Zonally Symmetric Hough Modes and Meridional Circulations in the Middle Atmosphere

R. ALAN PLUMB

CSIRO Division of Atmospheric Physics, Aspendale 3195, Australia

(Manuscript received 2 July 1981; in final form 11 January 1982)

ABSTRACT

The structure of the meridional circulation driven by steady, zonally symmetric thermal or mechanical forcing is analyzed via the techniques of tidal theory. The relevant Hough functions are derived and, together with consideration of their associated vertical structure, these are used as the basis for discussion of forced meridional circulations in the middle atmosphere. Thus, the global extent of the complementary cooling associated with stratospheric warmings in the high-latitude winter hemisphere is explained as a manifestation of the global nature of the associated (residual) mean circulation. However, the response to equatorial forcing is more localized and it appears that the high-latitude manifestations of the quasi-biennial oscillation cannot be accounted for as a result of coupling, via a mean circulation, with the tropical stratosphere.

1. Introduction

When a rotating stratified fluid is subjected to localized (thermal or mechanical) zonally uniform forcing the effects of such forcing may be manifested in regions remote from the forcing. In general, the forcing would by itself destroy thermal wind balance; this is restored on a time scale of the order of the inverse of the Coriolis parameter—and therefore "instantaneously" on a quasi-geostrophic viewpoint—by a meridional circulation [Eliassen (1951); see also Chapter 10.4 of Holton (1979)]. While maintaining this balance, the circulation distributes the effects of the forcing over a region which is more extensive than that in which the forcing is applied. Thus the process is, in principle, one capable of establishing inter-latitudinal communication in a zonal-mean sense.

Mathematical solution of the problem, the zonally-uniform "omega equation", is via a straightforward elliptic equation which can be solved by well-established methods; indeed, such solution is routine in many numerical models. However, it appears that apart from the work of Eliassen (1951) little attempt has been made to answer simple conceptual questions about the structure of the circulation. Eliassen discussed the qualitative aspects of this structure, including the differences between mechanically- and thermally-forced circulations. However, other questions remain to be clarified, in particular the scale (in height and latitude) of the response to any given forcing. Such basic insight is the aim of the work described here. To this end, the problem is solved here by the methods of tidal theory, whereby the solution is obtained as an expansion of Hough functions. These functions are the eigenvectors of Laplace's tidal equation for a given frequency and zonal wavenumber, both of which are zero for the present situation. The form of these functions and their eigenvalues or "equivalent depths", which determine the corresponding vertical structure, give substantial insight into the structure of forced meridional circulations.

This approach is used to investigate the response to localized forcing in the equatorial, mid-latitude and high-latitude stratosphere. The first example was inspired by the analyses of Tucker (1979) and Holton and Tan (1980) which revealed a substantial quasi-biennial oscillation at middle and high stratospheric latitudes. Since it seems well-established that the driving mechanism for this phenomenon is concentrated near the equator (Holton and Lindzen, 1972), it seems likely that some secondary process must be responsible for communicating the oscillation to higher latitudes. The mean circulation driven by high-latitude forcing is also of interest, partly because of the observation that high-latitude stratospheric warmings are associated with a complementary cooling at low latitudes, even extending into the opposite hemisphere. It is usually assumed that this is a manifestation of the (residual) mean meridional circulation (see Andrews and McIntyre, 1976) which is an integral part of the warming mechanism (e.g., Dunkerton *et al.*, 1981). These calculations show that high-latitude forcing is indeed capable of generating a large-scale circulation of the form implied by the requirements of the observed heat budget. Equatorial forcing, however, is inefficient at driving global-scale circulations and it therefore seems that

0022-4928/82/050983-09$06.25

some other mechanism must be responsible for the observed high-latitude quasi-biennial oscillation.

2. Mathematical development

Consider quasi-geostrophic motion driven by zonally uniform thermal and/or mechanical forcing in a rotating spherical atmosphere. The basic equations in spherical-log pressure co-ordinates are (e.g., Holton, 1979)

$$\frac{\partial u}{\partial t} - 2\Omega\mu v = F, \tag{2.1}$$

$$2\Omega\mu u = -\frac{1}{a}(1-\mu^2)^{1/2}\frac{\partial\phi}{\partial\mu}, \tag{2.2}$$

$$\frac{\partial\phi}{\partial z} = -\frac{gT}{T_0}, \tag{2.3}$$

$$\frac{\partial T}{\partial t} + wS = Q_*, \tag{2.4}$$

$$\frac{1}{a}\frac{\partial}{\partial\mu}[v\rho(1-\mu^2)^{1/2}] + \frac{\partial(w\rho)}{\partial z} = 0, \tag{2.5}$$

where μ = sin(latitude), (u, v, w) are eastward, northward and upward velocities, ϕ is geopotential, T temperature, Ω the Earth's rotation rate, a the Earth's radius, $S = (\partial T_0/\partial z) + (\kappa T_0/H)$ and T_0 is a constant reference temperature. F and Q_* are respectively the mechanical acceleration (e.g. due to external forcing or to friction) and the diabatic heating rate divided by c_p, where c_p is the specific heat at constant pressure. These quantities may also include eddy forcing terms. The quasi-geostrophic assumption implies the neglect of advection of zonal and latitudinal momentum in comparison with the Coriolis terms. This assumption is good except close to the equator where mean wind shears may become comparable with $2\Omega\mu$. The consequences of this neglect for equatorially-driven flows will be discussed below (Section 6b).

