Curvature Diminution in Equatorial Wave, Mean-Flow Interaction[1]

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(Manuscript received 15 May 1981, in final form 2 September 1981)

ABSTRACT

It is shown that slowly varying linear equatorial Rossby-gravity waves in a barotropically neutral mean-wind profile near the equator accelerate the mean flow in a stabilizing sense there. This indicates that the Rossby-gravity wave, believed to be the driving force in the easterly acceleration phase of the quasi-biennial oscillation, cannot force a barotropically unstable mean flow near the equator. Mean flows generated near the equator in the easterly phase of the oscillation in the context of these approximations will therefore resemble or be approximately bounded by a parabola of curvature β, where β is the planetary vorticity gradient. This result does not depend upon a "barotropic adjustment" process, although the latter has been suggested in the past and would yield the same result, but over a broader latitudinal area.

1. Introduction

The theory of equatorial waves and wave, mean-flow interaction has an interesting history in recent years. Probably the most notable achievement of the theory to date is a successful qualitative explanation of the quasi-biennial oscillation advanced by Holton and Lindzen (1972). According to these authors, the stratospheric mean-wind oscillation arises from the alternating absorption, *via* radiative damping, of the lowest order equatorial wave modes—the Kelvin and Rossby-gravity waves—which are believed to be excited in the troposphere. The Kelvin wave appears to drive the mean flow in a westerly direction when the Rossby-gravity wave is shielded by the low-level flow, while in the opposite phase the Rossby-gravity wave drives the mean flow in an easterly sense similarly. Holton and Lindzens' theory explains this phenomenon in terms of linear, slowly varying waves while accounting for their influence on the mean flow in a "quasi-linear" model. Despite these limitations the theory has been greatly supported by the laboratory simulation of Plumb and McEwan (1978), who were also able to explain their laboratory oscillation in terms of a similar theory.

The laboratory simulation involved a stratified, nonrotating fluid in a cylindrical annulus having internal waves forced from below in the form of a standing wave. This standing wave can be regarded as the sum of two internal waves of equal amplitude propagating in opposite directions. Plumb and McEwan demonstrated that this configuration is unstable with respect to a quasi-biennial-type oscillation. In the laboratory, viscous diffusion provided the necessary absorption mechanism. The resulting descending mean-shear zones formed a remarkable parallel to the observed oscillation.

However, there is at least one important difference between the observed and laboratory oscillations, and it is the purpose of this note to explore this difference theoretically. Because the atmosphere is in rotation, there is an equatorial waveguide formed by the Coriolis force near the equator. Unlike the laboratory simulation this leads to a significant meridional

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0022-4928/82/010182-05$05.25

structure across the waveguide. This structure manifests itself in two ways. First, there is a Gaussian envelope of the wave fields in latitude causing them to decay to zero at large distances from the equator. This effect was formally included in the Holton-Lindzen theory insofar as the model equations were first integrated over latitude. Second, the meridional structure in the wave fields causes the mean flow to have a meridional structure as well; in other words, latitudinal shear will result from equatorial wave accelerations. This shear, however, was not incorporated into the Holton-Lindzen model. While the success of their model might otherwise suggest the irrelevance of this shear, it is gradually becoming clear that the effect of latitudinal shear is not negligible, and may play a significant role in the oscillation (Boyd, 1978; Dunkerton, 1982). Although latitudinal shear can formally affect both Kelvin and Rossby-gravity waves, the effect seems to be much greater for the latter wave, as this wave has a shear-advecting perturbation meridional velocity centered on the equator.

