The Double-Diffusive Modes of Symmetric Instability on an Equatorial Beta-Plane

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ABSTRACT

The symmetric instability due to horizontal shear on an equatorial beta-plane exhibits two distinct modes of instability. The classical monotonic non-oscillatory instability exists for all Prandtl numbers but is favored when the Prandtl number is approximately less than $^3/_2$. For values of Prandtl number approximately larger than this we find that an oscillating "overstability" is the preferred mode of instability. This result contrasts with the baroclinic centrifugally stable case in which overstabilities exist but are never preferred. Similar results can be demonstrated analytically on an artificially bounded f-plane which mimics the finite latitudinal scale imposed by the equatorial beta-plane geometry. Radiative relaxation would favor the monotonic mode, but the effect might be insignificant if breaking internal gravity waves are present.

1. Introduction

The interpretation of certain mesoscale circulations in terms of inertial, or symmetric, instability has prompted a recent resurgence of interest in the subject (Bennets and Hoskins, 1979; Emanuel, 1979). Rayleigh's (1916) original discussion attributed the inertial instability to an imbalance of pressure gradient and centrifugal forces when the absolute value of angular momentum decreases with radius. Later authors described the effects of baroclinity in terms of a balance of forces along isentropic surfaces (Solberg, 1933; Kuo, 1954, 1956; Stone, 1966; McIntyre, 1970). Hoskins (1974) provided an important potential vorticity argument demonstrating that departures from conservative motion are essential to the inertial destabilization of extratropical flows. Recently, Busse and Chen (1981) have suggested that baroclinic inertial instabilities may not be exactly symmetric when the Prandtl number differs from unity.

Of particular interest is McIntyre's (1970) recognition of two distinct modes of double-diffusive inertial instability when the Prandtl number differs from unity. In addition to the classical nonoscillatory instability (CNI), McIntyre found an "overstability," or oscillatory instability. The latter owed its existence to temporary imbalances set up by differing diffusion rates of heat and momentum. The oscillatory instabilities were found to be weak, however, and marginal stability criteria were found to always be harder to satisfy for these modes relative to those of the CNI in this baroclinic case.

In contrast to the baroclinic problem, the classical horizontal shear inertial stability problem has not

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received its proper emphasis in the atmospheric sciences literature, although mention has been made in papers by Lindzen (1966), Stone (1971) and Holton (1972), among others. A possible reason for this has been the lack of suggested meteorological applications. Recently Dunkerton (1981) suggested that the horizontal type of inertial instability may be relevant to the winter tropical mesosphere since Rayleigh's inviscid criterion is formally violated by any cross-equatorial shear as is observed in the middle atmosphere at the solstices. A numerical study by Hunt (1981) has given some preliminary support to this hypothesis, although better observational coverage is clearly warranted.

Dunkerton (1981) discussed the marginal stability criteria for horizontal shear inertial instability on an equatorial beta-plane, i.e.,

$$\gamma_c = 2\sqrt{5}\nu^{1/5}\left(\frac{N\beta}{4}\right)^{2/5}, \tag{1.1a}$$

$$m_c^5 = \frac{N\beta}{4\nu^2}, \tag{1.1b}$$

where γ_c and m_c are critical horizontal shear and vertical wavenumber, respectively, and N, β and ν are static stability, planetary vorticity gradient and the momentum diffusion coefficient. In these criteria the Prandtl number σ is set equal to 1, where $\sigma = \nu/\kappa$, where κ is the thermal diffusion coefficient. Dunkerton argued that although the observed climatological cross-equatorial solstice shears are everywhere stabilized by the "observed" eddy diffusion in the middle atmosphere, it is possible that (i) local horizontal shears may be much stronger than the climatological values, and more consistent with the diabatic circulation; and (ii) eddy diffusion may be the *result* of a strong inertial instability. In any case breaking-gravity-wave-induced diffusion and deceleration appears to be crucial to the stability of this flow (Leovy, 1964; Lindzen, 1981; Dunkerton, 1982; Holton, 1982).

