NOTES AND CORRESPONDENCE

Wave-Permeable Lateral Boundary Conditions for Convective Cloud and Storm Simulations

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ABSTRACT

Linearized conditional instability theory is used to test the effects of lateral boundary conditions on convective elements. By this theory the outer environment of an amplifying convective element acts like an internal gravity wave with imaginary horizontal wavelength which propagates outward with a wave velocity slightly greater than that of hydrostatic modes. Lateral boundary conditions based on wave radiation principles are therefore appropriate and can eliminate the growth constraints produced by rigid or periodic boundaries.

Recent investigations by Klemp and Wilhelmson (1978), Clark (1979) and Miller and Thorpe (1981) are in agreement on the importance of the lateral boundary conditions to the results of simulation of convective storms, though they differ to some extent on the optimal form of such conditions. The purpose of this note is to suggest that simplified convective dynamics and gravity wave theory lead to a better understanding of the meaning of these conditions.

Ideally, one would avoid lateral boundary conditions in a convective storm model. Practical simulations must normally be bounded, however, so that the best we can expect of the boundary is that it be as inconspicuous as possible. It should have a minimal effect on dynamics of convective elements within it, and especially should introduce no computational instabilities. The importance of minimizing wave reflection off the boundary depends on whether the reflected modes interact noticeably with the primary storm dynamics. In this note I will show that the interaction is strong for the dominant mode.

The simplest boundary conditions in common use are those which completely isolate the computational domain from its environment, like periodicity and reflective symmetry. Such specifications severely limit the energetics of a convective storm, since its heat and moisture inflow and outflow must be confined to the integration domain. Less restrictive conditions, as discussed by Clark (1979, Section 2c) allow specification of tangential velocity components and scalars at inflow boundaries and their extrapolative transport through outflow boundaries. The crucial problem then becomes the determination of the normal velocity component, since its equation of motion includes the normal derivative of a non-conservative quantity, pressure. Recently, Raymond (1981) has proposed a more sophisticated approach, using Green's function integrals over both time and space. Its practical value is not yet known, however.

Two fairly closely related schemes for determination of the normal velocity component are being utilized in recently reported convective storm models. Both are versions of a radiation condition, based on the knowledge that gravity waves can exchange momentum laterally through pressure forces, and both involve attempts to reduce or eliminate the reflection of gravity wave energy across the artificial boundaries. Orlanski (1976) proposed a method of local analysis of wave propagation normal to the boundaries, followed by extrapolation of the analyzed wave through them. The scheme proposed by Klemp and Wilhelmson (1978) allows outward propagation of wave energy whenever the largest estimated internal wave propagation velocity is greater than any locally inward particle velocity present. The apparently arbitrary nature of this assumption is defended on the basis of sample calculations indicating that artificial wave reflection is negligible with its use and that simulations bounded by such conditions are similar to those contained within a much larger domain.

Clark (1979) shows results of a comparison of the Orlanski and Klemp-Wilhelmson schemes for a square boundary surrounding a developing simulated storm. He found that for both these schemes

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a net mass circulation developed across the boundaries, leading to a nonzero mean vertical velocity within the domain. The mean vertical velocity for the Klemp-Wilhelmson scheme developed faster and attained values substantially larger than those of the Orlanski scheme, however.

In this note I investigate analytically the effect of boundary conditions like those of Orlanski and of Klemp and Wilhelmson on the dynamic structure and evolution of convective elements inside the boundary. The analysis will be based on the linearized model of disturbances in a conditionally unstable atmosphere developed by Lilly (1960), in extension of the work of Haque (1952). The version of this model appropriate to present purposes contains a moist unstable ascent in a central band flanked by dry stable descent extending out indefinitely. After reviewing the assumptions and general results of the model, I will discuss the effects of a lateral boundary upon it and the circumstances under which those effects can be minimized. These results allow new interpretations to be placed on the tests of lateral boundary conditions made by Klemp and Wilhelmson and by Clark.

