An Objective Method for Determining the Generalized Transport Tensor for Two-Dimensional Eulerian Models

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ABSTRACT

An objective method for deriving the components of a generalized transport tensor for a two-dimensional model is presented. The method uses representative meridional and vertical velocities and thermodynamic scalars at a uniform grid to reduce the problem to solving two flux equations for two unknowns. One unknown is the streamfunction, coefficient of an antisymmetric tensor, which corrects the Eulerian mean motions for Stokes drift. The other is a time constant, which converts the deviatory velocity tensor (Reynold’s stress tensor for temporal averaging) to a symmetric transport tensor. The complete asymmetric tensor is called a transport rather than a diffusion tensor because its divergence yields both advection and diffusion by the deviatory velocities. Advantages and disadvantages of Lagrangian and Eulerian averages are also discussed.

1. Introduction

Atmospheric transport of trace species remains a very controversial and conceptually difficult subject despite the fact that only two processes are involved: transport by mean motions and by deviations from the means. Due primarily to fundamental observational limitations, this subdivision is unavoidable and it applies to any model of the atmosphere whether conceptual, analytical or numerical. However, the classification and relative importance of the two types of transport changes with the resolution and dimensions of the model. Transport by the (local) mean motions dominates in high-resolution, three-dimensional (3-D) models. In two-dimensional (2-D) models transports by both the (zonal) means and the (eddy) deviations from the means are comparable and, importantly, they also are competitive. Conversely, in a one-dimensional (1-D) model the (global) mean motions are vanishingly small and the transport is dominated completely by deviatory motions.

This change in classification is conceptually confusing because, in principle, the mean motions are observable or deterministically predictable, while the deviatory motions are not and, therefore, their effects must be described or predicted statistically. Since, in this case, the classification changes only as a result of spatial integrations, over all longitudes for the 2-D model and then over all latitudes for the 1-D model, we can eliminate the confusion by deriving the statistical descriptions directly from 3-D models. Anyone who has attempted to do this immediately confronts the fundamental source of controversy and conceptual difficulty: only the horizontal components are observed, not the mean motions themselves, and the horizontal components are not measured at a uniformly spaced grid of points. In fact, the distribution of observations is adequate only at extratropical latitudes in the Northern Hemisphere.

Our inability to measure the mean vertical motions is due to their small magnitudes, of the order of millimeters and centimeters per second. The measurable horizontal velocities are about three orders of magnitude larger. Obviously, if only their relative magnitudes are considered, the large-scale atmospheric motions are practically two-dimensional and horizontal. But the troposphere and stratosphere, which comprise approximately 80 and 20% (respectively) of the total atmospheric mass (>99.9%) are comparably two-dimensional and horizontal. In particular, an air parcel moving with a vertical speed of only 10 cm s⁻¹ would traverse the entire troposphere in 1–2 days, depending on latitude, thus the velocities and distances are commensurately scaled.

Also, because the vertical gradients of kinetic and potential energies greatly exceed the horizontal gradients, accelerations experienced by an air parcel moving three dimensionally differ dramatically from those the same parcel would experience if it moved horizontally. Consequently, the trajectories can have horizontal curvatures of opposite sign and the direction of transport can be reversed within 12–24 h (Danielsen, 1961). The difference in horizontal curvature depends on the product of vertical
velocity and the turning of the wind with height, a
geostrophic indicator of temperature advection.
Since the vertical velocities tend to increase with an
increase in advection, the difference in curvature
generally increases with the square of the vertical
velocity. Therefore, the 3-D trajectories and their
vertical projections onto a horizontal surface are
sensitive to the small vertical velocities we cannot
observe.

This sensitivity can be significantly reduced by
analyzing the motions on isentropic rather than
isobaric surfaces, thereby taking advantage of the
fact that, to a first approximation, the air moves
isentropically. Three-dimensional trajectories com-
puted from isentropic analyses include reasonable
approximations to the net vertical displacements
and produce only small errors in their horizontal
coordinates.

Since being introduced to isentropic analyses in
World War II, the author has developed hand and
computer methods for constructing isentropic tra-
jectories to study atmospheric dynamics and trans-
port processes. These quasi-Lagrangian trajectories,
which include the effects of ageostrophic acceler-
ations, are extremely well suited for conceptual
descriptions of transport by the large-scale wave-
like motions of the troposphere and stratosphere,
and the mass exchange between these two major
subdivisions of the atmosphere. They clarify the
importance of rapid transfers from the stratosphere
to the boundary layer and vice versa, and they aid
us in identifying irreversible transports. Unfor-
tunately they are extremely ill suited for a quanti-
tative description of the type we seek in a 2-D trans-
port model. Therefore, the method to be discussed
here is based on zonal-Eulerian rather than zonal-
Lagrangian integrals.

2. Open Eulerian systems versus semiclosed Lagran-
gian systems

Two-dimensional numerical models based on
zonal Eulerian integrals have the disadvantage that
the zonal mean circulations are not material circu-
lations. In particular, as shown by Mahlman (1975),
the mean transport often tends to be a small residual
between two opposing fluxes, one due to Eulerian
mean circulations, the other due to deviatory mo-
tions. The zonal-Lagrangian integral proposed by
Andrews and McIntyre (1978) attempts to eliminate
this dichotomy by assuming that the integrals are
macroscopically closed systems, i.e., all Reynolds's
stresses vanish because there is no elemental mass
transport across the boundary surfaces of the system.

However attractive this assumption seems theo-
retically, it is physically unrealistic and unattain-
able for the following reason. The fundamental
equations of fluid dynamics apply to small elemen-
tal volumes, such as a cubic millimeter or a cubic
centimeter. Obviously, the observations do not per-
mit our resolving these elemental volumes, nor our
tracing the motions of their center of gravity.

Instead, we are compelled by the measurements
available for large-scale analyses to consider bulk
volumes containing 10^{19}—10^{20} elemental volumes,
even if we use the larger (1 cm^3) for the elemental
volume. Integrals over these bulk volumes represent
local averages and yield local mean values. It will
be helpful to denote local Eulerian means by $\overline{y}$
and local Lagrangian means by $\langle \tilde{y} \rangle$. Since the raw-
sonde measurements are averaged to produce rea-
sonable approximations to the mean horizontal
winds of these bulk volumes, we can analyze for
the approximate trajectories of the center of gravity
of the local Lagrangian systems. However, since a
closed system is constrained to always contain the
same elemental masses, we cannot follow the de-
forming boundaries of such a system. Consequently,
we must recognize that both the local Eulerian and
local Lagrangian systems are open systems.

A more reasonable constraint for a Lagrangian
integral, an extension of the molecular constraint
for an elemental system, is that the integrated
deviatory mass flux across the boundary is zero
(see the Appendix for details). With this constraint
the bulk, open system conserves its total mass,
moves with the mean velocity (the velocity of its
center of mass) and deforms due to spatial gradients
of these same mean velocities. It also mixes with its
environment as a consequence of the deviatory
fluxes across the moving boundaries.

The larger scale zonal integrals are obtained by
summing 10—100 of these local means and dividing
by the total number $N$, where $N$ depends on latitude.
Consistent with our notation, the zonal Eulerian mean

$$\overline{y} = \sum_{i=1}^{N} (\overline{y})^i / N,$$

where the sum is at constant latitude. Similarly, the
zonal Lagrangian mean

$$\langle \tilde{y} \rangle = \sum_{q=1}^{N} (\tilde{y})^q / N,$$

where the sum spans all longitudes but is not re-
stricted to a given latitude. For example, at some
initial time $t = 0$ a could equal $i$, but at all sub-
sequent times the position of each local Lagrangian

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1 Mahlman, J. D., 1975: Some fundamental limitations of
simplified-transport models as implied by results from a three-
dimensional general-circulation/tracer model. Proc. 4th Conf.
Climatic Impact Assessment Program, DOT-TSC-OST-75-3B,
T. M. Hard and A. J. Broderick, Eds., U.S. Dept. of Transpor-
tation, 132—146.
mean, identified by the index $a$, would be determined by the trajectory of its center of mass.

At the initial time the two zonal integrals would be equal because the sum is taken over the same set of local integrals. However, as time increases the two zonal means will differ. The Eulerian integral, being stationary and completely open, will be formed from new air parcels. The Lagrangian integral will move with the mean velocity of the original set of $N$ local integrals. It is closed only to the large-scale motions, which can be resolved from the local mean velocities. Consequently, the Reynolds stresses are reduced but not eliminated.

Due to the ever present large-scale waves, the local Lagrangian integrals move coherently away from their initial positions. Some moving with a northward component, others with a southward component, cause the line of $N$ bulk parcels to undulate and the tendency for velocity cancellation produces small zonal Lagrangian velocities. However, as time increases the coherence breaks down as adjacent parcels diverge in deformation fields, following almost independent paths. Errors in the computed trajectories increase as do those in resolution caused by the increasing nonuniformity of the distribution of bulk parcels.