From the continuity equation (2.5), we may define a mass streamfunction χ for the meridional circulation where

$$\left.\begin{aligned} v &= -\frac{1}{\rho}\frac{\partial(\chi\rho)}{\partial z} \\ w &= \frac{1}{a}\frac{\partial}{\partial\mu}[\chi(1-\mu^2)^{1/2}] \end{aligned}\right\}, \tag{2.6}$$

while from (2.2) and (2.3) we obtain the thermal wind relation

$$2\Omega\mu\frac{\partial u}{\partial z} = -\frac{1}{a}(1-\mu^2)^{1/2}\frac{g}{T_0}\frac{\partial T}{\partial\mu}. \tag{2.7}$$

Differentiating (2.7) with respect to time and substituting from (2.1) and (2.4) gives the zonal component of the vorticity equation

$$4\Omega^2\mu^2\frac{\partial v}{\partial z} - \frac{N^2}{a}(1-\mu^2)^{1/2}\frac{\partial w}{\partial\mu} = -\frac{(1-\mu^2)^{1/2}}{a}\frac{\partial Q}{\partial\mu} - 2\Omega\mu\frac{\partial F}{\partial z}, \tag{2.8}$$

where $N^2 = gS/T_0$ is the square of the buoyancy frequency and $Q = gQ_*/T_0$. Note that under the quasi-geostrophic assumption N^2 may be a function of z but is independent of μ. Now, substituting from (2.6) for v and w leaves the equation

$$\frac{4\Omega^2a^2}{N^2}\frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial(\chi\rho)}{\partial z}\right] + \frac{(1-\mu^2)^{1/2}}{\mu^2}\frac{\partial^2}{\partial\mu^2}[\chi(1-\mu^2)] = \frac{a}{N^2}\frac{(1-\mu^2)^{1/2}}{\mu^2}\frac{\partial Q}{\partial\mu} + \frac{2\Omega a^2}{N^2\mu}\frac{\partial F}{\partial z}. \tag{2.9}$$

Alternatively, we may obtain an equation for w, *viz*:

$$\frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)}{\mu^2}\frac{\partial w}{\partial\mu}\right] + \frac{4\Omega^2a^2}{N^2}\frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial(w\rho)}{\partial z}\right] = \frac{1}{N^2}\frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)}{\mu^2}\frac{\partial Q}{\partial\mu}\right] + \frac{2\Omega a^2}{N^2}\frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)^{1/2}}{\mu}\frac{\partial F}{\partial z}\right]. \tag{2.10}$$

Unless the thermal and mechanical forcing terms are of such a form that their contributions to (2.9) exactly cancel each other everywhere—an unlikely possibility but one that is satisfied by models in which F and Q are completely specified by Rayleigh friction and Newtonian cooling respectively, with equal coefficients—then a meridional circulation is required to maintain thermal wind balance.

3. Homogeneous solutions

When the forcing terms on the rhs of (2.10) are zero, the equation becomes

$$\frac{\partial}{\partial\mu}\left[\frac{1-\mu^2}{\mu^2}\frac{\partial w}{\partial\mu}\right] + \frac{4\Omega^2a^2}{N^2}\frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial(w\rho)}{\partial z}\right] = 0. \tag{3.1}$$

We look for separable solutions of the form

$$w(\mu, z) = Z_n(z)\Theta_n(\mu), \tag{3.2}$$

where $\Theta_n(\mu)$ is a zero wavenumber, zero frequency Hough function satisfying the corresponding special case of Laplace's Tidal Equation

$$\frac{d}{d\mu}\left[\frac{(1-\mu^2)}{\mu^2}\frac{d\Theta_n}{d\mu}\right] - \epsilon_n\Theta_n = 0, \tag{3.3}$$

(e.g., Chapman and Lindzen, 1970), where the separation constant ϵ_n is an eigenvalue which is related to the equivalent depth, h_n, of tidal theory by the relation

$$\epsilon_n = 4\Omega^2a^2/gh_n.$$

It is straightforward to show, following Flattery (1967) and Kasahara (1977, 1978) that the Hough modes satisfy an orthogonality condition

$$(\epsilon_n - \epsilon_m)\int_{-1}^{1} \Theta_n(\mu)\Theta_m^*(\mu)d\mu = 0. \quad (3.4)$$

Now, the quasi-geostrophic modes of interest here are quite distinct from the geostrophic shallow-water modes discussed by Kasahara (1978); in particular, there is no degeneracy problem since, as will be seen below, the eigenvalues ϵ_n are distinct. Therefore the Hough functions are orthogonal and we may normalize so that

$$\int_{-1}^{1} \Theta_n(\mu)\Theta_m^*(\mu)d\mu = \delta_{nm}. \quad (3.5)$$

From (3.1), the vertical structure equation for Z_n now becomes

$$\frac{d}{dz}\left[\frac{1}{\rho}\frac{d(\rho Z_n)}{dz}\right] + \frac{N^2\epsilon_n}{4\Omega^2a^2}Z_n = 0. \quad (3.6)$$

