NOTES AND CORRESPONDENCE

The Geostrophic Coordinate Transformation

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ABSTRACT

The geostrophic coordinate transformation is a contact transformation that preserves the correspondence between the slopes of geopotential surfaces ϕ in physical space and the slopes of the surfaces Φ, the maps of ϕ, in transformed space. The transformation back to physical space may be accomplished by integration along characteristic surfaces. This technique may be used to determine the time and place where a discontinuity would form as a function of the initial conditions. A model solution is used to illustrate properties of the transformation.

1. Introduction

Coordinate transformations are frequently employed to cast the hydrodynamical equations, governing large-scale atmospheric and oceanic motions, into a more suitable form. Such a form is usually dictated by computational considerations associated, for example, with boundary conditions or with resolution enhancement. Transformations used for these purposes do not necessarily provide complementary analytical simplifications. Moreover, the introduction of new coordinates, to exploit certain dynamical properties of the basic equations, is a well-known analytical tool commonly employed in the mathematical theory of compressible gas dynamics. However, the dynamics of large-scale unsteady atmospheric and oceanic flow is governed by a more complex set of equations, than the equations of gas dynamics, due to the relative importance of the earth's rotation and density stratification. As a consequence, the primitive form of the basic equations of geophysical fluid dynamics has apparently not yielded to a simplifying coordinate transformation.

Yet approximate forms of the primitive equations have been further simplified by the use of coordinate transformations. A notable example is the transformation of the *geostrophic momentum equations*, introduced by Eliassen (1948), to a new system that is very similar in form to the quasi-geostrophic system of equations: Eliassen (1962) first introduced this coordinate transformation in his study of two-dimensional frontal dynamics. He referred to the transformed horizontal coordinate, normal to the front, as the *absolute momentum coordinate*. The absolute momentum in this case is associated with geostrophic flow in a Cartesian coordinate system that rotates with constant angular velocity about a vertical axis. The transformed system, referred to as the *semi-geostrophic equations* by Hoskins (1975), is more amenable to analysis than the original set which contains an explicit representation of the ageostrophic advecting velocity. However, the transformation back to physical space must be carried out to complete the solution. Hoskins and Bretherton (1972) employed a variant of the absolute momentum coordinate in a study of frontogenesis and, later, Hoskins (1975) extended the transformation to three dimensions. The horizontal coordinates introduced in Hoskins' analysis, which differ only by a constant factor from absolute momentum coordinates [see (7)], are referred to as *geostrophic coordinates*. This latter terminology will be used here.

The present analysis places the geostrophic coordinate transformation on a firmer theoretical foundation. It will be shown that this transformation is an example of a *contact transformation*; other examples have been introduced in the theory of gas dynamics. Then a strategy will be developed for dealing with the transformation from geostrophic coordinate space back to physical space, and finally, a relatively simple example will be examined.

2. Contact transformation

For simplicity, consider a curve C defined by the function of one variable $\phi = \phi(x)$. A *point transformation* maps C into a new curve Γ defined by

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the function $\Phi = \Phi(X)$, where $X = X(x)$. A *contact transformation*, in this example, is one that maps two curves C and C' which are tangent at point P into two curves Γ and Γ' which are tangent at point P', the map of P. In general, a *contact transformation* maps two multi-dimensional surfaces (hypersurfaces) $S(x_i)$ and $S'(x_i)$, $i = 0, 1, \ldots, m$, tangent at point P into two hypersurfaces Σ and Σ' which are tangent at P', the map of P (Jeffrey and Taniuti, 1964; Appendix F).

The mapping technique is most clearly demonstrated by restricting attention to a single independent variable x. Following Jeffrey and Taniuti, we consider a mapping of the points P of C by means of a transformation which depends not only on P, but also on the tangent to C at P. The appropriate transformation may be expressed as

$$X = X(x, \phi, p),\qquad \Phi = \Phi(x, \phi, p), \tag{1}$$

where $p \equiv d\phi/dx$. The slope of the transformed curve Γ at point P' in X space is

$$\left.\frac{d\Phi}{dX}\right|_{\Gamma(P')} = \left(\frac{\dfrac{\partial\Phi}{\partial x} + \dfrac{\partial\Phi}{\partial\phi}p + \dfrac{\partial\Phi}{\partial p}\dfrac{dp}{dx}}{\dfrac{\partial X}{\partial x} + \dfrac{\partial X}{\partial\phi}p + \dfrac{\partial X}{\partial p}\dfrac{dp}{dx}}\right)_{C(P)}. \tag{2}$$

