The Effects of Moisture on Trapped Mountain Lee Waves

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ABSTRACT

The effects of latent heat release on the dynamics of mountain lee waves are examined with the aid of two-dimensional numerical simulations, for several situations in which the Scorer parameter has a nearly two-layer vertical structure. Changes in the moisture in the lowest layer are found to produce three fundamentally different behaviors: 1) resonant waves in an absolutely stable environment are distorted and untrapped by an increase in moisture; 2) resonant waves in a conditionally unstable layer are destroyed by an increase in moisture; and 3) resonant waves in a moist environment are detuned by a decrease in moisture. Changes in the humidity in the upper layer are found to amplify or damp the wave response, depending on the depth of the lower layer. In most situations, the wave response is significantly more complicated than that predicted by simply replacing the dry stability with an equivalent moist stability in the saturated layer.

1. Introduction

Although dry mountain lee waves have been studied extensively for the last 40 years, the influence of moisture on the dynamics of these waves has received little attention. This lack of research is due in part to the qualitative success with which dry dynamical formulations have been able to reproduce lee-wave flows, and in part to the difficulty of including latent heating in theoretical models. Condensation might be expected to modify the dry dynamics by lowering the effective stability in saturated regions. Barninghill et al. (1979) have investigated the effects of reversible condensation on hydrostatic mountain waves. They found that the stability reduction produced in low-level clouds could decrease the mountain wave drag to as little as one-half the dry value. In this paper, we will examine the effects of moisture on smaller scale mountain waves, the so-called trapped lee waves.

Mountain waves are often generated when stably stratified air flows over a mountain ridge. Air parcels are vertically displaced during their passage over the ridge and, under appropriate conditions, may oscillate downstream about their equilibrium levels. According to linear theory, the character of the resulting gravity waves is principally governed by the size and shape of the ridge contour, and the Scorer parameter,

\[ I^2 = \frac{N^2}{\bar{u}^2} - \frac{\bar{u}\bar{z}}{\bar{u}}, \tag{1} \]

where \( N(z) \) is the Brunt-Väisälä frequency, \( \bar{u}(z) \) is the mean cross-mountain wind speed, and \( z \) the vertical coordinate. The Scorer parameter is the maximum horizontal wavenumber at which steady linear gravity waves can propagate in the vertical. Scorer (1949) observed that if \( I \) decreases with height, there will be a range of wavenumbers over which standing gravity waves can have a periodic vertical structure only near the ground. If this decrease in \( I \) is abrupt and sufficiently large, one or more resonant waves can develop in the lower atmosphere. These waves, called trapped lee waves, can extend many wavelengths downstream from the ridge crest, and can produce strong rotors and destructive winds along the lee slopes of the ridge.

If a portion of the atmosphere is saturated, its mean stability is lowered due to the release (absorption) of latent heat associated with condensation (evaporation) processes. Lala and Einaudi (1974) have demonstrated that in a saturated environment, the linear wave equation has exactly the same form as that for a dry atmosphere, if the stability parameter is appropriately altered to include the influence of moist processes. Thus, in principle, one could determine the effects of reversible condensation on linear mountain waves by solving dry equations in which the cloudy regions are replaced by dry regions of suitably reduced stability. This was essentially the approach adopted by Barcilon et al. (1979). However, since the cloud boundaries are not known \textit{a priori}, an iterative procedure is required to match the regions of reduced stability with the regions of upward displacement and obtain a self-consistent flow. Consequently, this method has only been applied to linear hydrostatic waves and simple atmospheric structures. The procedure is not well suited to the study of trapped lee
waves, which are inherently nonhydrostatic phenomena and usually associated with relatively complex atmospheric structures.

In order to investigate the influence of moisture on trapped lee waves, we have developed a model which numerically integrates the two-dimensional time-dependent equations of motion, governing the flow of moist air over a topographic barrier. The numerical modeling approach allows the investigator to study a wide variety of situations which are not amenable to analytic solution (i.e., nonlinear, nonhydrostatic, or time-dependent waves produced by nonuniform wind, temperature and moisture soundings) and, in particular, it allows a more accurate treatment of the moist processes. In Section 2, we will briefly describe and test this model. A detailed description of the model and its verification is being prepared for separate presentation. In the remainder of the paper, we will investigate the behavior of lee waves in the conceptually simple case of a two-layer atmosphere. In Sections 3 and 4, we consider the influence of moisture in the lower layer on the structure of linear and finite amplitude waves, respectively. In Section 5, we consider some effects of moisture in the upper layer.

2. The numerical model

The computer model is designed to calculate the two-dimensional airflow over an infinitely long, uniform mountain barrier. The Coriolis force is neglected, since only narrow mountains with widths much less than the mean wind speed divided by the Coriolis parameter will be considered. The model used in this study was derived from the convective cloud model of Klemp and Wilhelmson (1978). Three major modifications were made to that model: it was reduced from three to two dimensions; a terrain-following coordinate system was introduced; and a wave-absorbing layer was added to the top of the domain. Less significant changes were also made in the turbulent mixing parameterization, the small time-step differencing and the lateral boundary conditions.