Beginning from an initial state at rest, it is possible to determine the initial evolution of this latitudinal shear, by employing the known Rossby-gravity wave fields together with the slowly varying acceleration formula. A cogent presentation of this was given by Andrews and McIntyre (1976a) and the reader may consult their Fig. 1. For steady waves the acceleration is in the form of an easterly jet centered on the equator except when the ratio of mechanical-to-thermal damping is very small; in the latter range the easterly accelerations bifurcate, being maximum away from the equator and formally zero on the equator when the mechanical damping vanishes. Andrews and McIntyre recognized that transient, conservative waves induce a meridional flow profile identical to that caused by equal mechanical-thermal damping as further explained by Dunkerton (1980). It is presently believed that the transient wave result is relevant in the observed oscillation (Holton, 1979; Dunkerton, 1981). Holton's (1979) numerical integrations are the most general performed to date as the slowly varying approximation is not invoked.

Although it does not appear to have been noted previously, the formation of an easterly jet centered on the equator leads to what might be regarded as something of a paradox. This is because equatorial waves rely upon the equatorial waveguide formed by the β-effect for their existence, and in an easterly jet the β-effect vanishes. The question arises as to whether or not the transient Rossby-gravity wave can survive, as it were, its own apparent tendency to form such a mean-flow jet. A related question is whether or not barotropic instability can be generated in the easterly jet, and if such instabilities might preserve the β-effect by stabilizing the mean flow.

While these questions might call for a detailed numerical investigation, it recently became clear that there exists within the context of the linear, slowly varying wave approximations an important constraint on the mean flow evolution induced by the Rossby-gravity wave. As explained in the next section this result states that for this wave there exists a finite region about the equator in which the Rossby-gravity wave cannot force the mean flow beyond the point of neutral barotropic stability, defined by a vanishing mean-meridional relative vorticity gradient. This result is expressed in terms of a very simple expression for the Rossby-gravity wave acceleration in a barotropically neutral mean flow near the equator. It is found that in some finite region about the equator the Rossby-gravity wave acts to stabilize the mean flow. Hence, the conclusion follows that this wave cannot force the mean flow beyond the neutral stability boundary in this region in the context of these approximations.

Following Andrews and McIntyre (1976b) who discussed a shear enhancement effect, and later a shear diminution effect (D. G. Andrews, personal communication to J. R. Holton; see also Holton, 1979), we will refer to this as the *curvature diminution effect.*

Being an analytic result, this theorem places an important constraint on mean flows induced by Rossby-gravity waves. As these flows evolve in an overall easterly direction approaching the critical level, the latitudinal profile of the mean wind has a bounded curvature near the equator. From the overall momentum budget one may then estimate the shape of the wind profile as approximately bounded by a parabola of curvature β, where β is the planetary vorticity gradient. The meridional extent of this parabola is equal to the domain over which a stabilizing curvature is implied by the curvature diminution theorem. Outside of this domain, barotropically unstable profiles are not forbidden by the curvature diminution theorem, but the overall mean flow structure remains constrained by the requirement that the latitudinally integrated mean flow continue to accelerate in an easterly direction so long as there is significant growing wave transience or absorption. Significantly the curvature diminution theorem holds regardless of the relative importance of transience and dissipation hence the theorem remains valid throughout the easterly acceleration phase.[3]

Besides being directly applicable to the easterly phase of the observed quasi-biennial oscillation the curvature diminution theorem governs the occurrence of barotropic instability due to Rossby-gravity

[3] The curvature diminution theorem does not forbid the barotropic instability likely to occur when a Rossby-gravity wave is steady and dissipated thermally *only* as discussed by Dunkerton (1980); the weight of the argument here is directed at the transient or mechanically damped cases now believed more relevant.

wave accelerations. Andrews and McIntyre (1976) felt that this wave would generate an unstable mean flow almost immediately. We find here that a restriction must be placed on the occurrence of this instability; it may not necessarily occur early in the easterly acceleration phase, and it will definitely not occur at or around the equator. Its occurrence is limited to the region outside the domain of stabilizing curvature tendency.