Another curve C', tangent to C at P, will map into a curve Γ' that will have a point P' in common with Γ. However, Γ' will not necessarily be tangent to Γ at P' since dp/dx in (2) may be different for C and C'. The curves Γ and Γ' will be tangent at P' if $d\Phi/dX$ only depends on x, ϕ and p and is independent of dp/dx. Then (2) may be expressed as

$$\left.\frac{d\Phi}{dX}\right|_{\Gamma(P')} = \left(\frac{\dfrac{\partial\Phi}{\partial x} + \dfrac{\partial\Phi}{\partial\phi}p}{\dfrac{\partial X}{\partial x} + \dfrac{\partial X}{\partial\phi}p}\right)_{C(P)} \tag{3}$$

and

$$\frac{\partial X}{\partial p}\left(\frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial\phi}p\right) = \frac{\partial\Phi}{\partial p}\left(\frac{\partial X}{\partial x} + \frac{\partial X}{\partial\phi}p\right) \tag{4}$$

must be satisfied for (1) to represent a contact transformation.

Suppose we consider the transformation

$$\left.\begin{aligned} X &= x + p \\ \Phi &= \phi + p^2/2 \end{aligned}\right\}, \tag{5}$$

which satisfies condition (4) for a contact transformation, while (3) yields

$$\left.\frac{d\Phi}{dX}\right|_{\Gamma(P')} = \left.\frac{d\phi}{dx}\right|_{C(P)}. \tag{6}$$

This transformation (5) is a one-dimensional analog of the geostrophic coordinate transformation.

We now consider the complete geostrophic coordinate transformation[1] (Hoskins, 1975), expressed in nondimensional form as

$$\left.\begin{aligned} &X = x + \mathrm{Ro}v, \quad Y = y - \mathrm{Ro}u \\ &Z = z, \quad T = t \\ &\Phi = \phi + \tfrac{1}{2}\,\mathrm{Ro}(u^2 + v^2) \end{aligned}\right\}, \tag{7}$$

where (x, y, z) represent the usual Cartesian coordinates, t is time, ϕ is the geopotential, the geostrophic components (u, v) along the (x, y) coordinate axes may be expressed as $(-\partial\phi/\partial y, \partial\phi/\partial x)$ and Ro is the Rossby number. Eq. (7), assumes that the flow field can be completely specified by one dependent variable, in this case the geopotential ϕ. This assumption is valid for the system derived under the geostrophic momentum approximation. Moreover, any solution for the potential $\Phi(X, Y, Z, T)$ corresponds to a solution (ϕ, u, v) in the physical plane provided that the Jacobian defined as

$$J' \equiv 1 - \mathrm{Ro}(\Phi_{XX} + \Phi_{YY}) + \mathrm{Ro}^2(\Phi_{XX}\Phi_{YY} - \Phi_{XY}^2) \tag{8}$$

does not vanish.

Hoskins has shown that

$$(\Phi_X, \Phi_Y, \Phi_Z) = (\phi_x, \phi_y, \phi_z) \tag{9}$$

which may also be verified by substitution of Φ and either X, Y or Z, given by (7), into (3). It also may be verified that

$$\Phi_T = \phi_t. \tag{10}$$

Consequently, the geostrophic coordinate transformation is a contact transformation.

3. Transformation from geostrophic to physical space

The reverse transformation from geostrophic space back to physical space may be accomplished by application of (7) once $\Phi(X, Y, Z, T)$ has been determined. In special cases, such as the one treated by Andrews and Hoskins (1978), the transformation may be accomplished analytically; in general, the transformation would be carried out by means of a numerical algorithm.

