A Note on Rossby Wave Instability at Finite Amplitude

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1. Introduction

Loesch (1978, hereafter referred to as L) considered the problem of Rossby-wave instability at finite amplitude and found that the perturbation initially grew exponentially, then more slowly to a level determined by the nonlinearity, from which it decayed and never grew again. This disagrees with the result obtained by Deininger (1982, hereafter referred to as D) in which the behavior of the perturbation was found to be oscillatory. This discrepancy is caused by a combination of three properties of the analysis in L: the method used to handle phase changes of the Rossby wave, the special symmetry of the trun-

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cation used, and an improper closure of the perturbation field. The problem of handling the phase changes was discussed in D. The correct closure technique for the special truncation used in L will be discussed here as a special case of the results obtained in D. The notation used will be that of D.

2. Analysis

The appropriate special case of D is that for which

$$k_0 = l = 0, \tag{2.1}$$

in the three-mode truncation discussed in the appendix of D. When (2.1) is true, it is straightforward to determine the $O(|\Delta|)$ frequency and amplitude corrections to the Rossby wave as in D. They are

$$W_2 = \frac{2k^3(k^2 - l_0^2)}{\beta}|P_0|^2, \tag{2.2a}$$

$$\frac{dR^{(2)}}{dT} = -\frac{2}{R_c}\frac{d}{dT}|P_0|^2. \tag{2.2b}$$

Eqs. (2.2a,b) replace (4.42b,c) of L, respectively. Note the k, l_0, P_0, R_c, and $R^{(2)}$ of this paper correspond to N, l, X_0, A_c, and $2A^{(2)}$ of L. The difference between the two sets of equations is due to the different ways the phase change of the Rossby wave was handled.

It is not so straightforward, however, to obtain an equation for P_0. Let us examine the set of equations analogous to (4.37) of L from which the behavior of X_0 was determined in that work. They are

$$\frac{k^2R_cl_0}{\beta}Q_1^{(3)} - Q_0^{(3)} = \frac{2il_0}{k(k^2 - l_0^2)R_c}\left[K_1^2\frac{\partial P_1^{(2)}}{\partial T} + iK_1^2C_1W_2P_0 - ib(k^2 - l_0^2)\left(\frac{\Delta}{|\Delta|} + R^{(2)}\right)P_0 - 2ib(k^2 + 3l_0^2)f_1P_0|P_0|^2\right], \tag{2.3a}$$

$$Q_1^{(3)} = -Q_{-1}^{(3)}, \tag{2.3b}$$

$$-Q_0^{(3)} - \frac{k^2R_cl_0}{\beta}Q_{-1}^{(3)} = \frac{2il_0}{k(k^2 - l_0^2)R_c}\left[K_1^2\frac{\partial P_1^{(2)}}{\partial T} + iK_1^2C_1W_2P_0 - ib(k^2 - l_0^2)\left(\frac{\Delta}{|\Delta|} + R^{(2)}\right)P_0 - 2ib(k^2 + 3l_0^2)f_1P_0|P_0|^2\right]. \tag{2.3c}$$

In D, inhomogeneous terms analogous to those in (2.3) are resonant and their removal results in the determination of P_0. However, here, due to the symmetry of the truncation, the inhomogeneous terms in (2.3) are nonresonant (i.e., they are orthogonal to the adjoint operator), as are the corresponding terms in (4.37) of L. Therefore, these inhomogeneous terms give rise to the nonsecular particular solution

$$\psi_p^{(3)} = \sum_{n=-1}^{1} P_n^{(3)}e^{i(nkx+l_0y)} + *, \tag{2.4a}$$

where

$$P_0^{(3)} = 0, \tag{2.4b}$$

$$P_1^{(3)} = -P_{-1}^{(3)} = \frac{ik}{\beta l_0^2}\left[K_1^2\frac{\partial P_1^{(2)}}{\partial T} + iK_1^2C_1W_2P_0 - ib(k^2 - l_0^2)\left(\frac{\Delta}{|\Delta|} + R^{(2)}\right)\times P_0 - 2ib(k^2 + 3l_0^2)f_1P_0|P_0|^2\right]. \tag{2.4c}$$

This $O(|\Delta|^{3/2})$ solution adds new, nonsecular time behavior to the full solution and therefore should not be set equal to zero as was done in L by demanding that $\psi_h^{(1)}$ and $\psi_h^{(3)}$ (in L's notation) have identical structure (p. 936 of L). In L this led to setting the inhomogeneous terms analogous to (2.3) above to zero incorrectly and hence the wrong evolution equation for P_0 was obtained.

In this analysis the correct behavior of P_0, still undetermined to this order, can be determined at the next order. To go to $O(|\Delta|^2)$ formally requires the introduction of a longer time scale $T_1 = |\Delta|t$ and $O(|\Delta|^{3/2})$ amplitude and frequency corrections. However, this is unnecessary for the closure of the problem on the long time scale $T = |\Delta|^{1/2}t$, for the same reason we failed to achieve closure on the time scale T to $O(|\Delta|^{3/2})$. Again, this is the result of the special symmetry of the perturbation and basic state wave fields being considered. At the next order the only relevant inhomogeneous terms for obtaining a closed set of equations for the variables P_0, $R^{(2)}$, and W_2 on the time scale T are those resonant in the perturbation structure which are those proportional to

$$e^{i(kx+l_0y)},\ e^{il_0y},\ \text{and}\ e^{i(-kx+l_0y)}.$$

Terms resonant in the basic state structure e^{ikx} would only produce equations for higher-order quantities or on a longer time scale. Keeping these points in mind simplifies the $O(|\Delta|^2)$ problem. In its simplified form it can be written

