It was also shown that very small changes in $\tilde{u}$ at only one or two levels can, but need not, affect the existence and location of negative $\tilde{q}_i$ regions and associated $\tilde{q}_p$ zeros.

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REFERENCES

An Alternative Expression for the Eady Wave Growth Rate

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ABSTRACT

The energetics of Eady's (1949) model of baroclinic instability are used to express the wavenumber-dependent disturbance growth rate in terms of upward and northward fluxes of heat and momentum. This formulation leads to simple physical interpretations for the existence of the wavelength of maximum growth rate and the shortwave cutoff.

1. Introduction

The mechanism of baroclinic instability, in which perturbations amplify through the conversion of available potential energy to kinetic energy, was isolated theoretically by Charney (1947) and Eady (1949). This instability corresponds closely to the "self-development" process, identified observationally by Sutcliffe and Forsdyke (1950) and discussed by Petterssen (1956, pp. 334–338) and Pálmen and Newton (1969, pp. 324–326), which describes the deepening and intensification of midlatitude cyclones. Although the Eady model contains a number of restrictive assumptions such as the neglect of the

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beta effect, a zero north–south gradient of basic-state potential vorticity, and a rigid-lid upper boundary condition at the tropopause, it abstracts the essential features of the baroclinic instability mechanism with minimal mathematical complexity. In particular, the expression for the disturbance growth rate can be written in a simple closed form:

$$a \tilde{c} = [\text{coth} \alpha - \alpha (\alpha - \tan \alpha)]^{1/2}, \tag{1}$$

where $\alpha$ is nondimensional wavenumber and $\tilde{c}$ is the nondimensional imaginary part of the complex phase speed in the assumed waveform solution of the Eady model.

A shortcoming of (1) is that it does not reveal the physical processes and feedbacks accounting for the existence of a wavenumber of maximum growth rate and the shortwave cutoff. In this note, we re-express (1) in terms of domain-averaged upward and northward fluxes of heat and momentum, which arise from considering the energetics of the Eady model. The alternative expression for $a \tilde{c}$ emphasizes the conflicting effects of vertical motion on enhancing and diminishing the growth rate for varying wavenumber. The Eady model is reviewed very briefly in the following section. The alternative form for $a \tilde{c}$ is derived in Section 3 and interpreted physically in Section 4. The main results are summarized in Section 5.

2. Eady model overview

This section contains a sketch of the derivation of the Eady model in order to “set the stage” for the discussion relating the growth rate to the energetics in Section 3. The reader requiring additional background material as well as details of the derivation is referred to recent dynamic meteorology texts (e.g., Dutton, 1976, pp. 529–531; Pedlosky, 1979, pp. 456–464; Haltiner and Williams, 1980, pp. 83–86). The notation follows closely that of the Hoskins and Bretherton (1972) horizontal shear model of quasi-geostrophic frontogenesis, in which the Eady model is generalized to apply to finite-amplitude disturbances.

The Eady model is derived for an adiabatic, inviscid, Boussinesq atmosphere. The horizontal domain is infinite in the west–east ($x$) and south–north ($y$) directions, over which the Coriolis parameter $f$ is constant. The vertical domain is confined between flat, rigid lids at the surface ($z = 0$) and the tropopause ($z = H$), where

$$z = \frac{c_p \theta_0}{g} \left[ 1 - \left( \frac{p}{p_0} \right)^{R/c_p} \right]. \tag{2}$$

The lids are rigid in the sense that $w = dz/dt$ vanishes along them. In (2), $c_p$ is the specific heat at constant pressure for dry air, $\theta_0$ is a reference potential temperature, $g$ is gravity, $p$ is pressure, $p_0$ is a reference pressure equal to 1000 mb, and $R$ is the ideal gas constant for dry air.