Here we shall examine the underlying principle that is inherent in the transformation back to physical space. For this purpose it is necessary to use the derivative formulas presented by Hoskins (1975)

[1] Yudin (1955) (see Phillips *et al.*, 1960) first introduced the transformation $\xi = x + \mathrm{Ro}v'$, $\eta = y - \mathrm{Ro}u'$ for the horizontal coordinates, where (u', v') represents the horizontal velocity. Since a potential function was not defined, the practical use of this transformation is not apparent.

$$\left.\begin{aligned}\frac{\partial}{\partial x} &= \left(1+\mathrm{Ro}\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial X}+\mathrm{Ro}\frac{\partial v}{\partial y}\frac{\partial}{\partial Y}\\ \frac{\partial}{\partial y} &= -\mathrm{Ro}\frac{\partial u}{\partial x}\frac{\partial}{\partial X}+\left(1-\mathrm{Ro}\frac{\partial u}{\partial y}\right)\frac{\partial}{\partial Y}\\ \frac{\partial}{\partial z} &= \mathrm{Ro}\left(\frac{\partial\theta}{\partial x}\frac{\partial}{\partial X}+\frac{\partial\theta}{\partial y}\frac{\partial}{\partial Y}\right)+\frac{\partial}{\partial Z}\end{aligned}\right\}, \quad (11)$$

where $\theta = \phi_z$ represents potential temperature.

Applying the first two operators of (11) to (10) and making use of (9) yields

$$\left(\frac{\partial}{\partial t}+\mathrm{Ro}F\frac{\partial}{\partial x}+\mathrm{Ro}G\frac{\partial}{\partial y}\right)v=-F, \quad (12a)$$

$$\left(\frac{\partial}{\partial t}+\mathrm{Ro}F\frac{\partial}{\partial x}+\mathrm{Ro}G\frac{\partial}{\partial y}\right)u=G, \quad (12b)$$

where $F \equiv -v_T$ and $G \equiv u_T$. For present purposes, Φ is assumed to be a known real function of X, Y, Z and T, with continuous derivatives of all orders and the Jacobian, given by (8), does not vanish. Then F and G may be determined and expressed as $F(x + \mathrm{Ro}v,\ y - \mathrm{Ro}u,\ z,\ t)$ and $G(x + \mathrm{Ro}v,\ y - \mathrm{Ro}u,\ z,\ t)$. Consequently, (12a, b) are quasilinear. Further, this system may be rewritten in the characteristic form

$$\frac{dv}{dt}=-F,\quad \frac{du}{dt}=G, \quad (13)$$

where the characteristic curves are real and given by

$$\frac{dx}{dt}=\mathrm{Ro}F,\quad \frac{dy}{dt}=\mathrm{Ro}G. \quad (14)$$

Moreover, (13) and (14) may be combined to yield

$$\frac{d}{dt}(x+\mathrm{Ro}v)=0,\quad \frac{d}{dt}(y-\mathrm{Ro}u)=0. \quad (15)$$

These results show that the geostrophic coordinates X and Y are conserved along characteristic curves; equivalently, absolute momentum is conserved along characteristic curves.

Similarly, applying the third operator in (11) to (10) yields

$$\frac{\partial\theta}{\partial t}+\mathrm{Ro}F\frac{\partial\theta}{\partial x}+\mathrm{Ro}G\frac{\partial\theta}{\partial y}=H, \quad (16)$$

where $H = \theta_T$ may be determined from Φ. The potential temperature θ may then be determined by integration along the same characteristics.

The characteristic surfaces may be found by first integrating (15) to obtain

$$\left.\begin{aligned}x+\mathrm{Ro}v &= \xi+\mathrm{Ro}v_0(\xi,\eta,z,0)\\ y-\mathrm{Ro}u &= \eta-\mathrm{Ro}u_0(\xi,\eta,z,0)\end{aligned}\right\}, \quad (17)$$

where (ξ, η) represent (x, y) coordinates at the initial time and arbitrary height $Z = z$. In view of (17), $\Phi(X, Y, Z, T) = \Phi(\xi + \mathrm{Ro}v_0,\ \eta - \mathrm{Ro}u_0,\ z,\ t)$ on a characteristic surface. Consequently, the expressions in (14) may be integrated to find

$$\left.\begin{aligned}x &= \xi+\mathrm{Ro}\int_0^t F(\xi+\mathrm{Ro}v_0,\\ &\qquad \eta-\mathrm{Ro}u_0,z,t')dt'\\ y &= \eta+\mathrm{Ro}\int_0^t G(\xi+\mathrm{Ro}v_0,\\ &\qquad \eta-\mathrm{Ro}u_0,z,t')dt'\end{aligned}\right\}. \quad (18)$$

The velocity components (v, u) may then be determined by using (18) to evaluate the terms in (17). Since Φ is presumed to be known, the quadratures in (18) should present little difficulty. Then elimination of time t between the expressions for x and y provides a characteristic surface $S(x, y, z; \xi, \eta)$ for each pair of initial coordinates (ξ, η). If a two-dimensional model is investigated, say, $\partial/\partial y \equiv 0$, then the characteristic curve is provided by the first expression in (18).