Now, note from (2.6) that $\chi(\mu, z)$ may be similarly separated

$$\chi(\mu, z) = A_n(z)B_n(\mu), \quad (3.7)$$

where Θ_n and B_n are related by

$$\Theta_n(\mu) = \frac{d}{d\mu}[B_n(\mu)(1-\mu^2)^{1/2}]. \quad (3.8)$$

Because of the boundary conditions of $v = 0$ at the poles it is more convenient to solve an eigenvalue problem for $B_n(\mu)$ than $\Theta_n(\mu)$. The required equation may be obtained from (3.3) and (3.8) or, more directly, from the unforced Eq. (2.9), and is

$$\frac{(1-\mu^2)^{1/2}}{\mu^2}\frac{d^2}{d\mu^2}[B_n(1-\mu^2)^{1/2}] - \epsilon_n B_n = 0. \quad (3.9)$$

This equation (in its finite-differenced form) may then be solved by standard techniques subject to the boundary conditions $B_n(\pm 1) = 0$. Then $\Theta_n(\mu)$ may be obtained from (3.8).

It is worth noting here that all eigenvalues to (3.3) and (3.9) are real and negative. Consider the integral (3.5). Substituting for Θ_n and Θ_m^* from (3.8), integrating by parts and using (3.9) and the boundary conditions gives

$$-\epsilon_n\int_{-1}^{1} \mu^2 B_n(\mu)B_m^*(\mu)d\mu = \delta_{nm}. \quad (3.10)$$

For $n = m$ this implies

$$-\epsilon_n\int_{-1}^{1} \mu^2|B_n(\mu)|^2 d\mu = 1. \quad (3.11)$$

Since the integrand is positive definite (and finite) it follows that ϵ_n must be real and negative (and nonzero) and the eigenvectors are real. Further, for $n \neq m$,

$$\int_{-1}^{1} \mu^2 B_n(\mu)B_m(\mu)d\mu = 0. \quad (3.12)$$

4. Vertical structure

The vertical structure of solutions to the homogeneous problem (3.1) satisfies (3.6). Since $\epsilon_n < 0$ we may write

$$\Lambda_n^2 = -\frac{N^2\epsilon_n}{4\Omega^2a^2} = \frac{-N^2}{gh_n}, \quad (4.1)$$

where Λ_n^2 is real and positive. Therefore with

$$\rho = \rho_0\exp(-z/H),$$

(3.6) becomes

$$\frac{d^2Z_n}{dz^2} - \frac{1}{H}\frac{dZ_n}{dz} - \Lambda_n^2 Z_n = 0, \quad (4.2)$$

which has the solutions

$$Z_n(z) \propto \exp(\kappa_n z), \quad (4.3)$$

where

$$\kappa_n = \frac{1}{2H} \pm \left\{\frac{1}{4H^2} + \Lambda_n^2\right\}^{1/2}. \quad (4.4)$$

Note that one mode grows with height, the other decays. Because of the fact that the eigenvalues are all real and negative, and hence that the κ_n are real, there are of course no true normal modes of the problem.

5. Solution of the forced problem by expansion in Hough functions

In order to solve the general forced problem presented by (2.10) we follow the approach of tidal theory (e.g., Chapman and Lindzen, 1970), seeking a solution as an expansion of Hough functions

$$w(\mu, z) = \sum_n W_n(z)\Theta_n(\mu). \quad (5.1)$$

It is also necessary to expand the forcing. This is done by defining

$$Q(\mu, z) = \sum_n Q_n(z)\Theta_n(\mu), \quad (5.2)$$

where, from (3.5),

$$Q_n(z) = \int_{-1}^{1} Q(\mu, z)\Theta_n(\mu)d\mu. \quad (5.3)$$

Now, from (3.12) the functions $\mu B_n(\mu)$ are also orthogonal, though incomplete; we assume that F may be expanded

$$F(\mu, z) = \sum_n F_n(z)\mu B_n(\mu) \quad (5.4)$$

which assumes, at least, that F/μ is finite as $\mu \to 0$, since $B_n(0)$ is finite. Therefore this approach cannot be used if the mechanical forcing is nonzero on the equator. From (3.10),

$$F_n(z) = -\epsilon_n \int_{-1}^{1} F(\mu, z)\mu B_n(\mu)d\mu. \quad (5.5)$$

Then, using (3.3) and (3.8), the forcing term on the right-hand-side of (2.10) may be written

$$\frac{1}{N^2}\frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)}{\mu^2}\frac{\partial Q}{\partial\mu}\right] + \frac{2\Omega a^2}{N^2}\frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)^{1/2}}{\mu}\frac{\partial F}{\partial z}\right] = \sum_n G_n(z)\Theta_n(\mu), \quad (5.6)$$

where

$$G_n = \frac{1}{N^2}\left[\epsilon_n Q_n + 2\Omega a^2 \frac{dF_n}{dz}\right]. \quad (5.7)$$