In an attempt to overcome these difficulties, Andrews and McIntyre (1978) proposed an hybrid Lagrangian-Eulerian system called a "generalized Lagrangian mean", denoted by the symbol $(\bar{\cdot})^c$. Expressed in the above notation, each of the $N$ values of $(\bar{\cdot})^c$ would be evaluated by a Taylor series expansion of $(\bar{\cdot})^c$, where $(\bar{\cdot})^c$ and its spatial derivatives are functions of Eulerian coordinates. To overcome errors in resolution, the $N$ expansion points can be uniformly spaced longitudinally, but then $(\bar{\cdot})^c$ is not uniformly weighted with respect to arc length along the curve of displaced bulk parcels. The difficulty of locating these parcels is transferred to determining the $N$ expansion points and the $N$ displacement vectors from these points to the local Lagrangian integrals.

Simple solutions to the latter problem exist only for wave disturbances of small amplitude and for small time integrations, therefore, small vector displacements. Thus, following Andrews and McIntyre (1978), Plumb (1979), Holton (1980), Matsuno (1981) and others use only the linear terms in a Taylor expansion to express analytically the difference between the generalized Lagrangian mean and the zonal Eulerian mean. The difference between the two mean velocities, a wave induced term involving products of the linearized displacements and velocity gradients, is now generally called a Stokes drift, although in these theoretical computations it represents a nonmaterial drift.

This nonmaterial drift is caused by isentropic wave motions in the inhibiting presence of a vertical entropy gradient. Apparently, this isentropic component of the zonal mean Eulerian circulations accounts for the competition between the mean and eddy fluxes discussed by Mahlman (1975). By eliminating it, the generalized Lagrangian integrals would appear to be preferable to Eulerian integrals. But these results are drawn from theoretical models, including waves of small-amplitude and essentially reversible isentropic processes. It will be instructive, therefore, to examine some of the effects of finite-amplitude waves in the real atmosphere, where isentropic and anisentropic processes are inextricably mixed.

3. Three-dimensional transport based on quasi-Lagrangian trajectories

For a 3-D diagnostic or predictive model, the use of Lagrangian integrals is equivalent to generating three-dimensional trajectories which trace out the motion of the center of mass of the bulk systems. On the other hand, Eulerian integrals yield a succession of streamline (or the associated stream-function and velocity potential) analyses. If the grid volumes of both integrals are comparable in size, the trajectories and the streamlines describe the same local mean motions but from different viewpoints. Although a complete description would yield the same results, one's subjective concepts of the transport could be considerably different for the two systems.

The streamline analyses tend to emphasize wave motions and laminar-like flow, while the trajectories emphasize dispersion and turbulent-like flow. The latter can be seen in Fig. 1, which resembles a plate of tangled spaghetti. Actually, it represents 80 of 160 isentropic trajectories which were calculated by a computer using programs based on the methods developed by Danielsen (1961). Analyses of the isentropic streamfunction over the Northern Hemisphere north of 20°N were made by hand. After gridpoint values were read, coded and verified, the balance equation was solved for an ageostrophic streamfunction. Then trajectories were computed from the ageostrophic winds.

These trajectories have been designated here as quasi-Lagrangian because they were based on the isentropic approximation and because the contributions from a velocity potential were neglected. Also, the effects of deviatory boundary fluxes were not included in the trajectory computations and no attempt was made to trace their moving boundaries. Instead, the local mean velocities were used to derive trajectories of the centers of mass as if the bulk systems were completely closed. Obviously, errors are introduced by these approximations. For example, the dispersion along $\pm V\theta$ is really not zero as assumed, but the dispersion in three dimen-
activity of stratospheric origin (Danielsen, 1964, 1968). As expected, very large concentrations of strontium 90 and total $\beta$ activity were measured where $S^l$ is large. Farther north in Wyoming, where $S^l$ is small, equally small concentrations of radioactivity typical of the troposphere were measured. Therefore, from the midwest to the Rocky Mountains, the tropopause boundary in the isentropic surface is folded with tropospheric air both to the north and to the south of the extruded region of stratospheric air.

The heavy dashed line along latitude $60^\circ$N denotes the zonal monthly mean tropopause for April. It is apparent from Fig. 3 that the synoptic tropopause does not oscillate with small amplitude about the zonal monthly mean. Rather, the amplitudes produced by large-scale wave transports are very large and the boundary folds, creating a pattern similar to a breaking wave. The isentropic trajectories of Fig. 1 are consistent with this folding process, showing that the origin of the large $S^l$ over Colorado were at high latitudes close to the pole.

However, after the folding process develops, the maximum $S^l$ contours disappear from the folded structure. Similarly, the large $S^l$ values decrease along individual trajectories. In this experiment, air sampled over Colorado was resampled two days later over Ohio. Both the radioactivity and the potential vorticity decreased in a similar fashion (see Danielsen, 1967). The nonconservation of $S^l$ cannot, therefore, be attributed to gradients of diabatic heating; only mixing processes can maintain the correlation between the potential vorticity and the radioactivity.

Further evidence for the nonconservation of $S^l$ is obtained from the trajectory analyses. For each of the 160 bulk parcels, $S^l$ was computed every 12 h. Approximately 25% of these parcels exhibited fluctuations about a decreasing trend. An examination of the individual trajectories showed that they originated at high latitudes with stratospheric values of potential vorticity and terminated at much lower latitudes with slightly larger than typical tropospheric values. Since $S^l$ is the mean of some $10^{20}$ elemental values of $S$, the nonconservation of $S^l$ does not necessarily imply nor does it require irreversible processes.

However, the intermittent transfer of energy to smaller scales tends to proceed toward irreversibility. Thus, the large-scale deformations are the isentropic precursors to small-scale anisentropic processes.

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As stated earlier, on the basis of these hemispheric trajectory analyses and many other case studies, the author found Lagrangian methods advantageous for understanding the physical processes of transport, but abandoned them for Eulerian methods when attempting to develop an objective method for quantifying the transport in 2-D models. It is evident in Fig. 2b that the tendency of Lagrangian parcels to cluster and decluster makes it difficult, if not impossible, to maintain a uniform degree of resolution. Introducing new parcels intermittently also is self-defeating. When a Lagrangian integral is extended to include all longitudes, both the boundaries of the system and the fluxes through the boundaries must be determined. If the system is defined to minimize the flux, the boundary becomes folded and difficult to locate. Conversely, if the system is defined to minimize the deformations of the boundary, the flux becomes large and difficult to quantify. Using the hybrid Lagrangian-Eulerian method to evaluate the generalized Lagrangian integrals does not eliminate these problems.

4. Data sets for transport parameterization

Given the current and rapidly increasing interest in 2-D numerical models for predicting the effects of trace pollutants on atmospheric ozone, the effects of volcanic eruptions on aerosols and their radiative effects on climate, it is obvious that a reliable specification of the mean and deviatory transports is needed. Assuming that the advantages of Eulerian coordinates will outweigh their disadvantages, one expects Eulerian coordinates to be used in the models. A primary requisite for specifying the transport is a representative set of \( u, v \) and \( w \) velocity components, and their associated thermodynamic variables at a uniform grid over the globe or at least one hemisphere.

The thermodynamic variables, pressure and temperature, are required for computing the entropy gradients in the potential vorticity. Furthermore, the horizontal gradients, although much smaller than the vertical gradient, cannot be neglected because they are multiplied by the horizontal components of vorticity which are much larger than the vertical components. Thus, a representative data set implies a dynamically balanced, appropriately filtered set that spans the three-dimensional space.

J. Mahlman of the Geophysical Fluid Dynamics Laboratory (GFDL) has developed and tested such a set using the laboratory's General Circulation Model (GCM). The author and S. Hipskind are developing a numerical, diagnostic model based on isentropic coordinates, to derive another set from the radiosonde observations plus the grid-point analyses from the National Meteorological Center's numerical prediction model. This work, sponsored by the Federal Aviation Administration, is in progress and will be reported separately later. Using both data sets we will first compare the model's velocity statistics to those derived from observations and then compute from each set the grid-point values for the complete transport tensor.
5. 2-D model development and resultant problems

Two-dimensional models are based on integrating conservation equations over all longitudes. An additional integration over time, such as a month or a season, is often made but is not essential to this discussion. Thus equations typical of the model take the form

$$\frac{\partial}{\partial t} (\rho \dot{\chi}_i) = - \nabla \cdot (\rho \vec{V} \dot{\chi}_i) - \nabla \cdot (\rho \vec{V} \chi_i') + \dot{s}_i,$$  \hspace{1cm} (1)

where $\rho$, $\vec{V}$, $\chi$ and $s$ are the density, velocity, mixing ratio and all sources and sinks of species $i$. Since $\nabla \cdot (\rho \vec{V}) \approx 0$ the product $\rho \vec{V}$ can be expressed in terms of a streamfunction $\psi$. Also, since $\chi$ is a scalar it is customary to use first order closure and express the deviatory flux $\rho \vec{V} \chi'$ in terms of a scalar product of a diffusion tensor $\mathbf{K}$ and the gradient of $\chi$ (Stewart, 1945). Thus,

$$-\rho \vec{V} \chi' = \rho \left[ \begin{array}{cc} K_{uu} & K_{uz} \\ K_{zu} & K_{zz} \end{array} \right] \nabla \chi,$$  \hspace{1cm} (2)

where the components of the diffusion tensor must be specified as a function of $y$, $z$ and $t$.