In the original development oriented toward tropical cyclogenesis, Lilly (1960) used pressure for the vertical coordinate and mostly ignored non-hydrostatic motions. For present purposes I will assume that compressibility is not crucial and use non-hydrostatic Boussinesq equations. I will also use line $(x - z)$ symmetry for simplicity and because lateral boundary effects are probably most severe for a linelike element. Thus the perturbation equations on an assumed rest state are as follows:

$$\frac{\partial u}{\partial t} - fv + \frac{\partial p}{\partial x} = 0, \quad (1)$$

$$\frac{\partial v}{\partial t} + fu = 0, \quad (2)$$

$$\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} - b = 0, \quad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (4)$$

$$\frac{\partial b}{\partial t} + N^2 w = 0, \quad (5)$$

where p is pressure divided by a constant reference density, and b is a buoyancy variable, similar to $g \ln\theta$. The Brunt-Väisälä frequency N would be normally defined by $N^2 = g\partial \ln\bar{\theta}/\partial z$ in a dry atmosphere. Under the conditional instability assumption, the release of latent heat of condensation in ascent is simulated by defining a negative static stability for upward motion, i.e.,

$$\left.\begin{array}{ll} N^2 = N_m{}^2 < 0 & \text{for} \quad w > 0 \\ N^2 = N_d{}^2 > 0 & \text{for} \quad w < 0 \end{array}\right\} . \quad (6)$$

For simplicity it is assumed that $N_m{}^2$ and $N_d{}^2$ are constant with height. Solutions will be bounded by rigid lids ($w = 0$) at $z = 0$ and $z = H$.

Eqs. (1)–(5) may be reduced to a higher order expression in w, by eliminating v from (1) and (2), b from (3) and (5), p from the two remaining equations, and introducing (4) to eliminate u. The result may be written

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)\frac{\partial^2 w}{\partial z^2} + \left(\frac{\partial^2}{\partial t^2} + N^2\right)\frac{\partial^2 w}{\partial x^2} = 0. \quad (7)$$

The vanishing of w at the upper and lower boundaries suggests the assumption of sinusoidal profiles of vertical velocity with height. One seeks modes amplifying exponentially with time, so solutions are assumed in the form

$$w(x, z, t) = e^{qt} \sin(lz) W(x),$$

where $l = n\pi/H$ with n an integer. Inserting this into Eq. (7) allows the latter to be reduced to an ordinary differential equation, i.e.,

$$\frac{d^2 W}{dx^2} - \frac{q^2 + f^2}{q^2 + N^2} l^2 W = 0. \quad (8)$$

The coefficient of W on the right must be positive for dry descent and real q, since $N_d{}^2 > 0$. If the solution is bounded for all x and vanishes as $|x| \to \infty$, this coefficient must somewhere be negative, so that dW/dx can reverse sign. Therefore, $q^2 + N_w{}^2 < 0$ and solutions are periodic in the moist region. A boundary between the moist and dry regions at, say, $|x| = a$, must have pressure and the velocity component normal to that boundary continuous across it. The pressure condition requires [from use of (3) and (5)] that $(q^2 + N^2)W$ be continuous, and the normal velocity condition requires [from (1), (2), (3) and (5)] that $d[(q^2 + N^2)W]/dx$ also be continuous. If W is now taken to be proportional to $\cos kx$ within the moist region, $|x| < a$, and to $e^{-\kappa|x|}$ outside it (there being no outer boundary assumed), these interface conditions lead to a compatibility requirement of the form

$$k \tan ka = \kappa, \quad (9)$$

where from (8),

$$k^2/l^2 = -(q^2 + f^2)/(q^2 + N_m{}^2), \quad (10)$$

$$\kappa^2/l^2 = (q^2 + f^2)/(q^2 + N_d{}^2). \quad (11)$$

These can be solved together to determine the magnitude of k, κ and the amplification rate q, all of which become functions of the ratio of the moist and dry static stabilities and the moist cell aspect ratio la.

If we now insert a rigid lateral boundary in the outer dry downdraft region at, say, $|x| = b$, then the solution of (8) involves hyperbolic functions and the right side of (9) is replaced by $\kappa \tanh\kappa(b - a)$. This leads to a reduction of k and, consequently, of q with no amplifying solutions possible if the ratio of the area of descent, $(b - a)/a$, is less than the stability ratio $-N_d^2/N_m^2$. This result is essentially an extension of the Bjerknes "slice" method of forecasting convection.