Substituting (5.1) and (5.6) into (2.10), multiplying by $\Theta_m(\mu)$ and integrating from $\mu = -1$ to $\mu = 1$ gives the vertical structure equation

$$\frac{d}{dz}\left[\frac{1}{\rho}\frac{d(\rho W_n)}{dz}\right] + \frac{N^2\epsilon_n W_n}{4\Omega^2 a^2} = \frac{N^2 G_n}{4\Omega^2 a^2}. \quad (5.8)$$

From solutions to (5.8) (with suitable boundary conditions) it is straightforward to evaluate v and $\chi(\mu, z)$ since, from (2.6)

$$\left.\begin{aligned} v(\mu, z) &= \sum_n V_n(z)B_n(\mu) \\ \chi(\mu, z) &= \sum_n X_n(z)B_n(\mu) \end{aligned}\right\}, \quad (5.9)$$

where

$$\left.\begin{aligned} V_n(z) &= \frac{-a}{\rho}\frac{d}{dz}(\rho W_n) \\ X_n(z) &= aW_n \end{aligned}\right\}. \quad (5.10)$$

6. Results

a. Zonally uniform Hough functions and corresponding vertical structure

The Hough functions $\Theta_n(\mu)$, related functions $B_n(\mu)$ and corresponding eigenvalues ϵ_n were calculated by the method outlined in Section 3. It was found that adequate convergence was achieved with a grid increment $\Delta\mu = 1/30$. The structure of the first eight modes is shown in Fig. 1. [The modes are numbered according to the number of nodes in $\Theta_n(\mu)$ in $-1 < \mu < +1$. Thus even modes have symmetric Θ_n, antisymmetric B_n while the converse is true for odd n.] The eigenvalues ϵ_n and vertical scales $[\kappa_n^{(\pm)}]^{-1}$ of the circulation velocities appropriate to the stratosphere [derived from (4.4) and (4.1) and using $N^2 = 4.0 \times 10^{-4}$ s^{-2}, $\Omega = 7.27 \times 10^{-4}$ s^{-1}, $a = 6 \times 10^3$ km and $H = 7$ km] for the first ten modes are given in Table 1. Corresponding to the reduction in meridional scale with increasing n (and increasing ϵ_n) is a concomitant reduction in the vertical scale factor Λ_n^{-1}. Note the asymmetry arising from the density stratification; for all modes, the scale $(\kappa_n)^{-1}$ corresponding to downward influence (the positive root) is always smaller than H, and is much smaller for large n. For the graver modes, however, upward influence can extend much further, so that the e-folding height of the vertical velocity field for mode 1 is 38.9 km.

b. Response to equatorial thermal forcing

In a numerical study of the quasi-biennial oscillation (QBO), Plumb and Bell (1982) found the mean meridional circulation, driven by the diabatic relaxation of the quasi-biennial temperature oscillation, to be an important factor in communication of the equatorial forcing to higher latitudes. The analysis of Holton and Tan (1980) identified a significant component of the oscillation at middle and high latitudes; Plumb and Bell found that, in their beta-plane geometry, the influences of the meridional circulation were confined to within ~2000 km of the equator but recognized that the higher-latitude response might be larger if the effects of spherical geometry were taken into account.

To ascertain the effectiveness of the meridional circulation in linking equatorial forcing with the flow at high latitudes, consider the problem of the circulation driven by a localized thermal forcing

$$Q(\mu, z) = Q_0 \exp(-\mu^2/2\delta\mu^2) \times \exp[-(z-z_0)^2/2\delta z^2]. \quad (6.1)$$

To correspond with the diabatic heating of a cold thermal anomaly at 22 km associated with the quasi-biennial oscillation, the values

$$Q_0 = 5.0 \times 10^{-8} \text{ m s}^{-3}, \quad \delta_\mu = 0.2,$$
$$z_0 = 22 \text{ km}, \quad \delta_z = 2 \text{ km},$$

are adopted. The value of Q_0 approximately corresponds with a cold temperature anomaly of -1 K (Reed, 1964), warming according to a relaxation time of 10 days. The stratification is taken to be

$$N^2 = 1.5 \times 10^{-4} \text{ s}^{-2}, \quad z < 12 \text{ km},$$
$$N^2 = 4.0 \times 10^{-4} \text{ s}^{-2}, \quad 12 \text{ km} < z < 50 \text{ km},$$
$$N^2 = 1.5 \times 10^{-4} \text{ s}^{-2}, \quad z < 50 \text{ km}.$$

The boundary conditions are $w = 0$ on $z = 0$ and boundedness as $z \to \infty$.

The vertical velocity field is then solved by the method outlined in Section 5, through the expansion (5.1) and solution of (5.8) on a finite difference grid with increment 1 km. The boundary conditions on $z = 0$ become

$$W_n(0) = 0. \quad (6.2)$$

The upper boundary condition is applied at $z = 60$ km; it is assumed that the atmosphere is uniform and that $Q_n(z)$ is zero above that level. Therefore the solutions of Section 4 are appropriate above 60 km and boundedness requires that $W_n(z) \approx \exp(\kappa_n z)$ in this region, where κ_n is the negative root of (4.4). The upper boundary condition is then

$$\left(\frac{dW_n}{dz} - \kappa_n W_n\right)_{z=60\text{ km}} = 0. \qquad (6.3)$$

and solution of (5.8) becomes straightforward.