In one of the first 2-D numerical models, developed by the late B. Davidson, to predict the zonally-seasonally averaged distribution of radioactive aerosols (products of nuclear bomb tests conducted in the atmosphere) the mean circulations were ignored and numerical tests were conducted to determine the appropriate diffusion coefficients. This approach was in marked contrast to the earlier conceptual models of Brewer (1949) to explain the dry stratospheric observations, and of Wolf (1942) and Dobson (1956) to explain the excess ozone at high latitudes. Those conceptual models were focused on mean meridional circulations and, generally, diffusion was neglected.

It was soon discovered that Fickian diffusion, where the off-diagonal components of $\mathbf{K}$ are zero, would not simulate the observed negative slope of the radioactivity isolochths in the stratosphere. It was necessary to introduce negative off-diagonal components and to increase their magnitude until the $\chi$ isolochths had a larger negative slope than the potential temperature isotherms $\theta$. For details of the model, the resulting simulations and prediction, see Davidson et al. (1966).

The importance of diffusion in transporting radioactive tungsten 185 from the tropical stratosphere was emphasized, also, by Reed and German (1965), although they stressed that, in general, both mean and deviatory transports should be included. They used the mixing-length hypothesis, deduced from molecular diffusion or kinetic theory of ideal gases, to express the $K$’s of (2) as products of a mixing length (the $Y'$ or $Z'$ component of a displacement vector) and the appropriate velocity component. Thus, they assumed

$$\chi' = -\mathbf{L} \cdot \nabla \chi = - \frac{\partial \chi}{\partial y} Y' - \frac{\partial \chi}{\partial z} Z',$$  \hspace{1cm} (3)

and showed that

$$\left[ \begin{array}{cc} K_{uu} & K_{uz} \\ K_{zu} & K_{zz} \end{array} \right] = \left[ \begin{array}{cc} v'Y' & v'Z' \\ w'Y' & w'Z' \end{array} \right].$$  \hspace{1cm} (4)

Then they made the additional assumption

$$\frac{w'}{v'} = \frac{Z'}{Y'},$$  \hspace{1cm} (5)

which they considered permissible if the mixing length is small compared with the size of the eddies. As a direct result of this assumption, the off-diagonal components of (4) are equal and therefore the diffusion tensor is symmetric.

To obtain numerical estimates of and to provide a physical explanation of the three remaining terms, $K_{uu}$, $K_{uz}$ and $K_{zz}$ they introduced a third assumption
and expressed (4) as

$$\begin{bmatrix} K_{uu} & K_{uz} \\ K_{uz} & K_{zz} \end{bmatrix} = K_{uu} \begin{bmatrix} 1 & \tilde{\alpha} \\ \tilde{\alpha} & \tilde{\alpha}^2 + \tilde{\alpha}^2 \end{bmatrix},$$

(7)

where $\tilde{\alpha}$ is interpreted as the mean slope of the mixing surface and $\tilde{\alpha}^2$ is the variance of $\alpha$ about $\tilde{\alpha}$. For the last expression of (6) to be valid $\alpha$ must be a small angle. Later we shall see that $\alpha$ is not always small, therefore, assumptions (5) and (6) will be challenged and a more general expression for (4) will be derived.

Numerical values of $K_{uu}$ and $\tilde{\alpha}$ were determined from heat flux, temperature and meridional wind data while $\tilde{\alpha}^2$ was estimated from the vertical rate of spreading of tungsten 185 in tropical latitudes where $\tilde{\alpha} = 0$. Reed and German (1965) showed also that, by rotating the coordinates into the principle axis system (which eliminated $K_{uz}$), the rotation angle $\gamma = \tilde{\alpha}$ and the diagonal terms reduce to $K_{u} = K_{uu}$ and $K_{z} = \tilde{\alpha}^2 K_{uu}$. Since $K_{u} \gg K_{z}$, the former corresponds to the major axis and the latter to the minor axis of the ellipse associated with the matrix (7).

The techniques they introduced were widely accepted and their paper became the standard reference for most 2-D modelers. However, in some of these models, developed after ozone replaced radioactivity as the potential health hazard, little or no attention was paid to the interdependence of the mean and deviatoric motions. As shown by Kuo (1956), the mean circulations are forced by horizontal gradients of heating and vertical gradients of frictional forces, both of which are influenced and sometimes dominated by deviatoric motions. In particular, thermally indirect circulations, like the Ferrel cell, must be dominated by deviatoric motions. Thus, it was obvious to some that mean and deviatoric transports could not be specified independently, and it was also evident from the structure in tungsten 185 distribution that both transports were required.

Clearly, balancing these transports in a 2-D model would be challenging. One of the first attempts was made by Louis (1974),4 who completed a numerical model in which the mean circulations were first derived, and then the coefficients of the diffusion tensor were determined by cancelling the local change in ozone mixing ratio produced by these mean transports. This approach is valid wherever ozone has a chemical lifetime exceeding its mixing lifetime. Thus, it should be reasonably valid below 25 km and invalid above 30 km. In other words, in the upper stratosphere the last term in (1), the source-sink term, can cancel the net effects of the two transports, leaving both of them unconstrained by this approximation.

Two very interesting and perplexing results were obtained from Louis’ work. When the model was tested by simulating the transport from equatorial, mid- and high-latitude nuclear bomb tests, all of the transports were too fast. However, if both the mean and deviatoric transports were reduced by 50%, the results were excellent. Since a scalar reduction of both terms would not alter the assumption of a zero ozone tendency, there were two possible explanations. Either the mean circulations were a factor of 2 too fast, or there was a missing term in the transport parameterizations. Although the first explanation was certainly possible it was not highly probable. Furthermore, when Louis’ unmodified mean and turbulent transports were used in photochemical models, the poleward transport of ozone was much too slow. Reducing the transports by a factor of 2 to match the radioactivity transport would simply make the ozone transport even slower.

Faced by these conflicting results, modelers understandably adopted a pragmatic approach and adjusted the transports to fit their model’s predictions of either ozone or radioactive tracers. A comparison of these transport descriptions reveals large differences in the mean velocities and in the diffusion coefficients, not just a factor of 2. In some cases they approach a factor of 20. Why the transports can differ so greatly and still lead to comparable results for a particular trace constituent is another problem that needs explanation. Because the parameterized transports depend on the mean gradients of the species, it is highly improbable that comparable results could be achieved for all species.

In the meantime, Mahlman (1975)4 was taking a different approach. Having completed a tracer experiment in the GCM, he could easily generate the mean and deviations of the velocities and tracer mixing ratios, and directly evaluate the two transport terms in (1). As stressed earlier, his results showed a strong tendency for the two transports to oppose each other, yielding the net transport as a small residual of two large terms. Also, there were locations in the zonal-temporal means where the diffusion flux was not downgradient, thus assumption (2) appeared to be violated. Mahlman also used the two components of the deviatoric flux to evaluate the diffusion coefficients. If assumption (2) is made, there are two scalar equations and four unknowns. By adding assumption (3) the unknowns are reduced to three. To reduce them to two, Mahlman assumed

$$\frac{\rho u' \chi'}{\rho w' \chi} = \frac{K_{uu}}{K_{uz}}.$$

(8)

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With this assumption $K_{yy}$, $K_{xz}$ and $K_{zz}$ can be determined uniquely as ratios of fluxes and gradients.

$$K_{yy} = \frac{\nabla' \cdot \nabla' x}{\nabla' \cdot \nabla x}, \quad K_{xz} = \frac{\nabla' \cdot \nabla' x w}{\nabla' \cdot \nabla x}, \quad K_{zz} = \frac{\nabla' \cdot \nabla' x}{\nabla' \cdot \nabla x}, \quad (9)$$

Each of these ratios has a common denominator which must be positive if the deviatoric flux is downgradient. As indicated above, negative values were obtained at various locations from the GCM results. Although negative values could prove to be troublesome, the fact that the denominator of (9) must pass through zero is even more troublesome. Then at least one of the $K$'s must approach an infinite magnitude.

However, another consequence of assumption (8) is that $\alpha^2$ must be negligible compared to $\alpha$. In other words, the deviatoric flux must be effectively limited to an inclined mean stream surface, an assumption used by Fleagle (1955, 1957) to study baroclinic instability. Consistent with Eady's (1949) principle of energy conversions with virtual displacements, Fleagle found that a necessary condition for the amplification of a perturbation in a baroclinic fluid was that the slope $\delta$ of the stream surface must be positive but less than the slope $\delta_0$ of the mean isentropes, and that the maximum amplification occurred when $\delta = \frac{1}{2} \delta_0$. Conversely, if $\delta > \delta_0$ or if the negative slope of the stream surface exceeded the negative slope of the mean isentropes (as indicated above for the stratosphere) the perturbations were damped.