Multivalued solutions arise when characteristic surfaces intersect each other. The procedure for finding the time and place at which an intersection occurs will be demonstrated for the particular case examined in the following section.

4. Example

Solutions of the unstable Eady wave problem, expressed in geostrophic coordinate space, will be adopted to illustrate the theoretical development of Section 3. The two-dimensional problem will be examined first. The exact solution, derived by Hoskins and Bretherton (1972), may be represented in the form

$$\Phi(X,Z,T)=\hat{\Phi}(X,Z)e^{\sigma T}, \quad (19)$$

where σ denotes the relative growth rate. As a consequence, $F = -v_T = -\sigma v$ and $G = u_T = 0$, with $u = u(z)$. Substitution into (12a, b) yields

$$\left(\frac{\partial}{\partial t}-\mathrm{Ro}\sigma v\frac{\partial}{\partial x}\right)v=\sigma v. \quad (20)$$

Blumen (1980) has known that the velocity $v(x, z, t)$, derived by Andrews and Hoskins (1978) in terms of an infinite Fourier sine series, satisfies (20). However, suppose this solution is not known. Then integration of (13) gives

$$v=\hat{v}(\xi,z)e^{\sigma t}, \quad (21)$$

where $\hat{v}$ is specified by initial conditions. The characteristic curve, determined from (17), is

$$x=\xi-\mathrm{Ro}\hat{v}(\xi,z)(e^{\sigma t}-1), \quad (22)$$

The geostrophic velocity field $v(x, z, t)$ can be constructed from (21) and (22) by considering different values of ξ over one cycle of the wave.

The expression in (22) represents a family of characteristic curves with ξ as a parameter. Whitham (1974), for example, has shown that v develops an infinite slope, $\partial v/\partial x = \infty$, when characteristic curves intersect. These intersections form an envelope, $t = t(\xi)$, which may be determined by differentiating (22) with respect to the parameter ξ. The value $t = t_c$, at which a discontinuity in v first appears, is given by

$$t_c = \sigma^{-1}[\ln(1 + 1/\mathrm{Ro}\partial\hat{v}/\partial\xi)]_{\min}. \tag{23}$$

Some values of t_c have been displayed by Blumen (1979) for a particular case study.

An exact analytical solution of the three-dimensional nonlinear Eady problem has not been determined. However, Hoskins (1976) has shown that numerical solutions, for the cases that he examined, are reasonably well-represented by linear solutions of the form

$$\Phi(X, Y, Z, T) = \hat{\Phi}(X, Y, Z)e^{\sigma T}. \tag{24}$$

This solution (24) will be adopted for present purposes as a counterpart to (19). Later, the limitations involved will be examined. In this case, (12a, b) reduce to

$$\left[\frac{\partial}{\partial t} - \sigma\,\mathrm{Ro}\left(v\frac{\partial}{\partial x} - u\frac{\partial}{\partial y}\right)\right]v = \sigma v, \tag{25a}$$

$$\left[\frac{\partial}{\partial t} - \sigma\,\mathrm{Ro}\left(v\frac{\partial}{\partial x} - u\frac{\partial}{\partial y}\right)\right]u = \sigma u. \tag{25b}$$

Then use of (17) and (18) leads to the expressions

$$\left.\begin{aligned} x &= \xi - \mathrm{Ro}\gamma\hat{v} \\ y &= \eta + \mathrm{Ro}\gamma\hat{u} \end{aligned}\right\}, \tag{26}$$

where $(\hat{v}, \hat{u})$ are evaluated at (ξ, η, z) and $\gamma \equiv \exp(\sigma t) - 1$. As in the two-dimensional case, (26) may be used with (17) to construct the full solutions for v and u, and θ may then be determined from (16). Elimination of γ between the expressions in (26) provides the representation of the characteristic surfaces,

$$(x - \xi)\hat{u} + (y - \eta)\hat{v} = 0. \tag{27}$$

The critical time t_c, at which characteristic surfaces intersect, may be determined by following the procedure used to derive (23). Here an alternate and more direct approach provided by Whitham (1974, Section 2.1) will be followed. Consider the characteristic surfaces from (ξ, η), given by (26), and from $(\xi + \delta\xi, \eta + \delta\eta)$ given by

$$\left.\begin{aligned} x &= \xi + \delta\xi - \mathrm{Ro}\gamma\left(\hat{v} + \frac{\partial\hat{v}}{\partial\xi}\delta\xi + \frac{\partial\hat{v}}{\partial\eta}\delta\eta\right) \\ y &= \eta + \delta\eta + \mathrm{Ro}\gamma\left(\hat{u} + \frac{\partial\hat{u}}{\partial\xi}\delta\xi + \frac{\partial\hat{u}}{\partial\eta}\delta\eta\right) \end{aligned}\right\}. \tag{28}$$