The basic continuity, thermodynamic and momentum equations in the Klemp-Wilhelmson (KW) model may be written in two dimensions as follows:

\[
\frac{du}{dt} + c_u \beta_M \frac{\partial \pi}{\partial x} = D_u, \\
\frac{dw}{dt} + c_v \beta_M \frac{\partial \pi}{\partial z} = g \left( \frac{\theta_M - \theta_M}{\theta_M} \right) + D_w,
\]

\[
\frac{d\pi}{dt} + w \Pi_z + \frac{R}{c_v} (\Pi + \pi) \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \frac{d\theta}{d\pi} - \frac{R}{c_v} \left( \frac{\Pi + \pi}{\theta} \right) \frac{d\theta}{d\pi} = 0,
\]

\[
\frac{d\theta}{dt} = -L \frac{\Delta}{c_p \Pi} + D_\theta,
\]

where

\[
\Pi = \Pi \left( \frac{p}{p_0} \right)^{\rho \theta_M} = \left( \frac{R}{p_0} \rho \theta_M \right)^{\rho \theta_M},
\]

and

\[
\theta = \theta(1 + 0.61 q_v), \quad \theta_M = \theta(1 - q_v) \]

\[
\Pi + \pi = \left( \frac{p}{p_0} \right)^{\rho \theta_M} = \left( \frac{R}{p_0} \rho \theta_M \right)^{\rho \theta_M}
\]

\[
\frac{\partial \Pi}{\partial z} = -\frac{\theta}{c_p \Pi(z)}
\]

In the above, \( p \) is the pressure, \( p_0 = 1000 \text{ mb} \), \( \rho \) is the total density, \( R \) the gas constant for dry air, \( c_p \) the specific heat of dry air at constant pressure, \( \theta_v \) the specific heat at constant volume, \( L \) the latent heat of vaporization, \( \theta \) the potential temperature, and \( u \) and \( w \) are the horizontal and vertical velocity components. The term \( D \) contains the contributions from turbulent mixing. The mixing ratios of water vapor, cloud water, and the saturation mixing ratio are \( q_v \), \( q_c \) and \( q_{v,w} \), respectively. The saturation mixing ratio is calculated from Teten's formula:

\[
q_v = \frac{3.8}{\bar{p}} \exp \left( 17.27 \frac{\Pi \theta}{273} - 36 \right),
\]

where \( \bar{p} \) is expressed in millibars and \( q_v \) in grams per kilogram.

Ice processes are not included in the model, since they are not critical to this investigation and would significantly complicate the microphysical parameterizations. Although ice can be associated with mountain waves, in many cases of physical interest it is not present. Even when freezing does occur, the dynamical forcing produced is small compared to that produced by condensation, since the latent heat of vaporization is an order of magnitude larger than the latent heat of fusion. Furthermore, the inclusion of freezing processes would require a much more complex microphysical model since the amount of glaciation in a cloud depends on both the temperature and the availability of ice nuclei.

The actual computer code does include warm rain processes. However, since the model produced no precipitation in the cases discussed in this paper, a description of the rain parameterization will not be given here. The lack of precipitation is consistent with observational data which suggests that precipitation
is almost never produced in the short wavelength clouds which form in the crests of trapped lee waves, because the liquid water content in these clouds is too low. If rain or snow is associated with a mountain wave event, it usually falls from a cap cloud which impinges on the windward slopes of the mountain. In this paper we will consider only narrow ridges which, although they are ideal for generating trapped resonant waves, do not produce broad cap clouds. The liquid water concentrations generated by forced uplift, and the in-cloud residence time of the water droplets in the narrow cap clouds, are insufficient to produce precipitation. Consequently, the thermodynamic processes in these simulations are reversible.

When modeling phenomena on the scale of mountain lee waves, the effects of subgrid scale turbulence must be parameterized as a function of the larger scale flow. In the KW model, this is done by solving an additional prognostic equation for the subgrid scale kinetic energy, from which the eddy mixing coefficients are determined. While this approach is more sophisticated than a simple first-order closure scheme, it is not primarily designed for two-dimensional turbulence. In this two-dimensional model, we employ a conventional first-order closure formulation which depends on the relative strengths of stratification and shear (Lilly, 1962). Subgrid scale effects are introduced to the velocity field calculations through the terms $D_u$ and $D_w$:

$$D_u = \frac{\partial}{\partial x} (K_M A) + \frac{\partial}{\partial z} (K_M B),$$
$$D_w = \frac{\partial}{\partial x} (K_M B) - \frac{\partial}{\partial z} (K_M A),$$

(10)

where

$$A = \left( \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \right), \quad B = \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right),$$

(11)

$$K_M = k^2 \Delta x \Delta z (A^2 + B^2)^{1/2} \times \left[ \max \left( 1 - \frac{K_M}{K_M} \text{Ri}, 0 \right) \right]^{1/2},$$

(12)

$$\text{Ri} = \begin{cases} \frac{N^2}{A^2 + B^2} & \text{for } q_v < q_{vs} \\ \frac{N_m^2}{A^2 + B^2} & \text{for } q_v = q_{vs} \end{cases}$$

(13)

Here

$$N_m^2 = g \left( 1 + \frac{L q_v}{RT} \right) \left( 1 + \epsilon L^2 q_{vs} \right)^{-1} \times \left( \frac{\partial q_v}{\partial z} + \frac{L}{c_p T} \frac{\partial q_v}{\partial z} \right) - g \frac{\partial (q_v + q_c)}{\partial z},$$

(14)

and $\epsilon = 0.622$. The expression for the moist Brunt-Väisälä frequency $N_m$ is derived by Durran and Klemp (1982), and is shown to be a good approximation to the exact expression. The subgrid scale mixing terms in the scalar equations are of the form

$$D_\phi = \frac{\partial}{\partial x} \left( K_H \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial z} \left( K_H \frac{\partial \phi}{\partial z} \right),$$

(15)

where $\phi = \theta, q_v$ and $q_c$. In the model, $k = 0.21$ (Dear-dorff, 1971) and $K_H/K_M = 3$ (Dear-dorff, 1972). This ratio of $K_H/K_M$ allows turbulent mixing to begin when $Ri$ drops below $\frac{3}{4}$, which is slightly larger than the commonly accepted critical value for the stability of a shear flow, $Ri = \frac{3}{4}$.