Because of the symmetry of the forcing about the equator, only the even modes are excited. Results for

TABLE 1. Eigenvalues ϵ_n and related vertical scales $[\kappa_n]^{-1}$ corresponding to the first ten zonally symmetric Hough functions.

n	ϵ_n	$[\kappa_n{}^{(+)}]^{-1}$ (km)	$[\kappa_n{}^{(-)}]^{-1}$ (km)
1	−8.47	5.94	−38.9
2	−13.2	5.54	−26.7
3	−37.9	4.40	−11.8
4	−48.0	4.11	−9.94
5	−88.7	3.38	−6.53
6	−104	3.20	−5.89
7	−161	2.72	−4.45
8	−182	2.47	−3.81
9	−257	2.27	−3.35
10	−285	2.19	−3.18

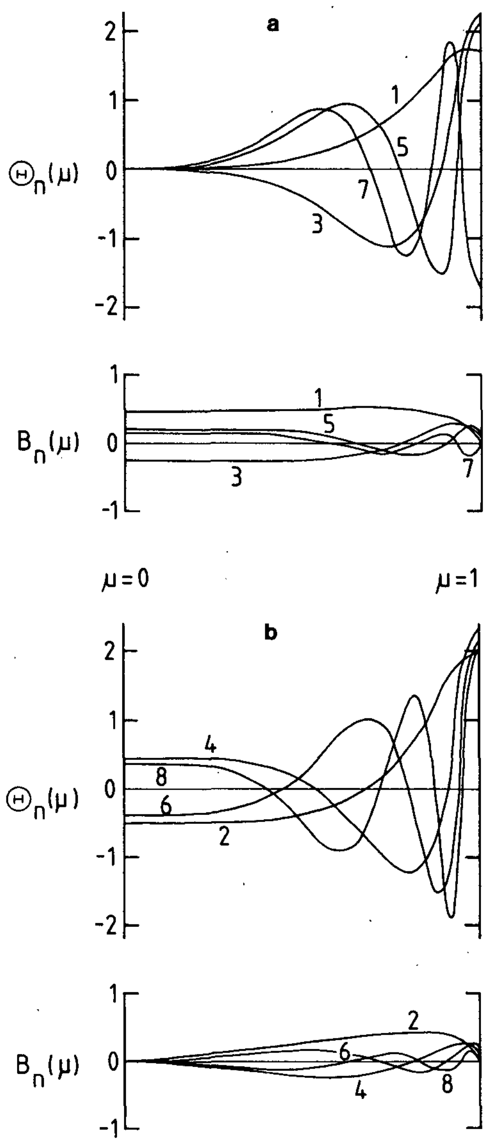

FIG. 1. Structure in $0 \leqslant \mu \leqslant 1$ of the first eight zonally symmetric Hough modes Θ_n and related functions B_n; (a) odd modes, (b) even modes.

$W_n(z)$ and the streamfunction $\chi(\mu, z)$ are shown in Figs. 2 and 3a, respectively. Fig. 2 shows the vertical structure of the first five individual modes as a function of height. Near the region of forcing, the maximum response is in those modes (8 and 10) whose vertical scale $\kappa_n{}^{-1}$ most closely matches that of the forcing. Modes with $n > 10$ have progressively smaller maximum amplitude and are of progressively narrower vertical extent. Above and below the forcing region, however, these modes decay more rapidly than modes of lower n and the latter dominate the response at large $|z - z_0|$. This is manifested in the circulation streamfunction $\chi(\mu, z)$ (Fig. 3a) as a poleward shift of the latitudinal maximum of χ above and, to a lesser extent, below the forcing region. Note, in accord with the results discussed in Section 6a, that downward penetration of the circulation is very weak; upward extension is stronger, especially in mid-latitudes. Despite the complexity of modal structure, the physical structure of χ is quite simple, a result of high-latitude cancellation between modes. The induced mean zonal acceleration,

$$\frac{\partial \bar{u}}{\partial t} = -\frac{2\Omega\mu}{\rho}\frac{\partial(\rho\chi)}{\partial z}, \qquad (6.4)$$

is shown in Fig. 3b. Note that thermal wind balance associates the temperature minimum at 22 km on the equator with easterly shear; therefore we may qualitatively associate this anomaly with the presence of easterly winds above and westerly winds below about 22 km within $|\mu| \leqslant 0.2$. Fig. 3b shows that the induced zonal acceleration maximum in the subtropics (at ~20°) has the opposite sign. This is in agreement with the principles discussed in Plumb and Bell (1982), viz. that the role of the mean circulation in the dynamics of the quasibiennial oscillation is a dissipative one in the subtropics. It is also in qualitative agreement with the reversed-phase QBO in subtropical latitudes evident in the results of Holton and Tan (1980). The low-to-middle latitude response is considerably enhanced compared with the circulations in the beta-plane model of

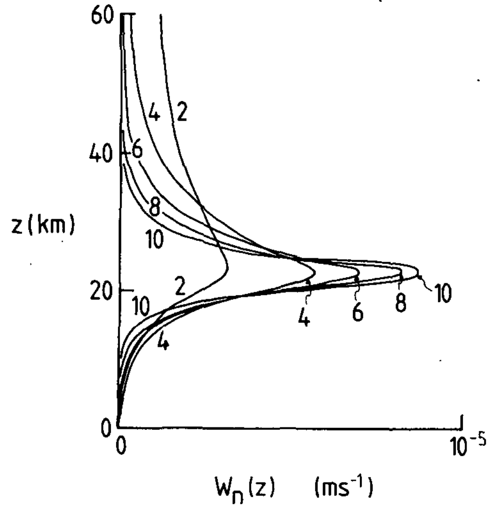

FIG. 2. Vertical structure of the first five (even) modes generated by the equatorial thermal forcing of Eq. (6.1). See text for discussion.