This concept was used by Newell (1961) to explain the countergradient heat flux in the stratosphere, by Reed and German (1965), and by Louis (1974) to parameterize the diffusion coefficients in the troposphere and stratosphere. It also figures prominently in the work of Green (1970). In the troposphere, where kinetic energy must be generated to account for both energy flux into the stratosphere and dissipation of energy at the earth's surface, $\delta$ must approach $\frac{1}{2} \delta_0$, i.e., the implied correlation between ascending motion and warm advection, descending motion and cold advection, must reduce the local change by advection. Indeed, at 500 mb the observed local change in temperature is about one-half of the observed advective change. Thus, there is observational and theoretical evidence to support assumption (8) which implies that the wave motions are linearly polarized, but Fleagle's (1957) results indicate that the unstable waves will become elliptically polarized and, therefore, $\alpha^2$ will grow. Again, we are confronted by an apparent inconsistency that remains to be resolved.

Last, but not least, a new challenge to the use of a diffusion tensor was raised by Wallace (1978). He assumed that the large-scale wave disturbances in the lower stratosphere are described by evanescent waves: amplitudes decreasing with height and with distance north and south of an axis of maximum amplitude. He then deduced that the trajectories of air parcels projected onto a meridional plane would describe counterclockwise ellipses sloping upward toward the pole. However, according to his model only the ellipse centered on the axis of maximum amplitude would close. Those to the north and south of the axis would not close, leading to a helical downward drift north of the axis and an upward drift south of the axis. He calls these drifts rectified Lagrangian parcel motions, Stokes drift and Lagrangian Stokes drift. He states that they are tending to steepen the negative slopes of the isentropic surfaces in the stratosphere and implies that they are responsible for the counter gradient heat and ozone flux in the lower stratosphere, even though the mean air trajectories in the waves slope upward toward the pole.

The present author has had difficulty accepting Wallace's (1978) arguments because they are based on isentropic motions and steady-state conditions relative to the moving wave. If the trajectories were traced on isentropic surfaces and their elliptical projections did not close, the isentropic surfaces would not tend to steepen—they would have to steepen. Conversely, it is this author's experience, from isentropic trajectory analyses, that no isentropic drift is evident as the air parcels move through a large-scale wave. As discussed in the next section, nonsentropic, diabatic drifts and irreversible mixing are implied, but not isentropic drifts.

Diabatic drifts certainly are implied by Wallace's (1978) Fig. 3, which shows helical trajectories crossing the entropy surfaces. However, if these are actually isentropic, nonmaterial drifts there must be equal and opposite drifts to cancel them. Clark and Rodgers (1979) show that in the presence of an inhibiting vertical entropy gradient, elliptically polarized velocities, which ascend at high latitudes move southward and descend at lower latitudes, generate a poleward entropy flux. Because this flux is orthogonal to the entropy gradient, it cannot be described by a symmetric diffusion tensor. Divergence (convergence) of this entropy flux south (north) of the axis of maximum wave amplitude will produce downward (upward) zonal Eulerian mean motions. These nonmaterial drifts, artifacts of zonal averaging, are likely candidates to cancel the drifts Wallace called Lagrangian Stokes drifts. If this interpretation is correct the helical paths do not describe the projections of Lagrangian trajectories. Instead, they must represent the projections of a
succession of different air parcels and are more analogous to streamlines.

6. Mean circulations: Trajectories or streamlines

In the lower stratosphere the axis of maximum wave amplitude usually corresponds to the axis of a jet stream. Now it is well known that when velocities are averaged longitudinally and temporally (not at constant latitude but at a constant distance from the axis of a jet) the resulting mean circulations are direct like the Hadley, not indirect like the Ferrel cell circulations. After Fultz and Riehl (1957) determined a direct circulation from their annular dishpan experiments, Krishnamurti (1961) showed that the subtropical jet had a similar circulation. Later, Mahlman (1973) found that averaging relative to the polar jet yielded similar results.

These jet rectified circulations include ascending motions on the anticyclonic side of the jet, poleward motions above the axis, descending motions on the cyclonic side and close with equatorward motions below the axis. The ascending and descending branches are similar in sense and location to the mean drifts deduced by Wallace (1978), but here they do not represent a correction to the Eulerian mean. Instead, they represent an approximation to the generalized Lagrangian mean of Andrews and McIntyre (1978).

However, in the lower stratosphere the jet axis is associated with a large gradient of potential vorticity, large values being located on the cyclonic side. Averaging relative to the jet tends to preserve this large gradient, while Eulerian averaging reduces it considerably. Therefore, if these jet rectified circulations are Lagrangian, the mean value of the potential vorticity must change rapidly as the center of mass of the bulk parcel crosses the jet axis. Of course, the mean value of the potential temperature or entropy must change also along the ascending and descending branches, but these changes are conceptually less challenging.

The mean value of potential vorticity can increase by a vertical gradient of mean diabatic heating and by mixing with larger environmental values. We can distinguish between these two processes by studying the correlation between potential vorticity and quasi-conservative trace constituents. Diabatic heating applies to elemental masses and must be evaluated by a mass or volume integral. On the other hand, mixing within the bulk system doesn’t change its mean value, therefore, only surface integrals are involved (see the Appendix).

It follows that when potential vorticity and the mixing ratio of a tracer are correlated mixing tends to change both mean values but to preserve the correlation. Conversely, when the potential vorticity changes by a gradient of diabatic heating the correlation will change. Therefore, if air with small values of potential vorticity, tagged by a tropospheric tracer, ascends and then crosses the axis of the jet, the potential vorticity could presumably increase by diabatic processes to the typically large, cyclonic side values, while the tracer retained its tropospheric value. The moving parcel would not generate a potential vorticity anomaly but it would generate either a tracer anomaly or there would be no tracer gradient across the jet axis.

Danielsen et al. (1962) participated in an aircraft experiment during the spring of 1960 to study cross-jet transport by means of radioactive aerosols (corrected for radioactive decay) which are conservative tracers at heights above the clouds. At the time of the experiment it was not known to us that the French had detonated their first bomb over the northern Sahara. This bomb provided two rather short-lived isotopes, strontium 89 and barium 140, which functioned as unique stratospheric tracers. The larger yield Russian and American bombs provided two longer-lived stratospheric tracers, strontium 90 and tungsten 185, while cosmic ray bombardment produced another predominantly stratospheric tracer, beryllium 7.

When these radioactive tracers were averaged relative to the jet axis, the results were unambiguous. Large values of the stratospheric tracers were on the cyclonic side, large values of the tropospheric tracer were on the anticyclonic side. No measurable values of strontium 89 and barium 140 were detected on the cyclonic side, despite the close proximity of very large values on the anticyclonic side. Thus, there was no evidence of strong small-scale mixing across the jet axis. Instead, diffusion perpendicular to the jet was apparently counteracted by a small convergence.

Also, when the tracers’ mixing ratios were plotted against potential vorticity, the proportionality factor did not change significantly over the entire range. Thus, there was no evidence to support a rapid change from tropospheric to stratospheric potential vorticities by a large gradient of diabatic heating. In general, therefore, the evidence opposed the interpretation of the jet rectified circulations as closed Lagrangian circulations—they were closed mathematically but not materially.

On the other hand, the evidence definitely supported the interpretation that these circulations consisted of two Lagrangian or quasi-Lagrangian components. We know that the tungsten 185 was injected into the tropical stratosphere at 11°N more
than 1.5 but less than 2 years earlier. On the basis of balloon and aircraft measurements, it was distributed between 16 and 20 km in height, between $\theta = 380$ and 480 K. During the 1960 experiment the tungsten mixing ratios were large in the stratospheric measurements and were almost independent of $\theta$ over the entire observational range, between 10 and 15 km and $\theta = 325$ to 390 K.

Obviously, the tungsten descended in elevation as it moved northward from tropical latitudes, but it did not descend isentropically. Like the descending branch of the jet rectified circulations, the tungsten crossed the $\theta$ surfaces toward decreasing values, equivalent to a net cooling of 60–80 K in potential temperature, on the cyclonic side of the jets. However, it is not a simple problem to separate the actual diabatic component from the effects of anisentropic diffusion. By completely ignoring all mean circulations, Davidson et al. (1966) and Reed and German (1965) produced qualitatively similar transports from purely diffusive models.

Probably the best evidence for diabatic cooling is obtained by comparing synoptic and statistical distributions. In Fig. 4, for example, the synoptic values of potential vorticity $S'$ or $S^*$ and the Eulerian zonal-seasonal mean values $\bar{S}$ are contoured along a meridional cross section close to $80^\circ$W for 24 April 1963. For the corresponding cross section of potential temperature and wind speed see Danielsen (1968); for detailed trajectory analyses see Danielsen (1967).

At high latitudes $S'$ oscillated about $\bar{S}$ so closely that only one set of isopleths is labeled. However, above and to the north of the almost merged polar (38°N) and subtropical jets (33°N), denoted by $J$'s in Fig. 4, oscillations are very large and, consequently, the deviations $S' = S^* - S$ are also large. Between Tallahassee (station 72214) and Bedford (station 72490) the $S^*$ = 800 isopleth drops 10 km in elevation and decreases 150 K in $\theta$ (from 476 to 325 K).