Terrain is incorporated in the model by a transformation of the vertical coordinate (Gal-Chen and Somerville, 1975)

$$\xi = \frac{z(z - z_t)}{z_i - z_s},$$

(16)

where $z(x)$ is the terrain elevation and $z_i$ is the uniform height of the top of the modeling region. This transformation has been used successfully to numerically simulate dry mountain waves by Clark and Peltier (1977). Eqs. (2)–(7) may be written in transformed form as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + (Gu + Hw) \frac{\partial u}{\partial \xi} = D_u,$$

(17)

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + (Gu + Hw) \frac{\partial w}{\partial \xi} = D_w + \frac{g}{\theta - \theta_M} \frac{\partial \theta}{\partial \xi},$$

(18)

$$\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + (Gu + Hw) \frac{\partial \theta}{\partial \xi} = \frac{\partial}{\partial \xi} \left( \frac{\partial \Pi + \chi}{\partial \xi} + \frac{R}{c_v} \frac{\partial \Pi + \chi}{\partial \xi} \right) = 0,$$

(19)

$$\frac{\partial q_v}{\partial t} + u \frac{\partial q_v}{\partial x} + (Gu + Hw) \frac{\partial q_v}{\partial \xi} = D_v,$$

(20)

$$\frac{\partial q_c}{\partial t} + u \frac{\partial q_c}{\partial x} + (Gu + Hw) \frac{\partial q_c}{\partial \xi} = -D_c,$$

(21)

where

$$G = \frac{\partial \xi}{\partial x} = \frac{\xi - z_t}{z_i - z_s},$$

$$H = \frac{\partial \xi}{\partial z} = \frac{z_i}{z_i - z_s},$$

(23)

The numerical model solves these equations using the technique of the KW model, which provides com-
Computational efficiency by treating sound wave modes separately on a shorter time step. The bulk of the computation is done on the large time step. In the large time step, the time differencing is leapfrog, horizontal advection is fourth order, and vertical advection is second order. Buoyancy, diffusion and coordinate transformation terms are computed to at least second-order accuracy. The microphysics are included through a two-step procedure proposed by Soong and Ogura (1973) and used in the KW model. In the first step, the temperature and moisture variables are advected and diffused; in the second step, they are adjusted to the correct thermodynamic balance. The details of the finite differencing are described by Durran (1981), and will not be given here.

The ground is the only physical boundary associated with the mountain wave problem. We require that the normal velocity vanish at the surface, a condition which is greatly simplified by the coordinate transformation. As a result $Gu + Hw = 0$ at $z = 0$, so the vertical flux terms in (17) through (22) vanish at the lower boundary. The formulations for the lateral and upper boundaries are also important, since they must allow disturbances to propagate freely out of the modeling region. The lateral boundary conditions specified in the model are similar to the wave-permeable boundary conditions used by Klemp and Lilly (1978), which are based on concepts proposed by Orlanski (1976). The phase speed of a gravity wave impinging on the boundary is estimated, and used to advect variables out the downstream and upstream boundaries. The procedure for estimating the phase speed is given by Durran (1981).

The radiation upper boundary condition, which specifies that all upward propagating wave energy pass through the boundary without reflection, is approximated by adding an artificial absorbing layer to the top of the model. The boundary condition at the top of the absorbing layer is $w = 0$, which is clearly reflective; however, this reflection is largely eliminated if waves entering the layer from below are sufficiently damped by the absorbing layer, so that they have negligible amplitude when they reach the upper boundary. Reflections, which might be produced by vertical variations in the artificial damping, are minimized by ensuring that the strength of the damping increases gradually as a function of height.

Klemp and Lilly (1978) have suggested that, for hydrostatic waves, a minimum depth of one vertical wavelength is necessary for an effective wave absorber. In mountain wave problems, this may require that as much as one-half of the computational grid be devoted to the absorbing layer. Both viscous and Rayleigh damping have been used in absorbing layers (Clark, 1977; Klemp and Lilly, 1978). Rayleigh damping has been chosen for this model because the second derivatives required for viscous damping have a more complicated finite difference representation due to the coordinate transformation. In the absorbing layer, only the perturbations of a variable from its undisturbed value are damped. The damping terms, which are added to the right-hand sides of the $u$, $w$ and $\theta$ equations, are

$$
R_u = \tau(z)(u - \bar{u}) \\
R_w = \tau(z)w \\
R_\theta = \tau(z)(\theta - \bar{\theta})
$$

The damping coefficient $\tau$ increases gradually with height throughout the absorbing layer with the magnitude chosen so that the dominant wavenumbers are damped most efficiently according to the criteria described by Klemp and Lilly (1978). However, if modes with significantly different horizontal scales are present, they cannot all be equally well damped.

The structure of the absorbing layer is described in detail by Durran (1981).

The model is initialized by slowly increasing the wind speed everywhere, from zero to its value in the mean upstream profile, over a time $t = 4a/\bar{u}$, where $a$ is the mountain half-width and $\bar{u}$ is a representative cross-mountain wind speed. The gradual start-up reduces the transients generated during initialization. However, since the lateral and upper boundary conditions allow the initial transients to pass out of the domain, the model is not highly sensitive to the initialization procedure. Most of the simulations presented in this paper have been obtained by integrating the model forward in time until the solution reaches an essentially steady state. The time-dependent approach guarantees that any steady state obtained is stable, and it allows the simulation of convective regimes in which the atmospheric response is inherently time dependent.