The purpose of this note is to derive marginal stability criteria when the Prandtl number differs from unity. Dunkerton (1981) suggested a weak increasing dependence of γ_c on σ. This is most easily seen when the CNI mode is neutral; one may remove σ from the equations by defining an effective static stability $\hat{N}^2 \equiv \sigma N^2$; hence

$$\gamma_c = 2\sqrt{5}(\nu\sigma)^{1/5}\left(\frac{N\beta}{4}\right)^{2/5}, \tag{1.2a}$$

$$m_c^5 = \frac{N\beta}{4\nu^2}\sqrt{\sigma}. \tag{1.2b}$$

Eq. (1.2a) has two interesting implications. When $\sigma \to 0$, as for example might occur with a large diffusion of heat, apparently less horizontal shear is required for instability. Physically this seems attributable to the fact that the vertical disturbance scale is increased as $\sigma \to 0$, so that a given amount of diffusion has a smaller effect.

By similar reasoning, (1.2a) implies that removal of the thermal diffusion hinders the instability. However, as $\sigma \to \infty$ the behavior of the marginal stability criteria is markedly changed. First, the neutral CNI is no longer the preferred non-oscillatory instability since the lowest marginally unstable shear for CNI is associated with a growing mode. Secondly, and more important, the overstability is actually the preferred mode of instability in this range, having a lower marginally unstable shear than any CNI.

Because the latter result may seem surprising insofar as it differs significantly from what McIntyre (1970) found in the baroclinic centrifugally stable problem, a separate explanation is warranted and will be given in this note. It has been found that a similar conclusion follows for horizontal inertial instability on an artificially bounded f-plane. At very large Prandtl number the overstability begins to resemble a symmetric inertia-gravity wave in which pressure perturbations are essential to the maintenance and vertical propagation of the overstability. This is why the overstability is no longer the preferred mode of instability when larger values of κ are introduced. It could be emphasized that this result is entirely in harmony with McIntyre's (1970) discussion of the underlying physics; in the present case destabilization is brought about by centrifugal-type forces acting in opposition to stabilizing or restoring buoyancy forces. The latter cannot be destabilizing in the absence of vertical shear at large Prandtl number, leaving the overstability as the only possibility here. By the same reasoning the effect of radiative relaxation is to favor the CNI by reducing the effective Prandtl number, but the quantitative significance of this fact seems doubtful, at least until better estimates of κ become available.

2. Double-diffusive instability criteria

For equations we adopt the linear, hydrostatic, Boussinesq, symmetric equatorial beta-plane set

$$u'_t + v'(\gamma - \beta y) = \nu u'_{zz}, \tag{2.1a}$$

$$v'_t + \beta y u' + \phi'_y = \nu v'_{zz}, \tag{2.1b}$$

$$\phi'_{zt} + N^2 w' = \kappa \phi'_{zzz}, \tag{2.1c}$$

$$v'_y + w'_z = 0, \tag{2.1d}$$

where u', v', w' and ϕ' are zonal, meridional, and vertical velocity, and geopotential, respectively. The substitution

$$\{u', v', w', \phi'\} = \mathrm{Re}\{u, iv, w, \phi\}\exp[i(mz + \omega t)] \tag{2.2}$$

yields the relations

$$u = \frac{(\beta y - \gamma)v}{\omega_M}, \quad \phi = -\frac{v_y}{\epsilon\omega_T}, \quad (2.3a, b)$$

where $\omega_M = \omega - i\nu m^2$, $\omega_T = \omega - i\kappa m^2$ and $\epsilon = m^2/N^2$. The meridional velocity equation is

$$v_{yy} - \epsilon v\{\beta y(\beta y - \gamma)\frac{\omega_T}{\omega_M} - \omega_T\omega_M\} = 0. \quad (2.4)$$