It is extremely difficult to account for the steeply inclined relative maximum between Tallahassee and Bedford, and the local maximum of 1600 at ~10 km over Bedford by excluding diabatic cooling. Note that the large positive $S'$ deviations are located above and slightly to the north of the two jets, where cold cirrus cloud shields can enhance radiative cooling in the lower stratosphere. With small radiative heating south of the jets and larger radiative cooling north of the jets, the isopleths of $S^*$ and of tracer mixing ratio will rotate in the proper sense relative
to the isopleths of \( \tilde{S} \) and tend to maintain an \( \tilde{S} \) gradient on the isentropic surfaces against small-scale diffusion.

However, an additional constraint is required. The ozone mixing-ratio profiles obtained from ozonesondes at Tallahassee and Bedford were found to be very well correlated with \( \tilde{S} \) in the stratosphere (Danielsen, 1968). It follows that the vertical gradient of diabatic cooling must be small to preserve the correlation during the air’s descent. As discussed above, conservation of the correlation between potential vorticity and the tracer’s mixing ratio, in this case ozone, is generally compatible with small-scale mixing processes, therefore, both anisentropic processes are probably involved.

The physical explanation offered by Danielsen et al. (1962) and the schematic diagram of the circulations (Fig. 16 of the same reference) still appear to be valid. They attributed the decreasing entropy transport primarily to radiative cooling over the cold cirrus cloud shields, above and to the north of the jet axis, and depicted the ascending and descending branches as separate Lagrangian flows which mix actively in certain regions, but which do not form a closed Lagrangian circulation.

In the atmosphere, as in the dishpan experiments, a poleward directed heat flux is required to compensate for the meridional gradient of cooling. With no rotation this flux is supplied by a direct thermal circulation which represents the trajectories of fluid parcels. Rotation deflects the meridional flows, storing energy in the fluid which then gives rise to wave-like oscillations. These wave generated velocities are both linearly and elliptically polarized. In effect, the mean circulations are then small residuals of inclined quasi-isentropic elliptical motions and are predominantly streamlines rather than trajectories. It is difficult to predict the polarity for individual waves in the stratosphere and to predict their combined effects on transport, therefore, one must rely on diagnostic studies of the atmosphere or of general circulation models. The latter have the advantage of dynamic consistency and include the integrated effects of diabatic heating.

7. Analyses of the deviatory velocity tensor

To determine the polarities, grid-point values of the velocities for one January realization of the GFDL’s general circulation model were analyzed. While one set is not sufficient it was thought to be necessary and provided a basis for program development and for future comparisons with objectively derived velocities from synoptic data. At four pressure levels (38, 65, 110, and 190 mb) and three latitudes (24, 48 and 74°N) the \( u, v \) and \( w \) grid-point values were averaged and subtracted from the grid values to determine the deviations and the following terms in the deviatory velocity tensor were computed:

\[
\begin{bmatrix}
\overline{v'v'} & \overline{v'w'}
\end{bmatrix} = \overline{v'v'} \begin{bmatrix}
1 & \gamma
\gamma^2 + \beta^2
\end{bmatrix} = \frac{1}{2}v_0 \left( \begin{array}{c}
w_0 \\
v_0
\end{array} \right) \cos \phi_0
\]

In the central matrix array \( \gamma = v'w'/v'v' \) represents the angle between the principle axis of the associated ellipse and the \( v \) or \( w \) axis, and \( \beta \) is the ratio of its minor to major axis. Note that the determinant of the central array is equal to \( \beta^2 \), and in the principle axis system the off-diagonal elements are zero and the diagonal elements are 1 and \( \beta^2 \).

The matrix array on the right has special significance, especially for computing and interpreting the deviatory velocity tensor. We can think of the associated ellipse as being generated by a single wave whose components are

\[
v = v_0 \cos \phi, \quad w = w_0 \cos (\phi + \phi_0),
\]

where \( \phi \) denotes the longitudinal angle and \( \phi_0 \) is a relative phase angle. By forming the appropriate products from (11a) and (11b), and integrating over all longitudes, it is readily shown that the matrix array on the right of (10) is generated.

The significance of this equivalent monochromatic wave is revealed by expanding \( v' \) and \( w' \) in a Fourier series, forming their products and integrating over all longitudes. Since only a relative phase angle is important, let

\[
v'(\phi) = \sum_{m=1} \overline{v(m)} \cos m\phi, \quad w'(\phi) = \sum_{\mu=1} \overline{w(\mu)} \cos (\mu \phi + \phi_0);
\]

then, for example,

\[
\overline{v'w'} = \sum_m \frac{1}{2} \overline{v(m)} \overline{w(m)} \cos m\phi_0 = \frac{1}{2} v_0 w_0 \cos \phi_0
\]

and all terms with \( \mu \neq m \) vanish. Similarly, \( \overline{\rho v'v'} \) can be represented by \( \frac{1}{2} \overline{\rho v_0 v_0} \) which is the kinetic energy density of the meridional deviatory motions. Also, since \( v_0 \gg w_0 \), we can consider it the kinetic energy density of the deviatory motions.

It is clear from (13) and (10) that the deviatory velocity tensor can be written as the sum of the wavenumber tensors. Also, the equivalent ellipse can be interpreted in terms of the individual ellipses associated with each wavenumber. This result is
very convenient because it is easy to evaluate the Fourier coefficients and the relative phase angles. The relative phase between \( v'(m) \) and \( w'(m) \) determines the properties of the ellipse. From the equality of the determinants of the central and right matrix arrays (10), it can be shown that

\[ \beta^2 = \left( \frac{w_0}{v_0} \right)^2 \sin^2 \phi_0, \]  

(14)

therefore, if \( \phi_0 = 0, \pi \) the minor axis of the ellipse vanishes and the velocities are linearly polarized. Furthermore, since \( \cos \phi_0 \) then equals 1, the right matrix depends only on \( v_0 \) and \( w_0 \). Thus, \( \phi_0 = 0, \pi \) is equivalent to Mahlman’s assumption (8), and \( \gamma = w_0/v_0 \). In this case, Reed and German’s interpretation of \( \alpha \) as the mean slope of the mixing surface is unambiguous.

However, when \( \beta^2 \neq 0 \) the velocities \( v_0 \) and \( w_0 \) are elliptically polarized. This mean that at some longitude, \( w' \) will be finite when \( v' = 0 \), thus violating Reed and German’s assumption that \( \alpha \) is a small angle. On the other hand, \( \gamma \) is a small angle and it, not \( \alpha \), defines the mean slope of the mixing surface. Therefore, \( \gamma \), the principle angle of the ellipse, is the relevant angle for the general case.

Four examples of the individual wave number ellipses and the total ellipse are presented in Fig. 5, along with the grid-point values from which they were derived. In each example, the \( v', w' \) points scatter about an ellipse whose major axis slopes downward toward the pole. Also, it can be shown that these slopes exceed the mean slope of the \( \theta \) surfaces at the same grid points, thus a countergradient heat flux is implied.

This evidence does not support Wallace’s assertion that the ellipses will have a positive slope in the lower cyclonic stratosphere, nor does it support Mahlman’s implicit assumption of linear polarization. Conversely, it does support Newell’s interpretation and the inclusion of \( \alpha^2 \) by Reed and German, and by Louis. Note, however, that \( \alpha^2 \) must be interpreted via \( \beta^2 \) as describing the departure from linear polarization. That such departures can be large can be seen in Fig. 6, which illustrates the ellipses at 65 and 190 mb and 24°N. Both ellipses have small positive \( \gamma \)'s and are quasi-circular in these plots. Of course, the \( w' \) scale is distorted by \( \sim 400 \) to 1, but this is a realistic distortion for the atmospheric asymmetries. The positive slope at 190 mb is expected because it is in the mean troposphere, south of the extratropical jet. The slope at 65 mb is too small to be significant, meaning that \( v'w' \) is small because \( \phi_0 \approx \pi/2 \). In this case it is 87°.

Figs. 5 and 6 clearly demonstrate the prevalence of elliptical polarity between \( v' \) and \( w' \). But they indicate, also, that the component wavenumber ellipses can differ significantly from each other. In general, when \( \gamma(m) \)’s include both positive and negative values, the scatter of the \( v', w' \) points about the equivalent ellipse is large and it has small eccentricity. There is evidence, also, of linear polarity for some wavenumbers. In this respect, the plot for Fig. 7, 100 mb and 24°N, is particularly interesting. Wavenumbers 2–5 are all linearly polarized and all have positive slopes. This grid-point is located in the lower, anticyclonic stratosphere where the \( \theta \) isotherms have a small negative slope. Therefore, the linear polarity and positive slopes for \( \gamma \) imply the local change in \( \theta \) exceeds that due to advection, and warm (cold) advection is correlated with descending (ascending) motions. It remains to be seen whether this result is an artifact of the model or if the atmospheric diagnostic analyses will confirm it. If it is confirmed, it has important implications for tropospheric-stratospheric exchange at subtropical latitudes.

