not be uniformly applicable over a wide range of σ_w because β actually varies with the energy containing scale L_1 as $\beta \approx \alpha_w^{-3/2}2\pi L_1^{-1}$ [Chen (1974)—here, α_w is the universal spectral constant for vertical wind fluctuations] and L_1 is not really

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