This equation is reduced to canonical form in the usual way:

$$y_1 = y - \gamma/2\beta, \quad \beta_1{}^2 = \beta^2\frac{\omega_T}{\omega_M}, \quad (2.5a, b)$$

$$\omega_1{}^2 = \omega_M\omega_T + \frac{\gamma^2}{4}\frac{\omega_T}{\omega_M}, \quad \xi = \frac{y_1}{y_0}, \quad (2.5c, d)$$

$$\epsilon\beta_1{}^2 y_0{}^4 = 1. \quad (2.5e)$$

Therefore,

$$v_{\xi\xi} - v\{\xi^2 - \epsilon\omega_1{}^2 y_0{}^2\} = 0, \quad (2.6)$$

so that

$$\epsilon\omega_1{}^2 y_0{}^2 = 2n + 1, \quad (2.7a)$$

$$v(\xi) = v_n H_n(\xi) \exp - \xi^2/2. \quad (2.7b)$$

From (2.5e) and (2.7a),

$$\omega_M{}^2 + \tfrac{1}{4}\gamma^2 = \frac{N\beta(2n+1)}{|m|}\left(\frac{\omega_M}{\omega_T}\right)^{1/2}, \quad (2.8a)$$

or more completely,

$$(\omega - i\nu m^2)^2 + \tfrac{1}{4}\gamma^2 = \frac{N\beta(2n+1)}{|m|}\left(\frac{\omega - i\nu m^2}{\omega - i\kappa m^2}\right)^{1/2}. \quad (2.8b)$$

It is understood that $\mathrm{Re}\sqrt{\ } > 0$ in order to satisfy boundedness of the Gaussian (2.7b) at infinity.

The dispersion relation may be nondimensionalized with the following scales:

$$m_0{}^5 = \frac{N\beta_2}{\nu^2}, \quad \gamma_0 = \nu^{1/5}(N\beta_2)^{2/5} = \omega_0,$$

$$\beta_2 = \beta(2n + 1). \quad (2.9a, b, c, d)$$

From now on, unless otherwise indicated, all quantities are nondimensional in accord with this scaling. Eq. (2.8b) becomes

$$(\omega - im^2)^2 + \tfrac{1}{4}\gamma^2 = \frac{1}{|m|}\left(\frac{\omega - im^2}{\omega - im^2\sigma^{-1}}\right)^{1/2}. \quad (2.10)$$

We immediately note from (2.9b, d) that the lowest order $n = 0$ mode is of greatest interest since dimensional marginally stable shears are minimized by this choice.

a. Classical nonoscillatory instability

Setting $\omega = i\omega_i$ in (2.10) yields

$$\tfrac{1}{4}\gamma^2 = \frac{1}{|m|}\left(\frac{\omega_i - m^2}{\omega_i - m^2\sigma^{-1}}\right)^{1/2} + (\omega_i - m^2)^2. \quad (2.11)$$

[When written in this way the marginally unstable shear is most easily derived; alternately, (2.10) might be written as a quintic polynomial in ω having solutions symmetric about the imaginary ω-axis]. The marginal stability criterion for a neutral mode which minimizes the shear γ is just (1.2a) in nondimensional form. However, we find that differentiation of (2.11) with respect to ω_i and m implies that when $\sigma > 2$ an extremum in γ exists below the real ω-axis:

$$\omega_i = m_c{}^2(2\sigma^{-1} - 1), \quad (2.12a)$$

$$4m_c{}^5 = [2\sqrt{2}(\sigma^{-1} - 1)^2]^{-1}, \quad (2.12b)$$

$$\gamma_c = 2^{3/5}\sqrt{5}(1 - \sigma^{-1})^{1/5}, \quad (2.12c)$$

valid for $\sigma \geqslant 2$. These agree with (1.2) when $\sigma = 2$. Asymptotic behavior at infinite σ is

$$\omega_i = -m_c{}^2, \quad 4m_c{}^5 = \{2\sqrt{2}\}^{-1},$$

$$\gamma_c = 2^{3/5}\sqrt{5}. \quad (2.13a, b, c)$$

These marginal stability criteria for the CNI are displayed in Fig. 1.