The geopotential field is assumed to take the form

$$\phi = \Phi(y, z) + \phi'(x, z, t), \quad (3a)$$

$$\Phi(y, z) = gz - S^2 y (z - H') + \frac{N^2}{2} z^2, \quad (3b)$$

where $\Phi$ and $\phi'$ refer to the basic-state and disturbance geopotential, respectively, $t$ is time, and $H' = H/2$. The terms $S^2$ and $N^2$ are positive constants defined by $S^2 = -(g/\theta_0)(\partial \Theta/\partial y)$ and $N^2 = (g/\theta_0)(\partial \Theta/\partial z)$, where $\Theta(y, z)$ is the basic-state component of the potential temperature, $\theta$, related hydrostatically to $\phi$ by

$$\frac{\partial \phi}{\partial z} = \frac{g}{\theta_0} \theta. \quad (4)$$

The basic-state zonal and disturbance meridional wind components are geostrophic, so that

$$U_g(z) = -\frac{1}{f} \frac{\partial \Phi}{\partial y}, \quad v_g(x, z, t) = \frac{1}{f} \frac{\partial \phi'}{\partial x}. \quad (5)$$

The Boussinesq form of the continuity equation allows the disturbance zonal wind component, which is ageostrophic, and the vertical velocity to be written in terms of a streamfunction $\psi(x, z, t)$, so that

$$u_{ag} = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x}. \quad (6)$$

When the above basic-state and disturbance quantities are substituted into the quasi-geostrophic thermodynamic and momentum equations, the results are

$$\frac{\partial \phi'}{\partial t} + U_g \frac{\partial \phi'}{\partial x} = \left( \frac{\theta_0 S^2}{g} \right) v_g - \left( \frac{\theta_0 N^2}{g} \right) w, \quad (7)$$

and

$$\frac{\partial v_g}{\partial t} + U_g \frac{\partial v_g}{\partial x} = -f u_{ag}. \quad (8)$$

Taking $\partial \phi'/\partial z$ of (7) and $\partial \phi'/\partial x$ of (8) leads to the prognostic equation for disturbance potential vorticity, which is conserved by the flow. If we assume that the magnitudes of disturbance quantities are vanishingly small at some time, which is consistent with exponential time dependence, then

$$\frac{1}{f^2} \frac{\partial^2 \phi'}{\partial x^2} + \frac{1}{N^2} \frac{\partial^2 \phi'}{\partial z^2} = 0. \quad (9)$$

Assuming normal-mode behavior for $\phi'$ in (9) and applying (7) at $z = 0$ and $z = H$ results in the expression for disturbance growth rate (1), which is expressed in nondimensional form. The maximum growth rate and short-wave cutoff in (1) occur at $\alpha = 0.803058$ and $\alpha = 1.19968$, respectively.
3. Eady model energetics

In order to facilitate the mathematical manipulations and permit comparison with other studies, the model variables are converted to nondimensional form, which is denoted by a tilde unless noted differently. The independent variables are given by \( x = (NH'/f)\tilde{z} \), \( y = (N^2H'/S)\tilde{y} \), \( z = H'(\tilde{z} + 1) \), and \( t = (N/S)\tilde{t} \). The mass variables are written as \( \phi = (N^2H')\tilde{\phi} \) and \( \theta = (\theta_0N^2H'/g)\tilde{\theta} \); the wind variables as \( v_\theta = (NH')\tilde{v}_\theta \), \( u_{ag} = (S^2H'/f)\tilde{u}_{ag} \), \( w = (S^2H'/N)\tilde{w} \) and \( \psi = (S^2H'/f)\tilde{\psi} \). In (1), \( c \), scales identically to \( u_{ag} \) and \( k = (f/NH')\alpha \), where \( k \) is dimensional wavenumber.