The condition that (26) and (28) hold simultaneously at (x, y, t_c) is

$$\left(1 - \mathrm{Ro}\gamma\frac{\partial v}{\partial\xi}\right)\delta\xi - \mathrm{Ro}\gamma\frac{\partial v}{\partial\eta}\delta\eta = 0, \tag{29a}$$

$$\mathrm{Ro}\gamma\frac{\partial u}{\partial\xi}\delta\xi + \left(1 + \mathrm{Ro}\gamma\frac{\partial u}{\partial\eta}\right)\delta\eta = 0, \tag{29b}$$

where the carets over u and v have been dropped for convenience. A nontrivial solution for u and v exists if

$$\left(1 - \mathrm{Ro}\frac{\partial v}{\partial\xi}\gamma\right)\left(1 + \mathrm{Ro}\frac{\partial u}{\partial\eta}\gamma\right) + \mathrm{Ro}^2\frac{\partial u}{\partial\xi}\frac{\partial v}{\partial\eta}\gamma^2 = 0. \tag{30}$$

For convenience, (30) may be rewritten as a quadratic equation for $\delta = \gamma^{-1}$ which may be solved to give

$$\delta = \frac{1}{2}\mathrm{Ro}\left\{\frac{\partial v}{\partial\xi} - \frac{\partial u}{\partial\eta} \pm \left[\left(\frac{\partial v}{\partial\xi} - \frac{\partial u}{\partial\eta}\right)^2 - 4\left(\frac{\partial v}{\partial\eta}\frac{\partial u}{\partial\xi} - \frac{\partial v}{\partial\xi}\frac{\partial u}{\partial\eta}\right)\right]^{1/2}\right\}. \tag{31}$$

The terms under the radical may be expressed in terms of the deformation D_0, as

$$D_0{}^2 \equiv \left(\frac{\partial v}{\partial\xi} + \frac{\partial u}{\partial\eta}\right)^2 + \left(\frac{\partial v}{\partial\eta} - \frac{\partial u}{\partial\xi}\right)^2, \tag{32}$$

by making use of $\partial v/\partial\eta = -\partial u/\partial\xi$. Then the critical time t_c is given by

$$t_c = \sigma^{-1}\left\{\ln\left[1 + \frac{2}{\mathrm{Ro}(\zeta_0 \pm D_0)}\right]\right\}_{\min}, \tag{33}$$

where $\zeta_0 \equiv \partial v/\partial\xi - \partial u/\partial\eta$ denotes the initial relative vorticity. The minimum value is achieved when the positive root is chosen, and (33) reduces to (23) when $\partial/\partial\eta = 0$. Moreover, $(\partial v/\partial\xi)^{-1} \geq 2(\zeta_0 + D_0)^{-1}$, so that the discontinuity occurs earlier for three-dimensional flow. An alternate derivation of (33) is given in the Appendix in order to provide a complementary perspective on the formation of the discontinuity.

Hoskins (1975, 1976) has shown that the linear solution of the Eady problem, formulated in geostrophic coordinate space, may be expressed as in (24). Under this circumstance the coefficient of γ^2

in (30) would not appear and (33) would reduce to

$$t_c = \sigma^{-1}\left\{\ln\left[1 + \frac{1}{\mathrm{Ro}\zeta_0}\right]\right\}_{\min}. \quad (34)$$

However, the usefulness of (34) is limited in view of the linearization employed.

5. Remarks

The geostrophic coordinate transformation, defined by (7), is a contact transformation that preserves the correspondence between the slopes of the geopotential surfaces ϕ in physical space (x, y, z, t) and the slopes of the transformed surfaces Φ in (X, Y, Z, T) space. It is also possible to take advantage of the relationship $\phi_t = \Phi_T$ to show that the transformation back to physical space may be accomplished by integration along characteristic surfaces. The value of this integration procedure, compared to other suggested algorithms (e.g., Hoskins, 1975), would depend on the particular model employed. However, the method of characteristics may be used to determine the time and the place where a discontinuity would form, as a function of initial conditions, without knowledge of the complete field of motion. This information could be used to advantage with other methods of integration.