Now we will consider imposition of a wave-permeable outer boundary condition. Either the Klemp-Wilhelmson or the Orlanski condition, valid for a vanishing mean flow, is given by

$$\frac{\partial u}{\partial t} = -c_* \frac{\partial u}{\partial x} \quad \text{at} \quad |x| = b, \tag{12}$$

where c_* is chosen in different ways, but always in an outward normal direction, thus having the same sign as x. If Eq. (12) is to be compatible with the conditional instability solution in the descent region, which is proportional to $\exp(qt - \kappa|x|)$, then the phase velocity c_* must be given by

$$c_* = \frac{q}{\kappa}\,\mathrm{sign}(x). \tag{13}$$

If $|c_*|$ is smaller than q/κ, application of (12) as a boundary condition on (8) leads to reduction of the amplification rate. The rigid boundary condition is attained for $c_* = 0$. If $|c_*| > q/\kappa$ a larger amplification rate results, with a maximum occurring when $|c_*| = \infty$, corresponding to vanishing vertical velocity at the boundary. One can regard the boundary in this extreme case as separating a normal convection cell and its descending environment from one in which the conditional instability is reversed, i.e., upward motions are statically stable and downward motions unstable. The outflow from the reversed cell then feeds the inflow from the normal cell, and vice versa.

For hydrostatic mesoscale circulations in which $f^2 \ll q^2 \ll N_d^2$, substitution of (11) into (13) shows that $c_* \sim N_d/l$. I have also evaluated (13) for the case when $la = \pi/2$, so that the moist region width is just equal to its depth. If the moist and dry stabilities are numerically equal, i.e., $N_m^2 = -N_d^2$, then $c_* = 1.08N_d/l$. For wider convective cells or smaller values of $(-N_m^2/N_d^2)$ the wave speed becomes even closer to the hydrostatic approximation. Klemp and Wilhelmson used this approximation with the assumption that $l = \pi/H$, with H the depth of the simulation model. They assumed that it was an upper bound to outgoing gravity waves, having found that transient wave reflection was less troublesome if the wave speed was overestimated than if it was underestimated. Provided Coriolis terms can be neglected, the above result shows that the effective phase speed of the evanescent outer environment of a growing convective system is always a little greater than the hydrostatic prediction based on the depth of convective instability. Assuming $b = 2a$ and the other parameters as above, the growth rate vanishes for $c_* = 0$, equals $0.41N_d$ for $c_* = q/\kappa$ (the correct value) and $0.51N_d$ for $c_* = \infty$. Here also it appears that an overestimate of c_* gives better results than an underestimate.

For the Orlanski scheme the wave speed c_* is determined objectively on the basis of time and space derivatives just inside the boundaries. Orlanski formulated these in a specific finite-difference framework, valid to second order in time and first order in space. If $x = b$ is the boundary location, with Δx and Δt the space and time mesh distances, Orlanski's estimated wave speed is given by

$$c_* = -\frac{\Delta x}{\Delta t}\,\frac{u(b - \Delta x, t) - u(b - \Delta x, t - 2\Delta t)}{[u(b - \Delta x, t) + u(b - \Delta x, t - 2\Delta t) - 2u(b - 2\Delta x, t - \Delta t)]}\,. \tag{14}$$

It should be noted that Orlanski originally proposed to evaluate c_* independently for each variable and to apply (12) for each variable based on its own c_*. Clark, like Klemp and Wilhelmson, applies the radiation condition only on the normal velocities. Also Orlanski proposed that if the calculated c_* exceeded the computationally stable maximum, $c_* = \Delta x/\Delta t$, the latter limit would be used, while if c_* were found to be directed inward from the boundary its value would be set equal to zero so that the boundary condition remains specified externally. Clark apparently applies both these precepts. Miller and Thorpe (1980) propose the use of several forms of the Orlanski condition having greater nominal accuracy than does (14).