To illustrate the ability of the model to simulate vertically propagating mountain waves, we consider an isothermal atmosphere in which the mean wind is constant with height. The Scorer parameter (see Eq. 1) for such an atmosphere is constant with height:

$$
l^2 = \frac{g^2}{c_p T \bar{u}^2} - \frac{g^2}{4 R^2 T^2}.
$$

Let the mountain contour be specified as

$$z_f(x) = \frac{ha^2}{x^2 + a^2}. 
$$

If the flow is hydrostatic, the expression for the displacement of a streamline from its height far upstream (derived in Alaka, 1960) is

$$
\delta(x, z) = \left(\frac{\bar{p}}{\rho_0}\right)^{1/2} \frac{ha}{x^2 + a^2} \cos z - \frac{x \sin z}{x^2 + a^2}. 
$$

The vertical velocity may be obtained from the relation $w = \bar{u} \delta_x$.
The streamlines and vertical velocity fields calculated from (26) for the case $T = 250$ K, $\bar{u} = 20$ m s$^{-1}$, $a = 10$ km, $h = 1$ m, are displayed in Figs. 1a and 2a. The corresponding fields obtained with the numerical model at $\bar{u}/a = 60$ are shown in Figs. 1b and 2b. In this simulation, the computational domain contains 80 points in the horizontal and 64 levels in the vertical; the absorbing layer occupies the top 32 levels. The grid intervals are $\Delta x = 2$ km, $\Delta z = 250$ m; the large and small time steps are 20 and 4 s, respectively. Only the central portion of the computational domain is shown, in which the grid indices run from 20 to 60 in the horizontal, and 1 to 32 in the vertical. In order to minimize the contribution from nonlinear terms, the model simulation is performed with a 1 m high mountain. The nonhydrostatic contribution to the model solution has also been minimized by choosing $Na/\bar{u} \gg 1$.

As evidenced in Figs. 1 and 2, the numerically generated mountain waves agree well with the linear analytic solution. Good agreement has also been obtained between numerical simulations of finite amplitude waves and analytic solutions to Long's equation. For more test results see Durran (1981).

3. Linear trapped waves

Scorer (1949) observed that the simplest atmospheric structure which will support linear resonant (trapped) waves consists of two horizontal layers, with a constant Scorer parameter [see Eq. (1)] in each layer. Resonant waves are nontrivial solutions of the linear equations of motion with homogeneous boundary conditions. Let the Scorer parameter in the upper and lower layers be $l_2$ and $l_1$, respectively, and suppose that the height of the mountain barrier is small so that linear theory applies. Note that $\bar{u}$ and $\bar{u}$ can be chosen so that $l$ is discontinuous without requiring the discontinuity of $\bar{u}$ and $\bar{u}$ themselves. Scorer showed that resonant waves can exist in the lower layer whenever

$$l_1^2 - l_2^2 \geq \pi^2 / 4H^2.$$  (27)

![Fig. 1. Streamlines for the flow over a 1 m high mountain obtained (a) from the steady-state analytic solution, (b) by numerical simulation at $\bar{u}/a = 60$. In order to better visualize the wave, the amplitudes of the streamline displacements have been multiplied by 600.](image)
Here $H$ is the depth of the lower layer, and $\pi$ is 3.1416. The resonant horizontal wavelength satisfies the condition

$$\tan \lambda_1 H = i \lambda_1 / \lambda_2,$$

where

$$\lambda_1(k) = (l_1^2 - k^2)^{1/2}, \quad \lambda_2(k) = (l_2^2 - k^2)^{1/2},$$

and $k$ is the horizontal wavenumber. If the lower layer is sufficiently deep, there may be multiple solutions to (28), which correspond to different vertical wavenumbers. If more than one resonant mode can exist, the one with the longest horizontal wavelength dominates. In most observed lee waves, only the lowest-order vertical mode is present (e.g., Smith 1976).

For a bell-shaped mountain contour specified by (25), the displacement of a streamline from its height far upstream will satisfy

$$\delta_{12}(x, z) = h a \left( \frac{\ddot{\rho}}{\rho_0} \right)^{-1/2} \frac{\ddot{u}(H)}{u(z)} (L_{12} + \pi R_{12}),$$

where

$$L_1 = \text{Re} \int_0^\infty \exp(ikx - ka)$$

$$\times \frac{\cos \lambda_1 z + (i \lambda_2 / \lambda_1) \sin \lambda_1 z}{\cos \lambda_1 H - i (\lambda_2 / \lambda_1) \sin \lambda_1 H} dk,$$

and

$$L_2 = \text{Re} \int_0^\infty \exp(ikx - ka)$$

$$\times \frac{\cos \lambda_2 z + i \sin \lambda_2 z}{\cos \lambda_1 H - i (\lambda_2 / \lambda_1) \sin \lambda_1 H} dk,$$

$$R_1 = -\left[ \frac{\lambda_1}{H + (1/i\lambda_2)} \right] e^{-ak} \left. \frac{k}{k_k} \right|_{k = k_k},$$

and

$$R_2 = -\left[ \frac{\lambda_1}{H + (1/i\lambda_2)} \right] \frac{\exp(-ka + i\lambda_2 z)}{k} \sin \lambda_1 H \sin kx \left. \frac{1}{k_k} \right|_{k = k_k}.$$

The subscripts 1 and 2 denote the lower and upper layers, respectively, $k_r$ is the resonant wavenumber which satisfies (28), $\ddot{\rho}$ is the mean density and $\rho_0$ a reference density. The imposed boundary conditions require that the disturbances vanish as $x \to -\infty$, and that energy transport be directed upward as $z \to \infty$. The derivation of these results can be found in Scorer (1949) or Alaka (1960).