8. Physical significance of an asymmetric tensor

In a review article on stratospheric transport, Danielsen and Louis (1977) showed that transport by mean circulations could be incorporated into the diffusion tensor, but that the generalized tensor would no longer be symmetric. This result follows from the condition mentioned earlier, that \( \nabla \cdot (\tilde{\rho} \tilde{V}) = 0 \) and, therefore

\[ \tilde{\rho} \tilde{V} = \nabla \times (\tilde{\psi} \tilde{u}), \]  

(15)

where \( \tilde{\psi} \) is a streamfunction for the mean momentum density and \( \tilde{u} \) is a unit vector in the zonal direction. Then, by use of vector identities, one can write

\[ \nabla \cdot (\tilde{\rho} \tilde{V} \tilde{\chi}) = \nabla \cdot \begin{bmatrix} 0 & \tilde{\psi} \\ -\tilde{\psi} & 0 \end{bmatrix} \nabla \tilde{\chi}. \]  

(16)

When this antisymmetric tensor is added to the diffusion tensor, the entire transport is specified by a single asymmetric tensor

\[ -\nabla \cdot (\tilde{\rho} \tilde{V} \tilde{\chi} + \tilde{\rho} \tilde{V}' \tilde{\chi}') \]

\[ = \nabla \cdot \tilde{\rho} \begin{bmatrix} K_{yy} & K_{yz} + \frac{\psi}{\beta} \\ K_{zy} - \frac{\psi}{\tilde{\rho}} & K_{zz} \end{bmatrix} \cdot \nabla \tilde{\chi}. \]  

(17)

The next and crucial question is whether \( K_{yy} = K_{yz} \). If they are not equal, their mean value will contribute to a symmetric tensor and their differences will contribute to the antisymmetric tensor. In other words, if they are not equal, the mean flow will be modified, since it is represented by the antisymmetric tensor (see Plumb, 1979).

Sommerfeld (1950), in discussing antisymmetric tensors, stresses that they can be represented by axial vectors. The curl operator (15) generates an
Fig. 5. Meridional-vertical velocity correlations from the Geophysical Fluid Dynamics Laboratories' General Circulation Model. The grid-point values are the small, open circles which scatter about the equivalent ellipse and the line denoting the slope of its principle axis. Individual wavenumber ellipses are plotted at the left with a minus sign indicating anticyclonic or clockwise rotation versus increasing east longitude.

axial vector, thus, the streamfunction and the vorticity associated with it should appear explicitly and implicitly, respectively, in the antisymmetric part of the tensor. Indeed, the mean circulations do represent vorticity in the zonal direction since

\[ \nabla \times (\hat{\rho} \hat{\mathbf{V}}) = -\nabla^2 \hat{\psi}. \]  

The mean flow expressed in terms of \( \hat{\psi} \) in (17) is
the Eulerian mean, part of which is derived by diabatic, irreversible processes and part by isentropic processes. If the latter are significant, the streamfunction associated with them must be subtracted from the former and, therefore, $K_{xu} \neq K_{yu}$.

In a recent work by Matsuno (1981), this conclusion follows as a necessary condition for consistency with his assumption. That is, he analyzes the highly simplified case of a single, vertically propagating planetary wave (wavenumber 1), which is locally stationary in a barotropic, channel-type mean flow. But this simplicity permits him to obtain analytical solutions to the transport which are very revealing. In this author's opinion, they provide the key to
Fig. 6. As in Fig. 5 except for 65 and 190 mb grids at subtropical latitudes.

the resolution of most (perhaps even all) of the above noted problems.

The key is Matsuno’s (1981) generalization of the mixing-length hypothesis. By introducing a mixing time \( \tau \) defined by

\[
\tau = \frac{\dot{X} - X}{dX/dt},
\]

(19)

assumption (3) is replaced by an integral

\[
X'(0) = \int_{-\infty}^{0} \frac{e^{it}}{\tau} \left( \frac{\partial X}{\partial y} Y' + \frac{\partial X}{\partial z} Z' \right) dt,
\]

(20)

where \( Y' \) and \( Z' \) are now functions of time.

In particular, if they are harmonic functions of time, oscillating with a single, circular frequency \( \omega \)
and describing elliptical paths, as they are in his idealized example, the flux vector he derives is

$$-\nabla \chi' = [\Phi_1 K' + \Phi_2 K''] \cdot \nabla \chi,$$  \hspace{1cm} (21)

where $K'$ denotes a symmetric tensor, similar in form to that of Reed and German (1965) except that it describes the effects of wavenumber 1 only, and $K''$ denotes an antisymmetric tensor whose divergence acts to oppose the mean flow. The most revealing parts of this result are the scalar weighting factors.

$$\Phi_1 = \frac{\omega_T}{1 + \omega_T^2}, \quad \Phi_2 = \frac{\omega_T^2}{1 + \omega_T^2}. \hspace{1cm} (22)$$

To be rigorously consistent with his model assumptions, which include no mixing processes and no diabatic heating, $\tau$ must be infinitely large. Then $\Phi_1$ vanishes, $\Phi_2 = 1$ and the trivial solution $\partial \chi / \partial t = 0$ is achieved. The vanishing of $\Phi_1$ is consistent with no mixing processes and when $\Phi_2 = 1$ the Stokes drift cancels the mean circulation. It should, for in the absence of frictional forces and diabatic heating there is no material drift.

On the other hand, if $\tau$ is finite due to chemical-photochemical reactions, then there will be a net transport which must be counteracted by the source-sink term in (1) to maintain steady state. By relaxing his initial assumptions, Matsumo estimated $\tau \approx 10^7$ s, attributing it to smaller scale mixing and/or chemical processes. Under these circumstances, $\Phi_1 \approx 10^{-2}$ and $\Phi_2 \approx 1$. Therefore, he concludes that, "the eddy transport is advective rather than diffusive in nature and, in effect, it represents transports due to Stokes drift."

We will return to this conclusion later because these are reasons for rejecting his value of $\tau$ as being much too large. Therefore, although we can disagree with his conclusions, we can profit by his generalizations of the diffusion approximation.

9. Method for determining asymmetric diffusion tensor

Given a set of $u$, $v$ and $w$ velocities, the zonal means $\bar{u}$, $\bar{v}$ and $\bar{w}$ can be readily computed and, from the meridional-height distribution of $\bar{p} \bar{v}$ and $\bar{p} \bar{w}$, a streamfunction can be derived. Then, from $\bar{u}' v'$, $\bar{v}' w'$ and $\bar{w}' w'$, as shown in (9) and (19), the equivalent velocities $v_0$ and $w_0$, plus the relative phase angle $\phi_0$ can be evaluated. Associated with these velocities will be some, as yet unknown, equivalent $\omega_0$. Therefore, if we generalize (11) as

$$v' = v_0 \cos(\omega_0 t + \phi_1), \quad w' = w_0 \cos(\omega_0 t + \phi_2), \hspace{1cm} (23)$$

the associated displacements from the origin are

$$Y' = \frac{v_0}{\omega_0} [\sin(\omega_0 t + \phi_1) - \sin \phi_1], \hspace{1cm} (24)$$

$$Z' = \frac{w_0}{\omega_0} [\sin(\omega_0 t + \phi_2) - \sin \phi_2].$$
where
\[ \phi_2 = \phi_1 + \phi_0. \] (25)

If Eqs. (24) are substituted into (20) and integrated,
\[ \chi'(0) = \frac{-v_0}{\omega_0} \frac{\partial \chi}{\partial y} [\Phi_1 \cos \phi_1 + \Phi_2 \sin \phi_1] \]
\[ - \frac{w_0}{\omega_0} \frac{\partial \chi}{\partial z} [\Phi_1 \cos \phi_2 + \Phi_2 \sin \phi_2]. \] (26)

Next, multiplying (26) by (23), after setting \( t = 0 \), and then integrating over all longitudes (here \( \phi_1 \) and \( \phi_2 \) are functions of longitude), one obtains
\[ -\rho V^\prime \chi' = \left( \rho \begin{bmatrix} \frac{1}{2} v_0 v_0 & \frac{1}{2} v_0 w_0 \cos \phi_0 \\ \frac{1}{2} w_0 v_0 \cos \phi_0 & \frac{1}{2} w_0 w_0 \end{bmatrix} \Phi_1 \right) \frac{1}{\omega_0} \]
\[ + \begin{bmatrix} 0 & \rho^{1/2} v_0 w_0 \sin \phi_0 \\ -\rho^{1/2} v_0 w_0 \sin \phi_0 & 0 \end{bmatrix} \Phi_2 \cdot \nabla \chi. \] (27)

The solutions are
\[ T_1 = \frac{-\rho V^\prime \chi' \frac{\partial \chi}{\partial y} - \rho w^\prime \chi' \frac{\partial \chi}{\partial z}}{\rho \left( \frac{1}{2} v_0 v_0 \frac{\partial \chi^2}{\partial y} + v_0 w_0 \cos \phi_0 \frac{\partial \chi}{\partial y} + \frac{1}{2} w_0 w_0 \frac{\partial \chi^2}{\partial z} \right)} = \frac{-V^\prime \chi' \cdot \nabla \chi}{\nabla \chi \cdot K_s \cdot \nabla \chi}, \] (30)
\[ \psi_0 = \frac{-\rho V^\prime \chi' \left( \frac{1}{2} v_0 v_0 \cos \phi_0 \frac{\partial \chi}{\partial y} + \frac{1}{2} w_0 w_0 \frac{\partial \chi}{\partial z} \right) + \rho w^\prime \chi' \left( \frac{1}{2} v_0 w_0 \cos \phi_0 \frac{\partial \chi}{\partial y} + \frac{1}{2} v_0 v_0 \frac{\partial \chi}{\partial y} \right)}{\nabla \chi \cdot K_s \cdot \nabla \chi}. \] (31)

where \( K_s \) is the symmetric deviatory velocity tensor defined by (10) and (29). It is analogous to the Reynolds’ stress tensor, the latter being defined for temporal rather than zonal averaging.