The nondimensional prognostic equations for the domain-averaged perturbation available potential energy, perturbation kinetic energy and meridional heat flux, derived from (7) and (8), are

\[
\frac{\partial}{\partial t} \int_{-1}^{1} \frac{1}{2} \tilde{\theta} d\tilde{z} = -\int_{-1}^{1} \tilde{v}_\theta d\tilde{z} + \int_{-1}^{1} \tilde{v}_\psi d\tilde{z},
\]

(10)

\[
\frac{\partial}{\partial t} \int_{-1}^{1} \frac{1}{2} (\tilde{v}_\phi)^2 d\tilde{z} = \int_{-1}^{1} \frac{1}{2} \tilde{w} d\tilde{z},
\]

(11)

\[
\frac{\partial}{\partial t} \int_{-1}^{1} \frac{1}{2} \tilde{u}_{ag} \tilde{\psi} d\tilde{z} = \int_{-1}^{1} (\tilde{v}_\phi)^2 d\tilde{z} - 2 \int_{-1}^{1} \tilde{w} \tilde{d} d\tilde{z},
\]

(12)

where

\[
\int_{-1}^{1} \Phi d\tilde{z} = \frac{\alpha}{2\pi} \int_{0}^{2\pi/a} \Phi \, d\phi,
\]

(13)

The derivation of (12) utilizes the equality

\[
\int_{-1}^{1} \tilde{w} \tilde{d} d\tilde{z} = \int_{-1}^{1} \tilde{u}_{ag} \tilde{\psi} d\tilde{z},
\]

(14)

which can be established by multiplying the nondimensional version of (9) by \( \tilde{\psi} \) and integrating by parts over the domain.

The expressions (10) and (11) can be re-expressed schematically as

\[
\frac{\partial \tilde{A}'}{\partial t} = \tilde{C}(\tilde{A}_n, \tilde{A}') (1 - \tilde{\Delta}_o),
\]

(15)

\[
\frac{\partial \tilde{K}'}{\partial t} = \tilde{C}(\tilde{A}_n, \tilde{A}') \tilde{\Delta}_o,
\]

(16)

where \( \tilde{C}(\tilde{A}_n, \tilde{A}') \) denotes the nondimensional conversion rate of basic-state (zonal) available potential energy to disturbance available potential energy. This conversion term physically corresponds to the domain-averaged meridional heat flux appearing in (10) and (12). In the above equations,

\[
\tilde{\Delta}_o = \int_{-1}^{1} \tilde{w} \tilde{d} d\tilde{z} / \int_{-1}^{1} \tilde{v}_\phi d\tilde{z},
\]

(17a)

\[
\Delta_o = \int_{0}^{H} \int_{0}^{L} w d\theta dx dz / \left( \frac{S}{N} \right)^2 \int_{0}^{H} \int_{0}^{L} u_{ag} d\theta dx dz,
\]

(17b)

are the ratios of the slope of the heat transport vector for the disturbance to that of the basic-state isentropes expressed in nondimensional and dimensional form (Mecho, 1980).

In an analogous manner, one can rewrite (12) schematically as

\[
\frac{\partial \tilde{C}(\tilde{A}_n, \tilde{A}')}{\partial t} = 4 \tilde{K}' (\frac{1}{2} - \tilde{\Delta}_o),
\]

(18)

where

\[
\tilde{\Delta}_o = \int_{-1}^{1} \tilde{w} \tilde{d} d\tilde{z} / \int_{-1}^{1} (\tilde{v}_\phi)^2 d\tilde{z},
\]

(19a)

\[
\Delta_o = \int_{0}^{H} \int_{0}^{L} w v_{ag} dx dz / \left( \frac{S}{N} \right)^2 \int_{0}^{H} \int_{0}^{L} (v_{ag})^2 dx dz,
\]

(19b)

may be interpreted as the ratio of the slope of the momentum transport vector to that of the basic-state isentropes.