The linear, steady, trapped wave solution can also be generated by the numerical model if the mountain height is sufficiently small. The potential temperature, windspeed and Scorer parameter profiles for a two-layer atmosphere in which $l^2$ decreases abruptly with height are shown in Fig. 3. Fig. 4 shows the streamlines and vertical velocity field produced when this flow encounters a small amplitude mountain of the shape given by (25) with $h = 1 \text{ m}$ and $a = 2.5 \text{ km}$. In this simulation, the domain contains 100 points in the horizontal and 48 levels in the vertical; the absorbing layer occupies the top 24 levels. The grid intervals are $\Delta x = 800 \text{ m}$, $\Delta z = 333 \text{ m}$; the large and small time steps are 12.5 and 21/12 s, respectively. The model is run until the solution reaches a nearly steady state. (This requires from 3 to 5 h of model time.) Fig. 4 includes only the central part of the domain which runs from grid points 10 to 70 in the horizontal
\( \bar{u} \) and \( \bar{\theta} \) which produce a single sharp discontinuity in Scorer parameter.

The streamline displacements and vertical velocity fields calculated numerically from (29), assuming a sharp interface at 3 km, appear in Fig. 5. The overall agreement between the analytic (Fig. 5) and numerical (Fig. 4) solutions is good, although there are small differences in the wavelength and amplitude of the trapped waves. As shown in Table 1, these differences are smaller than the changes in the analytic solution which would be produced by raising or lowering the interface height one vertical grid interval. Thus, it appears that the differences between the numerical and analytic solutions can be largely attributed to the difference in the Scorer parameter structures. Nevertheless, even with an approximate Scorer parameter

and 1 to 24 in the vertical. The mountain peak is located at grid point 25.5. Trapped lee waves are clearly visible, along with a weak, long-wavelength, vertically propagating wave.

If the Scorer parameter used in the numerical calculations had a perfect two-layer vertical structure, the flow field in Fig. 4 should be described by (29). However, as shown in Fig. 3, the two-layer structure is not perfect. The \( I^2 \) profile used in the calculations drops from a nearly constant value of \( 1.0 \times 10^{-6} \text{ m}^{-2} \) to a nearly constant value of \( 1.5 \times 10^{-7} \text{ m}^{-2} \) in a transition layer two grid points deep. The finite depth of this layer is due, in part, to the difficulty of resolving discontinuities on a numerical grid, and, in part, to the difficulty of specifying realistic vertical profiles of

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**Fig. 4.** (a) Streamlines, and (b) vertical velocities (\( \times 10^{-4} \text{ m s}^{-1} \)) produced by a 1 m high mountain when RH = 0% upstream. The streamline displacements are multiplied by 300 for display.

**Fig. 5.** As in Fig. 4, except that the flow is calculated from (29).
Table 1. The effect of variations in the lowest layer depth on resonant wavelength and vertical velocity.

<table>
<thead>
<tr>
<th>Lower layer depth $H$ (m)</th>
<th>Lee wavelength (km)</th>
<th>Maximum vertical velocity ($\times 10^{-3}$ m s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2667</td>
<td>10.3</td>
<td>7.8</td>
</tr>
<tr>
<td>2833</td>
<td>9.7</td>
<td>6.9</td>
</tr>
<tr>
<td>3000</td>
<td>9.3</td>
<td>6.1</td>
</tr>
<tr>
<td>3167</td>
<td>9.0</td>
<td>5.5</td>
</tr>
<tr>
<td>3333</td>
<td>8.7</td>
<td>4.9</td>
</tr>
</tbody>
</table>

structure, the model seems capable of simulating the results obtained via two-layer analytic theory with reasonable accuracy.

In the case just discussed, the dynamics of dry lee waves are determined by the vertical profiles of $\bar{u}$ and $\theta$, and the mountain contour. The dynamics of moist waves also depend on the sensible temperature and the vertical profile of moisture. Constructing idealized atmospheric profiles for moist waves is, thus, further complicated since the vertical profile of sensible temperature must be physically reasonable. Unrealistically high temperatures will allow the air to hold so much water vapor that latent heat effects will be exaggerated. Similarly, if the air is unrealistically cold, latent heat effects will be minimized. The physical cases described in this chapter are approximately representative of springtime air masses flowing over the Continental Divide in the Colorado Rockies.

Consider again the atmosphere shown in Fig. 3, and let the upstream flow contain moisture in the lowest layer (between 0 and 3 km) so that it always remains saturated. This is accomplished by adding a uniform cloud containing 0.2 g kg$^{-1}$ of liquid water to the lowest layer. Fig. 6 shows the streamlines and vertical velocity field for this case, which, except for the moisture, is identical to the one shown in Fig. 4. Since the waves generated by the 1 m high mountain have small amplitude, the cloud never dissipates in the wave troughs, and latent heat effects are symmetric in the troughs and crests. The practical effect of the latent heat released is to lower the stability (according to Eq. 14) in the lowest layer, which alters the Scorer parameter. The saturated Scorer parameter structure obtained by replacing $N^2$ by $N_m^2$ in (1) is also shown in Fig. 3b. Note that the two-layer structure is much weaker in the saturated case; in fact, the decrease in $l^2$ is too small to satisfy (27), so resonant trapped waves should not occur. This is indeed the case; as shown in Fig. 6, the waves are weak and vertically propagating. The addition of moisture has untrapped the waves.