Eqs. (27)–(31) have many interesting implications:

1) A solution exists unless \( \nabla \chi \cdot K_s \cdot \nabla \chi \) goes to zero faster than the numerator of (30) and (31). When integrated over the entire atmosphere this scalar product must be greater than zero because it equals \(-\partial \phi / \partial t (\frac{1}{2} \chi^2)\), which must be positive as \( \chi^2 \) decreases towards its minimum, a uniform distribution. It may, however, be very small or slightly negative at some latitude or height and, if so, a lower limit on the denominator must be used to maintain computational stability.

2) The numerator of (3) is the same as the common denominator for Mahlman’s \( K_{pp}, K_{pp}, \) and \( K_{sz} \) coefficients. Because \(-V^\prime \chi' \cdot \nabla \chi \) appears in the numerator instead of the denominator, the effect of a deviatory flux being approximately orthogonal to \( \nabla \chi \) apparently is not serious. Also, there appears to be no inherent difficulty with an upgradient flux \((V^\prime \chi' \cdot \nabla \chi > 0)\) providing that the divergence of the anti-symmetric tensor compensates for its local concentrating tendency.

3) Reed and German’s (1965) approximation is included also in (27) and (29) as the limiting case \( \omega_0 \rightarrow 0 \) for a finite \( \tau_0 \). This result is obtained from (22) by dividing \( \Phi _1 \) and \( \Phi _2 \) by \( \omega_0 \) and passing to the limit. Then
\[ -\rho V^\prime \chi' = \hat{\rho} K_s \tau_0 \cdot \nabla \chi. \] (32)

The same result is obtained by setting \( \omega_0 = 0 \) in (23) which yields \( Y' = v_0 \cos \phi \) and \( Z' = w_0 \cos \phi \) and then evaluating (20) with these linear displacements. Obviously this limiting result is unrealistic for the atmosphere but, of course, one need not pass to the limit for realistic applications of their approximation. Whenever the mixing period \( \tau_0 \) is small compared to the oscillatory period associated with \( \omega_0 \) the approximation is reasonably valid despite the prevalence of elliptical polarizations. From this viewpoint, it is a more general approximation than Mahlman’s, but because both methods eliminate \( \psi_0 \) there is reason to question their general applicability.

4) Adapting Matsuno’s (1981) concept of a mixing
time $\tau$ provides the additional insight that the mixing length hypothesis can be applied to 2-D models, but the vector displacements must be corrected for elliptically polarized wave motions. In other words, the history of the air parcel’s trajectory is important unless the effective mixing time is very small compared to the effective wave period. This correction for the parcel’s history is reminiscent of that required for elastic collisions in molecular diffusion (Jeans, 1925).

5) Eq. (27) may also resolve the perplexing results obtained by Louis (1974). Reference to (22) shows that $\Phi_1 = \Phi_2 = \frac{1}{2}$ when $\omega_0 \tau_0 = 1$. If, as indicated by Matsuno (1981), $\psi_0$ does tend to cancel $\psi$ then both the Eulerian mean circulations and the effects of the symmetric tensor would be reduced by $\sim 50\%$. On the other hand, in the middle and upper stratosphere, where the diabatic forcing increases in magnitude, $\psi$ could dominate and perhaps be reinforced rather than opposed by the deviatoric advects (see Section 11).

6) If $\tau$ becomes very small due to photochemical processes, both $\Phi_1$ and $\Phi_2 \to 0$. Then the local change in a trace species is comparably insensitive to transport. However, other than a loss of computer efficiency, there appears to be no logical difficulty in maintaining the transport computations. In principle, it seems preferable to do so and to consider only mixing processes in evaluating $\tau_0$, relying on proper descriptions of the chemical sources and sinks to account for the chemical lifetimes.

10. Physical significance of the divergence of the transport tensor

In the above discussion we have considered mainly the properties of the asymmetric tensor and how its components could be determined from a representative three-dimensional data set. However, in a 2-D model it is the divergence of the tensorial product that determines the net contribution by transport process to the local time rate of change, therefore, the spatial gradients of the tensor’s components are possibly as important as the components themselves. In particular, differences in the effects of the spatial gradients perhaps could explain why similar results are obtained from models whose diffusion coefficients differ drastically in magnitude.

To clarify the physical significance of the divergence of the tensor product, we will first write the flux in its general form

$$\nabla \cdot [\hat{\rho} \mathbf{V} \hat{\chi} + \hat{\rho} \mathbf{V} \hat{\chi}'] = -[\nabla \cdot \hat{\rho} K^* \cdot \nabla \hat{\chi}]$$

$$- [\hat{\rho} K^* \cdot \nabla \hat{\chi}].$$  \hspace{1cm} (34)

In this form, the first term on the right represents an advection and the second a diffusion of $\hat{\chi}$, with both processes being weighted by the mean density. Thus, for example, if $\hat{\rho}$ were constant the divergence of $K^*$ would describe an equivalent advective velocity which would include the rotational mean velocity $\mathbf{V}$ and an irrotational velocity associated with the deformations and divergence of the deviatoric motions.

If for simplicity of notation we expand (34) in Cartesian $y, z$ coordinates, the two terms including the density weighting can be identified as

$$\text{advective} = -\left[ \frac{\partial}{\partial y} (\hat{\rho} K_{yy}^*) + \frac{\partial}{\partial z} (\hat{\rho} K_{zz}^*) \right] \frac{\partial \hat{\chi}}{\partial y}$$

$$- \left[ \frac{\partial}{\partial y} (\hat{\rho} K_{zy}^*) + \frac{\partial}{\partial z} (\hat{\rho} K_{zx}^*) \right] \frac{\partial \hat{\chi}}{\partial z},$$  \hspace{1cm} (35)

$$\text{diffusive} = -\hat{\rho} \left[ K_{yy}^* \frac{\partial^2 \hat{\chi}}{\partial y^2} + (K_{zz}^* + K_{zy}^*) \frac{\partial^2 \hat{\chi}}{\partial y \partial z} + K_{zx}^* \frac{\partial^2 \hat{\chi}}{\partial z^2} \right].$$  \hspace{1cm} (36)

It is now clear from (36) that the streamfunction $\psi$ will cancel in the central term, therefore, only the symmetric tensor contributes to the simulated diffusion, but from (35) one can see that both antisymmetric and symmetric tensors contribute to the advection. In consideration of this dual role it seems appropriate to call $K^*$ a “transport” rather than a “diffusion” tensor.

Also, from (35) after the streamfunctions and the expression (29) for the symmetric tensor are introduced, the effective meridional advection velocity

$$v_{\text{eff}} = \bar{v} + v_s - \frac{1}{\hat{\rho}} \left[ \frac{\partial}{\partial y} (T_1 \bar{v} \sqrt{2} v_0 v_0) \right.$$ \hspace{1cm} (37)

$$\left. + \frac{\partial}{\partial z} (T_1 \bar{v} \sqrt{2} w_0 w_0 \cos \phi_0) \right]$$

and vertical advection velocity

$$w_{\text{eff}} = \bar{w} + w_s - \frac{1}{\hat{\rho}} \left[ \frac{\partial}{\partial z} (T_1 \bar{v} \sqrt{2} w_0 w_0) \right.$$ \hspace{1cm} (38)

$$\left. + \frac{\partial}{\partial y} (T_1 \bar{v} \sqrt{2} v_0 w_0 \cos \phi_0) \right].$$

Here, to refresh one’s memory, $\bar{v}$ and $\bar{w}$ are the Eulerian mean velocities, computed directly from the grid-point values of $v$ and $w$, while $v_s$ and $w_s$ symbolize the mean velocities attributed to Stokes
drift, those derived from gradients of $\psi_0$. Presumably, the latter velocities will have signs opposite to the former and, therefore, act to reduce the Eulerian mean circulations. However, (35)–(38) clearly imply that the divergence of the symmetric tensor also can effectively augment or oppose the advection by Eulerian mean motions.