$$\frac{\partial}{\partial t_*}\nabla^2\psi^{(4)} + \beta\frac{\partial}{\partial x}\psi^{(4)} + R_c\cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)\times(\nabla^2 + K^2)\psi^{(4)} = \left(K_1^2\frac{\partial P_1^{(3)}}{\partial T} + iK_1^2W_2P_1^{(2)}\right)e^{i(kx+l_0y)} + ikl_0^3\left[\left(\frac{\Delta}{|\Delta|} + R^{(2)}\right)P_1^{(2)} - 6f_1P_1^{(2)*}P_0^2\right]e^{il_0y} - \left(K_1^2\frac{\partial P_1^{(3)}}{\partial T} + iK_1^2W_2P_1^{(2)}\right)e^{i(-kx+l_0y)} + *. \tag{2.5}$$

Removing the resonance from (2.5), using (2.4) and

$$P_1^{(2)} = \frac{i}{kl_0R_c}\frac{\partial P_0}{\partial T},$$

yields the equation governing the behavior of P_0, i.e.,

$$\frac{d^3P_0}{dT^3} - \frac{\Delta}{|\Delta|}k^2l_0^2\frac{(k^2-l_0^2)}{k^2+l_0^2}R_c\frac{dP_0}{dT} - k^2\frac{l_0^2(k^2-l_0^2)}{2(k^2+l_0^2)}$$
$$\times R_c\left[\frac{d}{dT}(P_0R^{(2)}) + R^{(2)}\frac{dP_0}{dT}\right] + \frac{\beta l_0^2}{k(k^2+l_0^2)}$$
$$\times\left[W_2\frac{dP_0}{dT} + \frac{d}{dT}(W_2P_0)\right] - \frac{k^2l_0^2R_cf_1}{k^2+l_0^2}\left[(k^2+3l_0^2)\right.$$
$$\left.\times\frac{d}{dT}(P_0|P_0|^2) - 3(k^2-l_0^2)P_0^2\frac{dP_0^*}{dT}\right] = 0. \quad (2.6)$$

The desired closed set of equations consists of (2.2a,b) and (2.6). Using (2.2a,b) in (2.6), results in an equation for P_0 alone, which is

$$\frac{d^3P_0}{dT^3} - \frac{\Delta}{|\Delta|}k^2l_0^2\frac{(k^2-l_0^2)}{k^2+l_0^2}R_c\frac{dP_0}{dT} - 2k^2l_0^2$$
$$\times\frac{(k^2-l_0^2)}{k^2+l_0^2}|P_0(0)|^2\frac{dP_0}{dT} + \frac{k^2l_0^2}{2(k^2+l_0^2)^2}$$
$$\times(17k^4 - 18l_0^4 - 3k^2l_0^2)|P_0|^2\frac{dP_0}{dT} + k^2l_0^2$$
$$\times\frac{(7k^4 - 6l_0^4 - 3k^2l_0^2)}{2(k^2+l_0^2)^2}P_0^2\frac{dP_0^*}{dT} = 0. \quad (2.7)$$

The behavior of P_0 implied by (2.7) is an oscillatory one provided we are not in the region of wavenumber space for which the nonlinearity is destabilizing. This behavior is most easily demonstrated if $P_0 = P$ is assumed to be real. In this case (2.7) can be integrated once to obtain

$$\frac{d^2P}{dT^2} - \frac{\Delta}{|\Delta|}k^2l_0^2\frac{(k^2-l_0^2)}{k^2+l_0^2}R_cP - 2k^2l_0^2\frac{(k^2-l_0^2)}{k^2+l_0^2}$$
$$\times P^2(0)P + \frac{k^2l_0^2}{(k^2+l_0^2)^2}(4k^4 - k^2l_0^2 - 4l_0^2)P^3$$
$$= \frac{k^2l_0^2}{(k^2+l_0^2)^2}(2k^4 + k^2l_0^2 - 2l_0^4)P^3(0). \quad (2.8)$$

In deriving (2.8) we used the initial condition

$$\frac{d^2P(0)}{dT^2} - \frac{\Delta}{|\Delta|}k^2l_0^2\frac{(k^2-l_0^2)}{k^2+l_0^2}R_cP(0) = 0,$$

which corresponds to initially exponential growth as obtained from linear theory. Equation (2.8) is the same form as (5.1) in D which is known to possess oscillatory behavior provided the nonlinearity is stabilizing. This is the case when the coefficient of P^3 in (2.8) is greater than zero, i.e., for

$$4k^4 - k^2l_0^2 - 4l_0^4 > 0.$$

Here we have derived the behavior of the perturbation when it is truncated to three terms and $k_0 = l = 0$. One might ask: Can this symmetry occur when more than three terms are retained? The answer is yes if $\lambda_r = 0$ on the neutral curve when $k_0 = l = 0$. In this more general case an equation of the form (2.7) will still describe the evolution.

3. Conclusion

The behavior of the perturbation and basic state Rossby wave for the special symmetry of those fields considered by Loesch (1978) is that of an oscillatory exchange between the Rossby wave and perturbation fields. The erroneous result obtained by Loesch (1978) was due to an incorrect closure of the problem.

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REFERENCES

Deininger, R. C., 1982: Free Rossby wave instability at finite amplitude. *J. Atmos. Sci.*, **39**, 563–572.

Loesch, A. Z., 1978: Finite-amplitude stability of Rossby wave flow. *J. Atmos. Sci.*, **35**, 929–939.