The dry and everywhere saturated flows are limiting cases; consider a situation where the lower layer is saturated upstream but cloud-free. The streamlines and vertical velocity field for this flow are shown in Fig. 7. Condensation and evaporation now occur only in the wave crests, decreasing the local stability and increasing the local wavelength. The flow in the troughs, which remain unsaturated, is similar to the dry case. The waves develop broad flat crests and narrow troughs, producing the distinctive asymmetry in the vertical velocity field shown in Fig. 7b. The waves remain trapped although their overall horizontal wavelength is much longer than in the dry case. The maximum vertical velocities are slightly weaker than those in the dry waves, but much stronger than the maximums in the everywhere cloudy case.

When the lowest layer is cloudy everywhere, the effects of moisture can be approximated by replacing the moist layer with a dry layer of suitably reduced stability. No similar simple a priori approximation

![Figure 6](https://via.placeholder.com/150)

**Fig. 6.** As in Fig. 2, except that RH = 100% and there is 0.2 g kg$^{-1}$ of cloud in the lowest layer upstream. Cloudy regions are shaded.
can be made in the partially cloudy case. The need to determine the cloud boundaries makes analytic analyses of even the small amplitude problem very difficult (e.g., Barlon et al., 1979).

To illustrate the tendency for the local wavelength to be modified by the local stability, we consider the simpler case of channel flow in which a single layer is bounded above by a rigid lid. Assume that the atmosphere in the channel is just at saturation so that any regions of net upward displacement are cloudy, while regions of net downward displacement are cloud-free. Let the Scorer parameters in the cloudy and dry regions be constant with values of $l_m$ and $l_d$, respectively. If $H_c$ is the channel depth, and there is no mean wind shear, the equation for the streamline displacement of linear resonant waves is

$$\delta_{zz} + \delta_{xx} + l^2 \delta = 0, \quad l^2 = \begin{cases} l_m^2 & \delta > 0 \\ l_d^2 & \delta < 0, \end{cases} \tag{30}$$

subject to the boundary conditions $\delta(x, 0) = \delta(x, H_c) = 0$. Requiring continuity of the normal velocity and pressure at the cloud boundaries, the solution for a wave of unit amplitude becomes

$$\begin{align*}
\delta_d &= \sin \lambda_c z \sin k_d x \\
\delta_m &= \frac{k_d}{k_m} \sin \lambda_c z \sin k_m x,
\end{align*} \tag{31}$$

where

$$k_m^2 = l_m^2 - \lambda_c^2, \quad k_d^2 = l_d^2 - \lambda_c^2, \quad \lambda_c = \pi / H_c. \tag{32}$$

Here, the $x$-axis is arbitrarily positioned such that $\delta(0, z) = 0$. Note that the ratio of the wavelengths in the saturated and unsaturated regions increases as the ratio $l_m/l_d$ decreases. The local wavelength is thus modified by changes in the local stability. Although the solution defined by (31) is qualitatively similar to the saturated trapped waves in Fig. 7 (for a suitably chosen $H_c$), the actual solution for a two-layer atmosphere is greatly complicated by the matching requirements at the interface. Furthermore, it is impossible for both $N$ and $N_m$ to be constant with height in a realistic sounding. The assumption made in the derivation of (31), that $N_m$ is constant, is not particularly suited to the previous case in which $N_m$ varies between 0.0056 and 0.0084 s$^{-1}$. Consequently, the solutions differ in a number of details. Particularly in the channel flow the amplitude in the saturated region increases over that in the dry region, in proportion to the increase in wavelength, whereas the ratio of amplitudes, in the saturated and unsaturated regions, varies with height in Fig. 7.

Although small amplitude flows can provide useful insight, they are not the best situations in which to examine moisture effects. Their major limitation is that they respond identically to flows in which the relative humidity (RH) is 0 and 99% upstream. In the next section, we will remove this constraint by examining the effects of moisture on finite amplitude flows.

### 4. Finite amplitude trapped waves

#### a. Resonant wave distortion and untrapping due to the addition of moisture

Consider again the atmosphere shown in Fig. 3. The height of the mountain is increased to 300 m, and the simulations described in the previous section are repeated. All the computational parameters except the mountain height are unchanged. The streamlines produced by different amounts of upstream hu-
Fig. 8. Streamlines produced by a 300 m high mountain in the flow shown in Fig. 1, for (a) RH = 0%, (b) RH = 90%, (c) RH = 100%, (d) RH = 100% with 0.2 g kg⁻¹ of cloud, in the lowest layer upstream. Cloudy regions are shaded.

Mididity are shown in Fig. 8. The basic behavior is the same as in the linear case; the addition of moisture to the flow widens the wave crests (Fig. 8c) and, when enough water is present, untraps the lee waves (Fig. 8d). For RH = 90%, condensation still occurs in the wave crests and significantly alters the wave structure (Fig. 8b). This result is in marked contrast to its linear counterpart, which is unaffected by moisture for RH = 90%.