b. Overstability

Eq. (2.10) may be written as

$$\tfrac{1}{4}\gamma^2 = \left[\frac{|\Omega_M|\sin\theta_s}{\sqrt{|\Omega_T|}\sin 2\theta_M}\right]^{4/5} \cdot [\cot\theta_s \sin 2\theta_M - \cos 2\theta_M], \quad (2.14)$$

where $\Omega = \omega/m^2$, $\Omega_M = \Omega - i$, $\Omega_T = \Omega - i\sigma^{-1}$, $\theta_M = \arg(\Omega_M)$, $\theta_T = \arg(\Omega_T)$ and $\theta_s = \tfrac{1}{2}(\theta_M - \theta_T)$. For a given Ω, m must satisfy $\mathrm{Im}\gamma = 0$; this result has been used in (2.14). This rather cumbersome equation reveals the emergence of overstabilities which minimize γ away from the imaginary ω axis (along the real ω axis) when $\sigma \gtrsim 5/4$. When $\sigma \gtrsim 1.48$ these γ are actually less than those implied by (1.2a) and (2.12c). This preference for overstability, which increases with Prandtl number, contrasts with the baroclinic case (McIntyre, 1970). More consistent with the baroclinic problem, however, we find these overstabilities to be rather weak; for example, when $\sigma = \infty$ and $\Omega_r = 2.2$, a growth rate $\omega_i = -0.2617$ yields $\gamma^2/4 = 2.6994$, not much below the asymptotic CNI value [(2.13c)].

An alternate and less direct solution method involves solving the quintic polynomical in ω as a func-

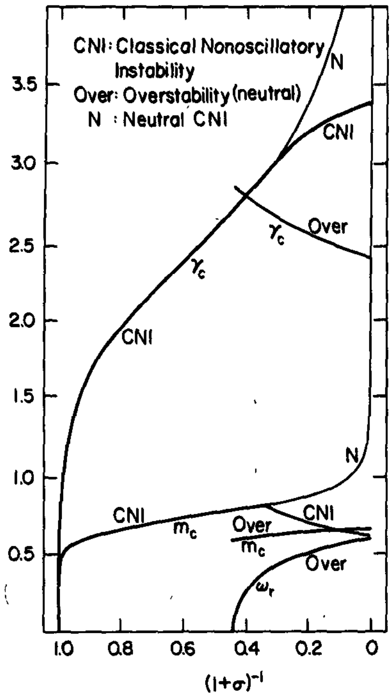

FIG. 1. Marginal stability criteria on the equatorial beta-plane for the classical non-oscillatory instability (CNI), purely neutral CNI (N), and overstability (OVER). Plotted are critical horizontal shear γ_c, critical vertical wavenumber m_c, and, for the overstability, the frequency of oscillation. All quantities are nondimensional in accord with the scalings quoted in the text.

tion of m, γ and σ. Squaring (2.10) introduces two spurious roots associated with the negative sign of the square root; of the remaining three roots one is always imaginary while the other two are either imaginary or a complex negative conjugate pair (overstabilities). When $\sigma < 1$ the former root is the CNI; when $\sigma > 1$, on the other hand, growth is associated with the remaining two roots. Numerical solution for these roots verifies the marginal stability criteria in Fig. 1. When $\sigma > 2$, the point of coalescence (where the overstable pair branches into two imaginary roots) which minimizes γ as a function of m occurs below the real ω axis, implying a non-neutral CNI. However, the overstability remains the preferred mode of instability in this range. When $\sigma < 1$ the coalescence point is found above the real axis.

The overstability propagates vertically and has a modest parabolic phase tilt in the latitude-height plane. At very large Prandtl number its structure begins to resemble an inertia-gravity wave and contrasts greatly with the CNI. In the latter, pressure perturbations oppose the horizontal redistribution of mass while in the overstability these pressure gradients maintain the vertical propagation of the disturbance. This explains *via* the hydrostatic relation the markedly different effect of thermal dissipation on the two modes (e.g., Fig. 1). The relative weakness of the overstability may be attributed to the smallness of the kinetic energy conversion which is proportional to $\overline{u'v'}$, with these velocities being somewhat out of phase in the overstability.