Jeffrey and Taniuti (1964) have noted that contact transformations are particularly useful for transforming *certain types* of partial differential equations to an essentially simpler form. The "certain types" of equations appear to be those for which the velocity field may be represented by either a velocity potential or a streamfunction, as in the present case. Although applications have been made to coupled equations containing both the velocity potential and streamfunction, e.g., von Mises (1958, article 17.1). On this basis, further application of contact transformations to systems of equations used in large-scale atmospheric and oceanic dynamics could prove to be rewarding.

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APPENDIX

Alternate Derivation of the Critical Time to Form a Discontinuity

The critical time t_c, given by (33) can be derived directly from (25a, b). The equations may be rewritten as

$$\left.\begin{aligned}\left(\frac{\partial}{\partial\tau} - V\frac{\partial}{\partial x} + U\frac{\partial}{\partial y}\right)V = 0\\ \left(\frac{\partial}{\partial\tau} - V\frac{\partial}{\partial x} + U\frac{\partial}{\partial y}\right)U = 0\end{aligned}\right\}, \quad (A1)$$

where $\tau = \exp(\sigma t)$ and $(V, U) = (\mathrm{Ro}v/\tau, \mathrm{Ro}u/\tau)$. Cross differentiation and subtraction of the second from the first equation of (A1) yields

$$\left(\frac{\partial}{\partial\tau} - V\frac{\partial}{\partial x} + U\frac{\partial}{\partial y}\right)\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)$$
$$= \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)^2 - 2\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial y} - \frac{\partial V}{\partial x}\frac{\partial U}{\partial y}\right)$$
$$= \frac{1}{2}\left[\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial y} - \frac{\partial U}{\partial x}\right)^2\right], \quad (A2)$$

where the relationship $U_x = -V_y$ has been used to rewrite the right-hand side of (A2). Then (A2) may be expressed as

$$\left(\frac{\partial}{\partial\tau} - V\frac{\partial}{\partial x} + U\frac{\partial}{\partial y}\right)\zeta = \frac{1}{2}(\zeta^2 + D^2), \quad (A3)$$

where $\zeta = V_x - U_y$ and $D^2 = (V_x + U_y)^2 + (V_y - U_x)^2$ represent the vorticity and deformation squared in terms of the variables (V, U).

The equation for D^2, derived from (A1), is given by

$$\left(\frac{\partial}{\partial\tau} - V\frac{\partial}{\partial x} + U\frac{\partial}{\partial y}\right)D^2/2 = \zeta D^2, \quad (A4)$$

which may be reduced to

$$\left(\frac{\partial}{\partial\tau} - V\frac{\partial}{\partial x} + U\frac{\partial}{\partial y}\right)D = \zeta D. \quad (A5)$$

Addition of (A3) and (A5) yields

$$\frac{d}{d\tau}(\zeta + D) = \frac{1}{2}(\zeta + D)^2, \quad (A6)$$

where $d/d\tau = \partial/\partial\tau - V\partial/\partial x + U\partial/\partial y$ denotes the total derivative following the motion, analogous to (13) and (14).

Integration of (A6) leads to

$$\zeta + D = [(\zeta_0 + D_0)^{-1} - \tfrac{1}{2}(\tau - \tau_0)]^{-1}, \quad (A7)$$

where (ζ_0, D_0) refer to initial values and $\tau - \tau_0 = \exp(\sigma t) - 1$. Inspection of (A7) shows that $\zeta + D$ becomes infinite at the critical time, defined by (33). This result, which complements the analysis presented in Sections 3 and 4, is in agreement with the property that discontinuities in the first derivatives of the velocity occur only along characteristic surfaces (Whitham, 1974).

Reduction of (A7), to the linear counterpart of the Eady problem in geostrophic coordinate space, is achieved by setting $D = \zeta$ in (A7), i.e., by neglecting $2(U_x V_y - V_x U_y)$ in (A2) in agreement with the neglect of the coefficient of γ^2 in (30). In this case relative vorticity, $v_x - u_y$, becomes infinite at the critical time given by (34).

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