The nonlinear trapped wave solutions for a 300 m mountain (Figs. 8a, c) are distinctly stronger than the linear solutions for the same mountain (Figs. 4a and 7a). This agrees with the observation of Smith (1976), who found that linear theory consistently underpredicted the amplitude of lee waves produced by the Blue Ridge, and the numerical results of Peltier and Clark (1979). The streamlines for the nonlinear waves also show more variation in the amplitude of successive waves than their linear counterparts. With further increases in the mountain height, the flow becomes more nonlinear, and the difference between successive waves increases.
b. Resonant wave breakdown due to the addition of moisture to a conditionally unstable layer

In the previous case, the lowest layer was absolutely stable, so that latent heat release reduced, but did not destroy, the local stability. Dry lee waves can also exist in a region which is conditionally unstable. The potential temperature, windspeed and $l^2$ profiles for such an atmosphere are shown in Fig. 9. Although the $\bar{u}$ and $\bar{\theta}$ profiles are different, they have been specified such that the dry Scorer parameter structure is nearly identical to that seen earlier in Fig. 3. As a result, the streamlines for the dry lee wave solution (Fig. 10a) are very similar to those for the previous run (Fig. 8a). Small differences may be expected since the mean density profiles are not identical. However, when the flow is initialized with $RH = 90\%$ in the lowest layer upstream, the two cases are very different.

As shown in Figs. 10b–d, steady waves are not produced in the conditionally unstable case. Condensation occurs in the crest of the first lee wave, but the cloudy regions are unstable and thus act as buoyant plumes. At first, their ascent is limited by the absence of moisture in the upper layer, but as more water mixes upward they rise higher and destroy the lee wave structure. This is most visible in Fig. 10d where the clouds are aligned with the updrafts, rather than the wave crests. The two-dimensional model probably does not accurately simulate the details of the moist wave breakdown, but we believe the basic influence of moisture on the model solution is properly represented.

c. Resonant wave detuning due to the removal of moisture

Suppose that the atmospheric structure is favorable for trapped resonant waves when the lowest layer is saturated and cloudy upstream. If the cloud is sufficiently dense that the lee wave troughs remain saturated, (14) can be used to compute an effective moist Scorer parameter. Fig. 11 shows the potential temperature, windspeed and $l^2$ profiles for such a case. Note that the moist Scorer parameter structure has been constructed to be nearly identical to the dry $l^2$ structure in Figs. 3 and 9. Consequently, the linear waves produced by a 1 m high mountain in the cloudy atmosphere should be almost identical to the dry linear waves discussed earlier. A comparison of Figs. 12a and 4a shows that the two solutions agree quite well.

In the finite amplitude case (Fig. 12b) the waves are similar, but distinctly stronger than their dry counterparts (Figs. 8a and 10a), and there is greater variation in amplitude between successive waves. This is not surprising since the nonlinear terms in the moist equations differ from those in the dry system. In fact, the nonlinear behaviors differ primarily because the saturated adiabatic lapse rate $\Gamma_m$ is not independent of height. As a parcel rises, the latent heat released per unit mass decreases with each successive meter of its vertical displacement. Thus, in a saturated gravity wave nonlinear effects act to increase the buoyancy restoring force in the wave crests beyond that predicted by linear theory, while decreasing it in the troughs. The importance of this asymmetry depends on the amplitude of the wave. As shown by the difference between Figs. 8a and 12b, the difference in the behavior of linearly equivalent wet and dry systems becomes significant in moderately strong waves.
When the lowest upstream layer is saturated but cloud-free, condensation does not occur in the wave troughs, and the overall wavelength is decreased. Trapped waves still occur as shown in Fig. 12c, but since the mountain does not force this shorter wavelength as efficiently, the wave amplitude is reduced. When the lowest layer is completely dry, the wave amplitude and resonant wavelength are further reduced as shown in Fig. 12d. Thus, when strong waves exist in a cloudy atmosphere, the removal of moisture may reduce the forcing at the resonant wavenumber, reducing the amplitude of the lee waves.

5. Moisture in the upper layer

What effect does the introduction of moisture above a wave trapping interface have on the flow? If the upper layer is conditionally unstable and rather
moist, lee waves could conceivably trigger convective storms. Booker (1963) has suggested that mountain lee waves might sometimes encourage the development of thunderstorms in the Allegheny Mountains. Although the possibilities are interesting, the study of lee wave induced thunderstorms is beyond the scope of this investigation. If the upper level is absolutely stable, the introduction of moisture reduces the stability and Scorer parameter in that layer, and might trap lee waves which could propagate vertically if the air were dry. However, the upper troposphere, being rather cold, does not contain much water vapor, so latent heat effects are reduced and the impact of moisture on the Scorer parameter decreases accordingly. Therefore, the only waves which could be trapped by changes in moisture in the upper layer would have a horizontal wavenumber very close to the critical value which satisfies (27) by equality. This situation requires a very special atmospheric structure and probably has little physical significance.

A more common situation in which moisture at mid-tropospheric levels might affect trapped lee waves occurs when the lower layer, in which \( I^2 \) is large, is rather deep. If moisture is introduced to the top of the lower layer, it can reduce the stability and Scorer parameter in that region to a value similar to that in the upper layer. The practical effect is to move the wave trapping interface down, increasing the lee wavelength and amplitude. The \( I^2 \) profiles for such a case are shown in Fig. 13a. The dry linear lee wave flow is shown in Fig. 14a; weak trapped waves are visible. When the flow is saturated and contains 0.2 g kg\(^{-1}\) of cloud between the heights of 2.3 and 4.0 km far upstream, the \( I^2 \) profile is altered as shown by the dashed line in Fig. 13a. The effective height of the wave trapping interface is thus lowered and, as shown in Fig. 14b, a stronger wave develops.