Furthermore, although (37) and (38) contain many variables, some of the dominant terms can be easily deduced and interpreted. For example, when $\partial T_1/\partial y$ is small, the meridional gradient of deviatoric kinetic energy density will produce advections that are directed away from the energy density maximum. Since the low wavenumbers dominate the energy density, they are mainly responsible for the meridional deviatoric advection. Also, in the tropics, where $w_0$ and $\cos\phi_0$ are probably small, the vertical gradient of the vertical deviatoric kinetic energy density can modify the vertical advection. Thus, the amplitudes of Kelvin and gravity modified Rossby waves become important to tropospheric-stratospheric exchange.

Finally, both the advective and diffusive effects of the complete spectrum of internal waves depends directly on $T_1$. If $T_1$ is small, these irreversible effects are negligible and the wave transports are essentially reversible. Since $T_1$ is small when the product $\omega_2 \tau_0$ is large, the effective mixing time $\tau_0$ plays a critical role in the parameterizations. It is difficult to estimate the magnitude and the spatial gradients of $\tau_0$. On the basis of the measurements of strontium 89 and barium 140, products of the first French nuclear bomb test in the Sahara,Danielsen et al. (1962) deduced a value of the order of a few days. A similar range of values was deduced from the dilution of strontium 90 and potential vorticity during Project Springfield (Danielsen, 1968). However, Staley (1957) and more recently Shapiro (1978) have estimated smaller values, approaching the order of a day. Certainly, in regions of wave-induced turbulence or cumulonimbus entrainment, the mixing times can be of the order of minutes. Since the effective $\tau_0$ can be computed from $T_1$ and $\psi_0$, it will be interesting to see what values are obtained from this objective method.

In any case, it is intuitively satisfying to discover from the method that the large-scale waves are predominantly responsible for transport potentials, through their amplitudes and relative phases of meridional and vertical velocities, while the small-scale waves, which are more directly responsible for irreversible mixing, control $\tau$ and, therefore, the realization of transport potentials.

11. Summary

With the growing interest in and development of zonally averaged models, there is a corroborative need for an objective, statistical description of the fully three-dimensional transports. In general, these models will be based on Eulerian coordinates. The major disadvantage of averaging over fixed spatial coordinates is that the grid volumes represent completely open systems. Thus, integrals which extend over all longitudes completely eliminate all nonzero Fourier wave modes. That is, transport by the complete wave spectrum must be described statistically.

On the other hand, it is shown that Lagrangian integrals suffer from a much more serious difficulty. Data limitations preclude completely closed systems, but they can be partially closed, i.e., closed to all resolvable scales of motion. However, then the boundaries of the system deform, twist and fold. The deformations start with large-scale motions and proceed intermittently to smaller and smaller scales. Consequently, either the boundary is difficult to locate or the deviatory fluxes across the boundary are difficult to objectively determine.

A major advantage of the Eulerian average is that the local change in $\chi$ due to transport reduces to two terms, $\nabla \cdot (\rho V \hat{\chi} + \rho V' \chi')$, the convergence of the mean flux and of the mean of the deviatory flux, which can be objectively determined from representative meteorological data. If the velocity and $\chi$ are expanded in a Fourier series, the products of wave-number 0 determine the mean flux and the mean of the deviatory flux reduces to the sum of the products for each nonzero wavenumber. All products between nonequal wavenumbers vanish.

Taking advantage of this orthonormal property, the density weighted, deviatory velocity tensor $\rho V V'$ can be expressed in terms of an effective ellipse whose major axis squared is the mean kinetic energy density of the meridional deviations and whose minor axis squared is the mean kinetic energy density of the vertical velocity deviations multiplied by the sine of the mean phase angle $\phi_0$ between $\nu'$ and $w'$. Also, the slope of the major axis is proportional to the cosine of the same angle $\phi_0$.

This analysis method applied to velocity data from the GFDL's GCM demonstrates that the slopes of the major axes are negative in the cyclonic middle and lower stratosphere, in support of Newell's (1964) interpretation of a countergradient transport for ozone and heat. The same analyses show that elliptical polarization is characteristic of the small wavenumber modes, those which have the largest amplitudes, and that the amplitudes vary with latitude, so that a contribution to the mean circulations by Stokes drift is probable. As shown by Matsuno (1981) this contribution would appear as an antisymmetric tensor in the statistical description of the transports.

The symmetric part of the tensor is essentially the product of the density, a time constant, and the
Reynold’s stress tensor. The antisymmetric parts include the streamfunctions for the mean circulations and for contributions due to Stokes drift.

To evaluate the components of these tensors requires a representative set of \( u, v \) and \( w \) velocity components and an appropriately balanced set of thermodynamic variables to permit representative computations of potential vorticity. Given such a set, either from numerical diagnostic or predictive analyses, a method for reducing the problem to two equations and two unknowns is presented and discussed. Work has begun to analyze both types of data to obtain solutions, twice daily or four times daily. Later these solutions will be temporally averaged.

The asymmetric tensor has been called simply a transport tensor because the divergence of the tensor-vector inner product includes both advection and diffusion of \( \chi \). Of course, it includes advection by the mean \( \bar{V} \) velocities, the rotational velocities, but it includes also advections by irrotational deviatory velocities. Thus, for example, important advectons result from spatial gradients of the deviatory kinetic energy density.

In the absence of the essential data set there are a large number of degrees of freedom for tuning a model’s transport coefficients. Consequently, the current lack of unanimity among 2-D modelers relative to transport specifications is understandable. However, for a generally valid set, applicable to many tracers, the degrees of freedom should be effectively eliminated as the data sets are generated and processed.

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APPENDIX

**Bulk Systems: Eulerian and Lagrangian**

A generalization of Leibnitz’ formula for transforming from the time derivative of a moving integral to the integral of a partial time derivative can be written in the following form:

\[
\frac{D}{Dt} \int xdV = \left( \frac{\partial}{\partial t} + \bar{V} \cdot \nabla \right) \int xdV
\]

\[
= \int \frac{\partial x}{\partial t} dV + \oint \bar{V}_b x \cdot d\alpha, \quad (A1)
\]

where \( \bar{V}_b \) is the velocity of the bulk system, \( \bar{V}_b \) is the local velocity of its boundary and \( dV \) and \( d\alpha \) are its differential volume and surface area, respectively.

For Eulerian integrals, \( \bar{V}_b \) and \( \bar{V}_b = 0 \), therefore, the bulk systems are stationary, completely open systems of constant volume. The conservation equations for mass, momentum and energy density can be integrated over the volume to yield the appropriate bulk equations.

For Lagrangian integrals the total or bulk mass is constant, but there are two possible solutions. If \( x = \rho \) substitution of the mass conservation equation into (1.1) yields

\[
\frac{DM}{Dt} = -\oint \left( \bar{V} - \bar{V}_b \right) \rho \cdot d\alpha = 0. \quad (A2)
\]

If \( \bar{V} = \bar{V}_b \) the system is closed for all but the molecular motions, but it is physically impossible to determine the three-dimensional velocities of the boundary. An alternative approach is to set \( \bar{V}_b = \bar{V} \), a continuous function of the spatial coordinates, where \( \bar{V} \) is the density-weighted mean for the bulk volume

\[
\bar{V} = \frac{\int \rho \bar{V} dV}{\int \rho dV}. \quad (A3)
\]

Then with \( \bar{V} = \bar{V} + \bar{V}' \) Eq. (A2) becomes

\[
\frac{DM}{Dt} = -\oint \bar{V}' \rho \cdot d\alpha = -\oint \nabla \cdot \rho \bar{V}' dV
\]

\[
= -\nabla \cdot \left( \int \rho \bar{V}' dV \right) = 0. \quad (A4)
\]

In this case the system moves with the velocity of its center of mass, remains constant in mass, but is open for all scales of motion smaller than that defined by the averaging volume. Thus, for example, the bulk or mean momentum equations for the Eulerian integrals are

\[
\frac{\partial \bar{V}}{\partial t} + \bar{V} \cdot \nabla \bar{V} = -\rho^{-1} \nabla \rho - g \int \mathbf{K} \times \bar{V} - \rho^{-1} \nabla [\rho \bar{V} \bar{V}'] = \frac{\partial}{\partial t} \int \rho \bar{V} \bar{V}' dV + \frac{\partial }{\partial t} \int \rho \bar{V}' dV \quad (A5)
\]

and for Lagrangian integrals
\[ \frac{D\mathbf{V}}{Dt} = -\frac{\hat{\rho}}{\rho} \nabla p - \mathbf{g} - \mathbf{f} \mathbf{K} \times \mathbf{V} - \frac{\hat{\rho}}{\rho} \mathbf{V} \cdot [\rho \nabla \mathbf{V}' + \tilde{\sigma}], \quad (A6) \]

where \( p, \mathbf{g} \) and \( f \) are the pressure, vector acceleration of gravity and the vertical component of the earth’s vorticity, and \( \sigma \) is the molecular stress tensor.

Eqs. (A5) and (A6) are equivalent in the sense that to follow the bulk systems one needs to know the spatial, temporal distributions of \( \mathbf{V} \) and, in general, the correlations between the deviatory velocities, although different for the two systems, remain unknowns.

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