Plumb and Bell. However, the second phase reversal observed by Holton and Tan in winter, with a high-latitude signal of amplitude about 5 m s^{-1}, in phase with the equatorial wind, has no obvious explanation in terms of Fig. 3b.

Of course, this simple calculation falls short of reality for many reasons. As noted earlier, the quasi-geostrophic approximation may break down in the tropics. At the extrema of the quasi-biennial cycle, the latitudinal wind shear $a^{-1}\partial u/\partial \mu$ becomes comparable with or greater than the Coriolis parameter within $\sim$10° latitude of the equator. Therefore the actual structure of the circulation within this region may be significantly different from that derived for a windless atmosphere. Outside this region, however, the quasi-geostrophic assumption is fairly reliable and therefore it seems the weak penetration of the circulation into middle latitudes would also be a characteristic of a more realistic calculation. This property is, indeed, likely to be reinforced by another neglected aspect of the complete problem, namely the dissipation of the thermal response to the vertical motion. Except very close to the equator, thermal dissipation will severely limit the magnitude of any departures from radiative equilibrium (Wallace, 1967; Plumb and Bell, 1982); in a steady state, where diabatic heating and adiabatic cooling must balance, this implies that the vertical motion will be similarly restricted. Thus, the fully developed circulation may therefore be even more confined to low latitudes than suggested by the above results. [This was pointed out to me by a referee.]

Overall, then, it seems that the QBO-induced mean meridional circulation cannot explain Holton and Tan's high-latitude winter signal, which serves

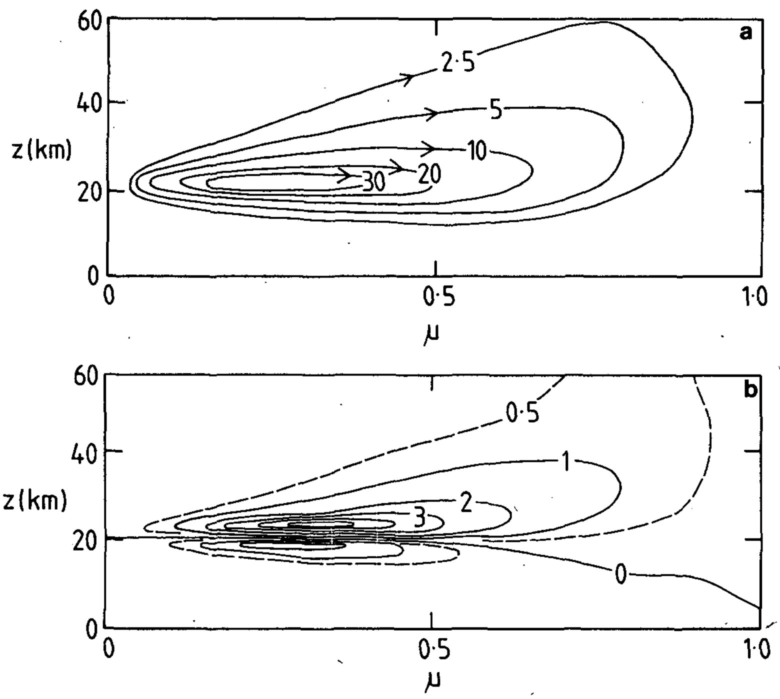

FIG. 3. Latitude–height structure (Northern Hemisphere) of the response to the equatorial thermal forcing of Eq. (6.1); (a) streamfunction χ (m^2 s^{-1}), (b) zonal acceleration (10^{-7} m s^{-2}).

to reinforce their suggestion that planetary waves may provide the implied coupling between low and high latitudes. Note, also, in line with the discussion in Section 6a of vertical structure, that downward penetration of the circulation is very weak and therefore there can be no significant coupling with the troposphere via such a circulation; there have been a number of reports of a tropospheric component of the QBO (e.g., Trenberth, 1980).

c. *Response to high-latitude forcing*

In order to illustrate the circulation driven by high latitude forcing, consider now the response to an isolated diabatic heating in the middle stratosphere

$$\left.\begin{aligned} Q &= Q_0 \exp\left\{-\frac{(\mu-1)^2}{2\delta\mu^2}\right\} \\ &\quad\times \exp\left\{-\frac{(z-z_0)^2}{2\delta z^2}\right\}, \quad \mu > 0 \\ Q &= 0, \quad \mu < 0 \end{aligned}\right\}, \qquad (6.5)$$

where $\delta\mu = 0.2$, $z_0 = 30$ km, $\delta z = 5$ km and where, for definiteness, we specify $Q_0 = 3 \times 10^{-7}$ m s^{-3}, although we are interested only in the latitudinal distribution of the response and, since the problem is linear, the amplitude is arbitrary.