Of course, if barotropic instability does occur, and if its net effect is to erase unstable curvatures as Andrews and McIntyre suggested, then an additional constraint governs the mean flow evolution overall, *viz.*, that the meridional profile of mean wind cannot exceed that of a parabola of curvature β. In other words, "barotropic adjustment" implies the same result as the curvature diminution theorem, but over a larger area. It is too early, however, to insist on the correctness of such a concept as barotropic adjustment, whereas the curvature diminution theorem is as accurate as the linear, slowly varying approximations upon which the Holton-Lindzen theory is based.

2. The curvature diminution effect

For reference the perturbation equatorial wave equations are

$$(D_t + \alpha_M)u' + v'(\bar{u}_y - \beta y) + w'\bar{u}_z + \phi_x' = 0, \quad (2.1a)$$

$$(D_t + \alpha_M)v' + \beta y u' + \phi_y' = 0, \quad (2.1b)$$

$$(D_t + \alpha_T)\phi_z' + v'\bar{\phi}_{zy} + w'N^2 = 0, \quad (2.1c)$$

$$u_x' + v_y' + w_z' = 0, \quad (2.1d)$$

where $D_t = (\partial/\partial t) + \bar{u}(\partial/\partial x)$, an overbar is a zonal average, $\bar{u}$ is the mean flow, u', v', w' are zonal, meridional and vertical velocity, ϕ' is geopotential, N^2 is the static stability, and α_M and α_T are mechanical and thermal damping coefficients. The waves are assumed hydrostatic and locally Boussinesq. Here we make the slowly varying approximation insofar as the transience, dissipation and vertical shear are formally small. The lowest order set of equations derived from (2.1a)–(2.1d) are the familiar equatorial eigenequations (Boyd, 1978) and imply waves of some slowly varying vertical wavenumber m and phase speed c. The vertical wavenumber is related to the eigenvalue ϵ as

$$\epsilon = m^2/N^2. \quad (2.2)$$

The equations are reducable to a single equation for the meridional velocity, i.e.,

$$\left(\frac{\partial}{\partial y} + \frac{\beta y k}{\hat{\omega}}\right)Q^{-1}\left(\frac{\partial}{\partial y} + \frac{k}{\hat{\omega}}(\bar{u}_y - \beta y)\right)v = \Delta v/\hat{\omega}, \quad (2.3)$$

where k is the zonal wavenumber, $\hat{\omega} = k(c - \bar{u})$, and

$$Q = \epsilon\hat{\omega} - k^2/\hat{\omega}, \quad (2.4a)$$

$$\Delta = \beta y(\beta y - \bar{u}_y) - \hat{\omega}^2. \quad (2.4b)$$

Symmetric disturbances in longitude ($k = 0$) satisfy the simpler equation

$$v_{yy} - \epsilon\Delta v = 0, \quad (2.5)$$

which is also the lowest order equation in the γ-plane approximation of Boyd (1978). The curvature diminution theorem does not depend in any way on the γ-plane approximation, although the theorem is slightly easier to verify in this case.

Derivation of the curvature diminution theorem may be accomplished in one of two ways. If in addition to assuming the vertical shear small, we assume that it is identically zero, the argument may proceed using vorticity considerations. It is assumed that in some region about the equator the mean flow is barotropically neutral, i.e.,

$$\beta - \bar{u}_{yy} = 0. \quad (2.6a)$$

It will be assumed that the mean flow is symmetric about the equator. This follows whenever the initial mean flow is symmetric (Takahashi and Uryu, 1982) but also would be supported by the shear diminution theorem quoted earlier (Holton, 1979). From (2.6a) and symmetry,

$$\beta y - \bar{u}_y = 0. \quad (2.6b)$$

This implies that the effective Coriolis torque in (2.1a) vanishes in this region. Hence in the following discussion the reader may be assured that latitudinal shear is not neglected; in this region its effect is restricted to variations in intrinsic frequency. The perturbation vorticity equation formed from (2.1a,b) when $\bar{u}_z = 0$ and (2.6b) holds is simply

$$(D_t + \alpha_M)\zeta' = 0, \quad (2.7)$$

where $\zeta' = v_x' - u_y'$, immediately implying that at lowest order the perturbation vorticity vanishes.