c. Bounded f-plane analysis

Setting $\beta y = f =$ constant in (2.4) and assuming rigid walls at $y = 0, L$ yields the dimensional eigencondition

$$f(\gamma - f) = \left(\frac{Nn\pi}{Lm}\right)^2 \frac{\omega - i\nu m^2}{\omega - i\kappa m^2} - (\omega - i\nu m^2)^2, \quad (2.15)$$

with sinusoidal eigenfunctions. Eq. (2.15) is simpler than its equatorial beta-plane counterpart (note, however, their underlying similarity) and is found to define marginal stability criteria analytically for the overstable modes as well. These criteria (and those of the CNI) closely resemble Fig. 1 but are somewhat more accentuated; for example, the CNI critical excess shear at infinite Prandtl number is four times the overstable value. (We omit further f-plane discussion but formulas are available from the author.) In summary, the f-plane analysis confirms the numerical beta-plane findings and clarifies that these features do not depend on the specific equatorial beta-plane geometry. Of course, *containment* of the eigenfunction, whether real, as in the beta-plane case, or artificial, as in the f-plane case, is the common and essential thread running through both problems.

While the preference for overstability is slightly surprising at first sight, in view of the baroclinic result, it should not be forgotten that a crude analogy exists between this problem and the so-called thermohaline convection; the emergence of overstabilities in the latter is well-known (e.g., Baines and Gill, 1969).

d. Effect of Newtonian Cooling

In view of the possible application of this result to the middle atmosphere it is desirable to study the effect of radiative relaxation. An effective Prandtl number is

$$\sigma_{\text{eff}} = \frac{\nu m^2}{\kappa m^2 + \alpha_R(m)}, \quad (2.16)$$

allowing for the scale-dependent acceleration of the radiative damping rate (Fels, 1982). According to Fels there is a square-root dependence at large m, and values of α_R may be five or more times greater than the $m = 0$ values when $m \geq 1$ km^{-1}. The general remark could be made that since σ_{eff} is reduced from σ, the effect of Newtonian cooling is to favor the CNI and militate against the overstability (Dunkerton, 1981). However, this ultimately depends upon the eddy diffusion coefficient, and as already remarked by Dunkerton (1981) the diffusive damping rate at the critical vertical wavenumber is much greater than Dickinson's (1973) radiative damping rates at upper levels. The scale-dependent acceleration would not alter this conclusion. Thus given such large values of $\kappa = O(\nu) = O(100 \text{ m}^2 \text{ s}^{-1})$, as in Lindzen's (1981) model of breaking internal gravity waves, Newtonian cooling would not affect the critical parameters more than a few percent.

Clearly, better estimates of eddy diffusion, particularly thermal diffusion, will be crucial to this discussion. Perhaps it should be noted that in an atmosphere where wavebreaking does *not* occur, it is conceivable that Newtonian cooling could provide a significant destabilizing influence with respect to the CNI (but *not* the overstability). The scale-dependent acceleration of the damping rate could then play an equally significant role in the destabilization process.

3. Conclusion

While more observational evidence is needed to fully assess the implications of equatorial inertial instability, this note suggests a worthwhile problem for GCM experimentation. Because ratios of thermal to mechanical diffusion can be controlled to a certain degree in numerical models, the Prandtl number dependences exhibited in Fig. 1 can be explicitly tested, together with radiative damping effects.

In conclusion, the remark should be made that the problem under investigation here has not yet been fully explored. In effect we are touching upon the field of equatorial wave instability. The author has seen a recent paper by Boyd (1982) claiming to have found a Kelvin wave instability in linear cross-equatorial shear, involving a "critical layer" in which the Doppler-shifted phase speed vanishes. Our investigation together with Boyd's suggests that equatorial waves may exhibit instabilities which are physically related to their simpler midlatitude counterparts, but in some way are modified by the equatorial beta-plane geometry. For example, Boyd's Kelvin wave instability is reminiscent of the barotropic and baroclinic instabilities involving critical layers imbedded in regions of reversed vorticity gradient, except that Boyd's instability involves a Kelvin wave unique to the equatorial area and the Rayleigh-Kuo condition is not violated in the linear shear flow Boyd considered. The inertial instability considered here is again similar to inertial instability observed elsewhere but is significantly modified by the containing equatorial beta-plane geometry. We suspect that further inquiry into this problem in the near future may be somewhat difficult, but may yield new and exciting results.

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