As will be seen, the upward penetration of the circulation is much greater than in the previous example and so the boundary condition (6.3) is now applied at $z = 98$ km; the vertical grid increment is 2 km. Downward penetration is still weak, however; therefore any errors introduced by this boundary condition will not affect the circulation far below.

The streamfunction $\chi(\mu, z)$ of the forced meridional circulation is depicted in Fig. 4. While there is some cross-equatorial penetration the circulation is rather weak at high latitudes of the unforced hemisphere. This can be understood in terms of the properties of the Hough functions as follows. Note from Fig. 1 that the structure of corresponding symmetric and antisymmetric modes are very similar at high latitudes. Therefore, from (5.3), these corresponding modes are excited with almost equal efficiency and, in the forcing hemisphere, with the same sign. By the same token, however, these modes must almost cancel at high latitudes of the opposite hemisphere. Nearer the equator, of course, the symmetric modes begin to dominate and the degree of cancellation diminishes, resulting in the meridional structure shown in Fig. 4. The strong vertical penetration evident in the figure is a result of the relatively high efficiency with which the low order modes are generated.

These deductions are reinforced by comparison with the structure of the response to midlatitude forcing. Fig. 5 shows the circulation driven by a thermal forcing

$$\left.\begin{aligned} Q &= Q_0 \mu \sin^2(\pi\mu) \\ &\quad\times \exp\left\{-\frac{(z-z_0)^2}{2\delta z^2}\right\}, \quad \mu > 0 \\ Q &= 0, \quad \mu < 0 \end{aligned}\right\}, \qquad (6.6)$$

where $Q_0 = 3 \times 10^{-7}$ m s^{-3}, $z_0 = 30$ km and $\delta z = 5$ km. The latitudinal structure function maximizes at 43°. The forcing of symmetric and antisymmetric modes is now unequal; in fact the symmetric modes are more strongly forced, increasing the penetration of the circulation into the opposite hemisphere. Vertical penetration, however, is reduced, a consequence of the less efficient forcing of the graver modes.

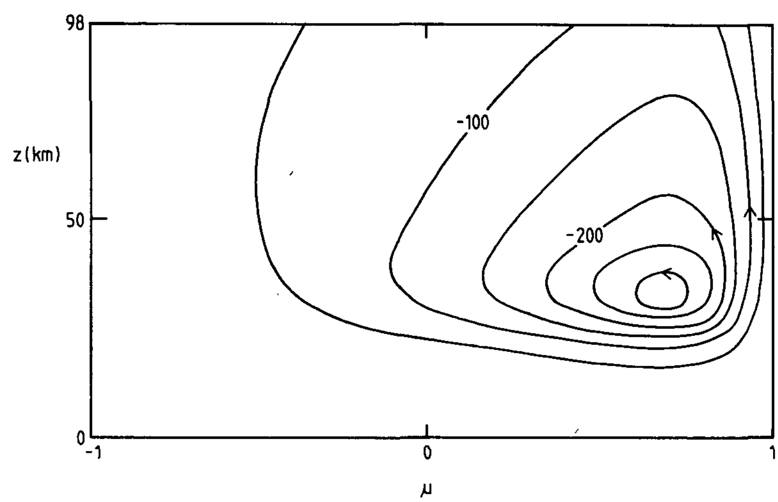

FIG. 4. Global circulation generated by the high-latitude thermal forcing of Eq. (6.5). Contours of streamfunction χ (m^2 s^{-1}).

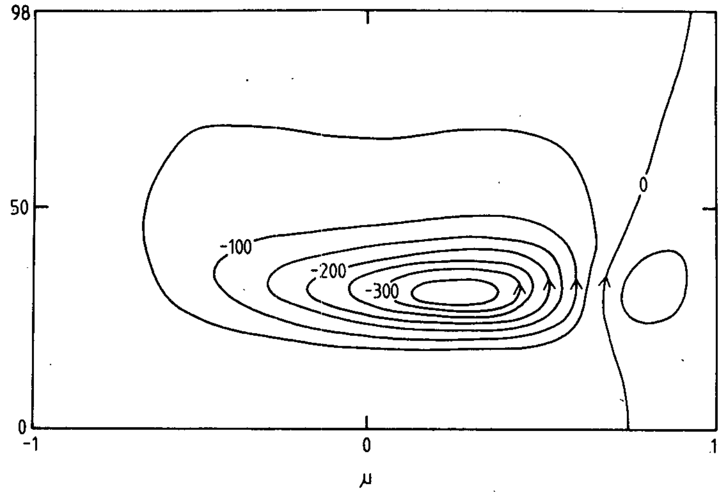

FIG. 5. Global circulation generated by the midlatitude thermal forcing of Eq. (6.6). Contours of streamfunction χ ($m^2 s^{-1}$).