The mean flow acceleration is simplified in this case, being equal to the convergence of eddy momentum flux (an otherwise dangerously incorrect assumption in view of the mean meridional circulation advection), i.e.,

$$\bar{u}_t = -\overline{v'u_y'} - \overline{w'u_z'}, \quad (2.8a)$$

since we take $\bar{w}\bar{u}_z = 0$ and assume that $\beta y - \bar{u}_y = 0$ locally. Following Dunkerton (1980), the acceleration may be written

$$\bar{u}_t = \overline{v'\zeta'} - \overline{w'u_z'}. \quad (2.8b)$$

Therefore, at lowest order

$$\bar{u}_t = -\overline{w'u_z'}, \quad (2.8c)$$

which may be evaluated from (2.1a,c) with $\bar{\phi}_{yz} = 0$, to give

$$\bar{u}_t = \left(\frac{\partial}{\partial t} + \alpha_M + \alpha_T\right)\frac{\overline{u'_z\phi'_z}}{N^2}, \tag{2.9}$$

which may, in turn, be evaluated with the lowest order fields

$$u' \approx \frac{\phi'}{c - \bar{u}}, \tag{2.10}$$

to give

$$\bar{u}_t = \left(\frac{\partial}{\partial t} + \alpha_M + \alpha_T\right)\frac{\overline{\phi_z'^2}}{(c - \bar{u})N^2}. \tag{2.11}$$

By reason of the Rossby-gravity wave symmetry about the equator, together with the negative sign of $(c - \bar{u})$, it is apparent that the acceleration vanishes at the equator, and implies a stabilizing curvature tendency near the equator, so long as the wave quantity in (2.11) is either growing, dissipated, or both. The special case of conservative decay is not likely to be relevant.

When the vertical shear is nonzero but formally small, the vorticity argument is more difficult, though possible. It is simplest to go directly to the linear, slowly varying formula for the mean flow acceleration derived by Andrews and McIntyre (1976a):

$$\bar{u}_t = -\alpha_M(\overline{\eta'u'})_y + \frac{1}{c - \bar{u}}\{\alpha_M[\overline{u'(u' + \eta'\bar{u}_y)}] + \alpha_M\overline{v'^2} + \alpha_T\epsilon\overline{\phi'^2}\} + \frac{1}{2}\frac{\partial}{\partial t}\left\{-(\overline{\eta'u'})_y + \frac{1}{c - \bar{u}}[\overline{u'(u' + \eta'\bar{u}_y)} + \overline{v'^2} + \epsilon\overline{\phi'^2}]\right\}, \tag{2.12}$$

where $D_t\eta' \equiv v'$. Eq. (2.3) in the neutral region is simply

$$\left(\frac{\partial}{\partial y} + \frac{\beta yk}{\hat{\omega}}\right)\frac{v_y}{Q} = -\hat{\omega}v. \tag{2.13}$$

Substitution of this result into (2.12) yields (2.11) after a few steps taking advantage of the peculiar form of the wave equipartition law in the neutral region [Andrews and McIntyre, 1976a, Eq. (7.5)]. It follows that curvature diminution is not dependent on zero vertical shear, but only a small shear. In other words, the theorem applies directly to the Holton-Lindzen model.

The curvature diminution theorem (2.11) implies that in a barotropically neutral mean flow about the equator, some finite domain exists in which the acceleration acts to reduce the magnitude of the mean-flow curvature. The point where no curvature tendency is implied marks the outer boundary of this domain. To determine this point depends on solving the eigenproblem (2.3) which is well beyond the scope of this note; however, a few qualitative remarks can be made. First, the effect of small easterly curvature is to expand the latitudinal scale of the Rossby-gravity wave and thereby help to insure the relevance of the curvature diminution theorem initially. This point is implicit in Boyd's (1978) γ-plane approximation and is a result that the author has verified with a numerical Hermite spectral model. For the observed wave, the latitudinal scale in the basic state at rest is on the order of 10° latitude. However, the latitudinal scale presumably shrinks to zero as the critical level is approached (Holton, 1979), thus ultimately implying that the domain governed by (2.11) would be very small. Nevertheless by this time the wave's effect is likely to be very small anyway.