We now evaluate c_* on the basis of the solutions discussed above, i.e., that

$$u(x, z, t) = e^{qt - \kappa|x|} fct(z), \tag{15}$$

where $fct(z)$ is sinusoidal. When (15) is substituted into (14), the wave speed is correctly predicted to the first order in Δx, provided that $(\Delta t)^2 \ll 2\kappa\Delta x/q^2$. Thus it would appear that the Klemp-Wilhelmson, Orlanski, and Miller-Thorpe algorithms should provide similar results.

The above analysis allows some apparently straightforward interpretations of the boundary condition tests made by Klemp and Wilhelmson. They carried out simulations of strong two-dimensional convective cells in a non-sheared environment, using both periodic boundary conditions (equivalent to $c_* = 0$) and Eq. (12), with $c_* = 10$ m s^{-1}, 30 m s^{-1}, and 50 m s^{-1}, with the boundaries located close to the edges of the simulated cloud. The results were then compared with a benchmark simulation carried out within a domain four times as wide. For $c_* = 0$ and 10 m s^{-1} the close-in boundaries were found to severely restrict the cloud efficiency, especially as measured by rain production, while the $c_* = 50$ m s^{-1} choice enhanced it a little. The use of $c_* = 30$ m s^{-1} produced, for the close-in boundaries, a nearly identical cloud evolution to that obtained for the extended domain. In the larger box the results did not depend strongly on the choice of boundary conditions. The value of 30 m s^{-1} corresponded approximately to the propagation velocity (phase and group) of a hydrostatic mode with a depth equal to the height of the convective cells. Allowing for some uncertainty in the appropriate depth (linear theory would predict a somewhat shallower cloud) these results are consistent with the predictions made from the present analysis.

Clark also made comparisons of the Klemp-Wilhelmson scheme and that of Orlanski. While he also used an extended domain, the extension was only by 50%, and was accompanied by a decrease in horizontal resolution. These limitations were necessary because his calculations were made in a fully three-dimensional domain for which computer memories severely limited the available domain size and resolution. His principal comparisons were for the mean vertical velocity inside a fixed size box, equal to the entire domain for the finer resolution experiments. He found that the use of the Klemp-Wilhelmson condition led to a substantially greater and faster growing mean mass exchange through the box than did the Orlanski condition. The fixed wave velocity used by Clark (45 m s^{-1}) was possibly as much as 50% too large because the total domain height, which he used as the characteristic dimension, was substantially greater than the depth of the moist cloud region. That may not be sufficient to explain the discrepancy, however, since by the above analysis even an infinite overprediction of c_* produces a growth rate only 25% too large.

The analysis is admittedly incomplete in several respects, but perhaps the most significant defect is the neglect of a mean flow. Addition of a constant mean flow is a rather trivial extension, in that c_* will simply be increased on the downwind side and decreased on the upwind side by the magnitude of that mean flow. The fact that the interface between positive and negative static stability is approaching toward or receding from a boundary does not matter, as long as it has not passed the boundary. Inclusion of a realistically sheared mean flow leads to a more complex situation, not easily investigated by the present technique. Provided the Richardson number is >0.25, all the propagating modes will have phase speeds outside the range of the mean flow, but probably not as far outside as for the stationary mean state. In this respect it should be noted that Klemp and Wilhelmson have found smaller values of c_* appropriate for strongly sheared flows, at least after the initiation stage (Klemp *et al.*, 1981).

In summary, I believe that application of the linear theory of conditional instability leads to a better qualitative understanding of the role of the lateral boundary condition in the simulation of thermal convection. From this viewpoint, the critical problem addressed by use of a wave-permeable boundary is its effect on the mass flow produced by the convective elements, rather than the avoidance of reflection of transient waves. An algorithm effective for the latter purpose is, however, found to be nearly optimal for the former. The general agreement between the Klemp-Wilhelmson tests and the present theoretical predictions lends confidence to each approach, although the apparent discrepancy of the Clark result remains partly unexplained. These results also suggest that application of wave-permeable boundary conditions also would be appropriate in hydrostatic simulations and predictions, especially when convective interactions are important.

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