To return to the example of Fig. 4, the form of the circulation explains why sudden warmings in the polar winter stratosphere are accompanied by stratospheric cooling not only at low latitudes, but spilling over into the summer hemisphere. In warmings, of course, the primary forcing of the *residual* mean circulation is mechanical, via the Eliassen-Palm flux convergence (Andrews and McIntyre, 1976; see also Edmon *et al.*, 1980). Because F appears in (5.7) as a vertical derivative, a localized mechanical forcing is equivalent to a dipolar thermal forcing, resulting in a circulation which has dipole structure (e.g., Eliassen, 1951; Matsuno and Nakamura, 1979) in the vertical but latitudinal structure of much the same form as shown in Fig. 4. The broad meridional nature of the response explains the global nature of the compensatory stratospheric cooling; because of the dipolar vertical structure, the circulation and temperature tendency change sign at high levels. This is consistent with the observation that high latitude warming/low latitude cooling in the stratosphere is accompanied by high latitude cooling/low latitude warming in the mesosphere (e.g., Quiroz, 1979).

7. Conclusions

The general form of the Hough functions as derived in this paper shows that downward penetration of meridional circulations is inefficient, the *e*-folding height for downward decaying modes being less than 6 km in the stratosphere, even for the mode of gravest latitudinal structure. Therefore this mechanism is ineffective as a means of downward communication of high-level activity. Upward communication is much more effective, the corresponding height scale being ~40 km for the gravest mode. How efficiently the low-order modes are excited depends on the horizontal and vertical length scales of the forcing, and on its latitude.

Equatorial forcing of limited vertical extent is an inefficient means of forcing the gravest modes and consequently the resulting circulation is confined to subtropical latitudes. Therefore it seems that the mean meridional circulation is incapable of providing sufficient interlatitudinal coupling to account for the high-latitude component of the quasi-biennial oscillation revealed by the analysis of Holton and Tan (1980); this result reinforces their suggestion that planetary wave activity may provide the required coupling. The meridional circulation probably is important at lower latitudes, however (Plumb and Bell, 1982).

Thermal forcing at middle and high latitudes is much more effective at generating global circulations, and it seems likely that this is characteristic of the residual circulation generated at the time of a stratospheric warming, thus supporting the suggestion (e.g., Dunkerton *et al.*, 1981) that this circulation is responsible for the cooling observed at such times in low latitudes and into the summer hemisphere.

REFERENCES

Andrews, D. G., and M. E. McIntyre, 1976: Planetary waves in horizontal and vertical shear: the generalized Eliassen-Palm relation and the mean zonal acceleration. *J. Atmos. Sci.*, **33**, 2031–2048.

Chapman, S., and R. S. Lindzen, 1970: *Atmospheric Tides*. D. Reidel, 200 pp.

Dunkerton, T., C.-P. F. Hsu and M. E. McIntyre, 1981: Some Eulerian and Lagrangian diagnostics for a model stratospheric warming. *J. Atmos. Sci.*, **38**, 819–843.

Edmon, H. J., B. J. Hoskins and M. E. McIntyre, 1980: Eliassen-Palm cross-sections for the troposphere. *J. Atmos. Sci.*, **37**, 2600–2616.

Eliassen, A., 1951: Slow thermally or frictionally controlled meridional circulation in a circular vortex. *Astrophys. Norv.*, **5,** 19–60.

Flattery, T. W., 1967: Hough functions. Tech. Rep. 21, Dept. Geophys. Sci., The University of Chicago.

Holton, J. R., 1979: *An Introduction to Dynamic Meteorology* 2nd ed. Academic Press, 391 pp.

——, and R. S. Lindzen, 1972: An updated theory for the quasi-biennial oscillation of the tropical stratosphere. *J. Atmos. Sci.*, **29,** 1076–1080.

——, and H.-C. Tan, 1980: The influence of the equatorial quasi-biennial oscillation on the global circulation at 50 mb. *J. Atmos. Sci.*, **37,** 2200–2208.

Kasahara, A., 1977: Numerical integration of the global barotropic primitive equations with Hough harmonic expansions. *J. Atmos. Sci.*, **34,** 687–701.

——, 1978: Further studies on a spectral model of the global barotropic primitive equations with Hough harmonic expansions. *J. Atmos. Sci.*, **35,** 2043–2051.

Matsuno, T., and K. Nakamura, 1979: The Eulerian and Lagrangian-mean meridional circulations in the stratosphere at the time of a sudden warming. *J. Atmos. Sci.*, **36,** 640–654.

Plumb, R. A., and R. C. Bell, 1982: A model of the quasi-biennial oscillation on an equatorial beta-plane. *Quart. J. Roy. Meteor. Soc.*, **108** (April).

Quiroz, R. S., 1979: The contribution of satellite remote-sounding data in stratospheric research. *Remote Sounding of the Atmosphere from Space*, H. J. Bolle, Ed., Pergamon Press, 69–82.

Reed, R. J., 1964: A tentative model of the 26-month oscillation in tropical latitudes. *Quart. J. Roy. Meteor. Soc.*, **90,** 441–466.

Trenberth, K. E., 1980: Atmospheric quasi-biennial oscillations. *Mon. Wea. Rev.*, **108,** 1370–1377.

Tucker, G. B., 1979: The observed zonal wind cycle in the southern hemisphere stratosphere. *Quart. J. Roy. Meteor. Soc.*, **105,** 263–273.

Wallace, J. M., 1967: A note on the role of radiation in the biennial oscillation. *J. Atmos. Sci.*, **24,** 598–599.