In between the initial evolution and the final approach to critical-level conditions we see that the possibility of barotropic instability is forbidden inside the domain governed by (2.11). By this we mean that the generation of barotropic instability, involving wave overreflection from critical lines inside a region of reversed relative vorticity gradient, cannot occur within this domain. (Of course, barotropic instabilities may or may not be *observed* here.) This constraint does not seem to have been previously appreciated; for example, Andrews and McIntyre (1976) speculated that the Rossby-gravity wave should lead to barotropic instability almost immediately in view of their Fig. 1. Our result implies for the transient Rossby-gravity wave the possibility of a longer elapsed time before such instability, and as already mentioned the instability cannot originate near the equator.

As stated in the Introduction, if the net effect of barotropic instability is to stabilize the mean flow as Andrews and McIntyre (1976a) speculated, this barotropic adjustment leads to the same result as curvature diminution, except that the mean flow would be everywhere bounded by an easterly parabola of curvature β. It is too early, however, to assume that such an adjustment is correct, and more investigation is needed.

3. Conclusion

Future investigations of slowly varying equatorial Rossby-gravity waves in latitudinal shear now appear to have an important constraint expressed by the curvature diminution theorem. Also, the occurrence of barotropic instability is governed by this theorem and is not allowed in some region near the equator determined by (2.11) together with (2.3). The overall constraint that the latitudinally integrated mean flow have an easterly acceleration during this phase of the quasi-biennial oscillation thus indicates that the final mean flow beyond which absorption reduces the wave amplitude exponentially to zero cannot have a curvature in excess of β near the equator and cannot exceed the phase speed c at the equator in the context of these approximations.

In an attempt to crudely assess the relevance of the curvature diminution theorem it is useful to compare the latitudinal scale of the Rossby-gravity wave with the latitudinal extent of the neutral parabola bounding the final mean flow. The latter is

$$\bar{u}_f = c + \tfrac{1}{2}\beta y^2, \qquad (3.1)$$

and vanishes at the distance from the equator

$$y_f = (-2c/\beta)^{1/2}. \qquad (3.2a)$$

The latitudinal scale of the Rossby-gravity wave without latitudinal shear, which is also the e-folding scale of the Gaussian envelope of this wave, is

$$y_0 = \frac{|\hat{\omega}|}{\beta(1 + k\hat{\omega}/\beta)^{1/2}}. \qquad (3.2b)$$

Crudely speaking, the relevance of the curvature diminution theorem is indicated by the ratio of y_0 to y_f:

$$y_0/y_f = [\tfrac{1}{2}|\chi|/(1 + \chi)]^{1/2}, \qquad (3.3)$$

where $\chi = k\hat{\omega}/\beta$. For the observed wave in the basic state at rest $\chi \approx -\tfrac{1}{2}$ and thus the theorem will be important.

Although more work needs to be done numerically at this point, there is some indication in Holton's (1979) symmetric mean flow integration that (3.1) is indeed a relevant constraint. Holton's time integrations were terminated somewhat early in part because of barotropic instability associated with the model's artificial sidewalls (Holton, personal communication).

The likelihood or otherwise of barotropic instability in the equatorial stratosphere in the easterly phase of the quasi-biennial oscillation remains perhaps the most interesting observational question having some bearing on this paper. A possibly related issue is whether or not the wavelike temperature anomalies observed in the lower stratosphere by Stanford and Short (1981) have any correlation with the quasi-biennial oscillation. Future study of these questions is likely to be interesting.

Acknowledgment. This research was supported by the Atmospheric Research Section, National Science Foundation, under Grant ATM79264-87.

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