One might conclude from Table 1 that decreasing the height of the wave-trapping interface always produces stronger lee waves. Fig. 15 shows the effects of changes in the interface height on the wavelength and amplitude of the two lowest-order trapped wave modes. As predicted by (27), a minimum lower layer depth of 1.6 km is required to support mode-one waves. A minimum depth of 5.1 km is necessary to support mode-two waves. The amplitude of each mode peaks sharply at a depth just slightly greater than the cutoff value. When the atmosphere is capable of supporting both modes, the higher-order mode dominates. Further discussion of the effects of changes in the dry atmospheric structure on resonant trapped waves may be found in Corby and Wallington (1956). If moisture is added to the top of a deep wave trapping layer in which the dry lee waves have a mode-two structure, the interface depth can be lowered, thereby eliminating the higher-order mode and producing a weaker, shorter wavelength response. Fig. 13b shows the \( I^2 \) structure in such a case, and Fig. 16a shows the dry linear lee waves produced by this atmosphere. Note that the second mode dominates. When the upstream flow is saturated and contains 0.2 g kg\(^{-1}\) of cloud between the heights of 3 and 6 km, as shown in Fig. 16b, the second mode disappears, and much weaker mode-one waves form in the lowest 3 km. The atmosphere appears to have a three-layer structure, with very weak short waves in the lowest layer, and stronger, longer waves in the middle moist layer. In the top layer, the waves decay with height, but only 2 km of this region are shown in the figure, so the decay is not obvious to the eye.

A deep layer of high \( I^2 \), like the one just discussed (Fig. 13b), would be encountered only rarely in the real atmosphere. The shallower layer (Fig. 13a) is
probably much more common, so, in most cases, moisture at mid-tropospheric levels might be expected to increase the amplitude of trapped lee waves.

6. Conclusions

We have seen that the presence of moisture affects trapped lee waves in a variety of ways. When moisture is introduced into the lowest layer of a two-layer tropospheric structure favorable for the development of trapped lee waves, three different behaviors were documented. If the effective moist stability in the wave environment is weak, but positive, the waves are distorted and untrapped as the upstream humidity increases (Fig. 8). If the atmosphere is convectively unstable, any clouds which form act like buoyant plumes and destroy the lee waves (Fig. 10). If there is strong moist stability in the lowest layer, changes

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Fig. 12. Streamlines produced by the flow shown in Fig. 8: (a) by a 1 m high mountain when RH = 100% and there is 0.2 g kg\(^{-1}\) of cloud in the lowest layer upstream; and by a 300 m high mountain when (b) RH = 100% and there is 0.2 g kg\(^{-1}\) of cloud, (c) RH = 100%, (d) RH = 0%, in the lowest layer upstream. Cloudy regions are shaded. In (a) only, the streamline displacements are multiplied by 300, producing a good approximation to the linear solution for a 300 m high mountain.
pospheric moisture can exert a significant influence on trapped lee waves. In the first, weak dry waves can be amplified by the addition of mid-level moisture, which effectively reduces the height of the wave trapping interface (Fig. 14). In the second case, a deep lower layer may exhibit a mode-two trapped wave of significant amplitude in the absence of moisture. If mid-level moisture is introduced, the deep two-layer structure is destroyed, and the mode-two wave disappears with only weaker mode-one waves remaining (Fig. 16).

This investigation of trapped waves has concen-

FIG. 13. Scorer parameter profiles for atmosphere favorable for the development of dry trapped lee waves when the lowest layer is (a) 4 km and (b) 6 km deep.

in the upstream humidity change the tuning properties of the trapped waves (Fig. 12). In the last instance, moisture can amplify or damp the wave, depending on the wavenumber spectrum of the orographic forcing.

In the real atmosphere, the potential influence of latent heating decreases, along with the available water vapor, as the temperature decreases with height. As a consequence, variations in the lower tropospheric humidity probably have a greater practical impact on lee wave dynamics than changes in the moisture content at higher levels. Nevertheless, there are at least two situations where changes in mid-tro-

FIG. 14. Streamlines produced by a 1 m high mountain by the atmosphere shown in Fig. 10a when the upstream flow is (a) dry and (b) saturated between the heights of 2.3 and 4 km. Cloudy regions are shaded. The streamline displacements are multiplied by 300 for display.
trated on atmospheres with an essentially two-layer Scorer parameter structure. Although the wind speeds and stabilities in these atmospheres are realistic, the exact Scorer parameter structure in any real sounding would certainly be more complicated. Nevertheless, when trapped lee waves are observed in the atmosphere, the Scorer parameter often has a rough two- or three-layer structure. The third layer is sometimes produced by a well mixed layer near the ground. Corby and Wallington (1956) have examined the effects of a low-level adiabatic layer on dry linear trapped waves and found that the general conclusions reached with a two-layer model are also applicable to this three-layer situation. Thus, one might expect

![Graph showing interface depth vs. resonant wavelength for two modes.](image)

**Fig. 15.** The influence of interface height on trapped lee wave wavelength (a), and amplitude (b), for the two lowest-order modes. \( l^2 \) is \( 1.5 \times 10^{-7} \) m\(^2\) and \( 1.0 \times 10^{-6} \) m\(^2\) in the upper and lower layers, respectively. The amplitude calculations are for a Witch of Agnesi mountain with half-width 2.5 km. (After Corby and Wallington, 1956.)

![Streamlines produced by a 1 m high mountain in an atmosphere with and without a low-level adiabatic layer.](image)

**Fig. 16.** Streamlines produced by a 1 m high mountain by the atmosphere shown in Fig. 10b when the upstream flow is (a) dry, and (b) saturated between the heights of 3 and 6 km. Cloudy regions are shaded. The streamline displacements are multiplied by 300 for display.

that in many instances the influence of moisture on trapped waves in the real atmosphere should be qualitatively similar to the behaviors in the two-level cases described here.

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