On account of the \( \exp(\alpha\tilde{c}_D t) \) temporal dependence of the disturbance quantities, one can express the growth rate as

\[
(\alpha\tilde{c}_D)^2 = \frac{1}{4} \left[ \frac{\partial \ln \tilde{K}'}{\partial t} + \frac{\partial \ln \tilde{C}(\tilde{A}_n, \tilde{A}')}{\partial t} \right].
\]

(20)

Use of (16) and (18) in (20) results in

\[
\alpha\tilde{c}_D = [\tilde{\Delta}_o (\frac{1}{2} - \tilde{\Delta}_o)]^{1/2}.
\]

(21)

The ratios \( \tilde{\Delta}_o \) and \( \tilde{\Delta}_o \) may be evaluated for the Eady model from the analytic solutions for \( w, \tilde{\theta}, \) and \( v_{ag} \) (not shown; see Keyser, 1981, pp. 46–48):

\[
\tilde{\Delta}_o = \frac{2\alpha \coth \alpha - 1}{2},
\]

(22)

\[
\tilde{\Delta}_o = \frac{2\alpha^2}{2\alpha \coth \alpha - 1} - \frac{3}{2}.
\]

(23)

Substituting (22) and (23) into (21) produces the expression for \( \alpha\tilde{c}_D \) given in (1). Fig. 1 contains graphs illustrating the dependence of (21)–(23) on \( \alpha \).

4. Discussion and physical interpretation

The energy flux in an amplifying Eady wave can be considered a two-step process in which: 1) disturbance available potential energy \( \tilde{A}' \) is extracted from an infinite reservoir \( \partial \tilde{\theta}/\partial y \) is constant) of zonal available potential energy \( A_n \) by a net transport of warm air northward \( [10] \) and (15)]; and then 2) converted to disturbance kinetic energy \( K' \) by a net transport of warm air upward \( [11] \) and (16)]. This process is the same as that illustrated schematically
wavenumber) maximizes its production of $\hat{K}'$ through a thermally direct ageostrophic circulation, while simultaneously minimizing the opposing influence of the ageostrophic circulation on the temporal rate of increase of $\hat{C}(\hat{A}_n, \hat{A}')$. This interpretation is consistent with (21), in which $\Delta_\theta$ and $\Delta_\nu$ must be simultaneously maximized and minimized, respectively, to achieve the maximum growth rate with respect to wavenumber. Fig. 1 shows that both $\Delta_\theta$ and $\Delta_\nu$ increase with $\alpha$ as the relative importance of vertical motions increases (Holton, 1979; pp. 235–236). At the short-wave cutoff, $\Delta_\nu = 0.5$; at this wavenumber the production of $\hat{C}(\hat{A}_n, \hat{A}')$ due to geostrophic motions balances the dissipation of $\hat{C}(\hat{A}_n, \hat{A}')$ due to ageostrophic motions (12). According to this analysis, the short-wave cutoff for baroclinic instability in the Eady model does not result from the depletion of $\hat{A}'$ by the ageostrophic circulations, but is a consequence of the inhibition of the development of the mechanism that generates $\hat{A}'$.

5. Summary

The advantage of the alternative expression for the Eady model growth rate (21), which is derived from energetics considerations, over the standard mathematical expression (1), is that the former leads to simple physical interpretations for the existence of the wavenumber of maximum growth rate and the shortwave cutoff. It is apparent from the discussion of (21) that the ageostrophic circulations and their associated vertical motions are necessary for wave growth because they effect the conversion from disturbance available potential energy to disturbance kinetic energy. Nevertheless, stronger vertical motions associated with shorter waves interfere with the development of the northward disturbance heat flux, which is also necessary for wave growth. At the wavenumber of maximum growth rate the vertical motions are "optimal" in the sense that the conversion from disturbance available potential energy to disturbance kinetic energy is maximized at the same time that the inhibiting effect of vertical motions on the development of the northward heat flux is minimized. At the shortwave cutoff, the vertical motions completely suppress the growth process by eliminating the development of the northward heat flux required for wave amplification.

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Bretherton (1966) discusses the shortwave cutoff in the Eady model in terms of the local balance between the advection and generation of disturbance potential vorticity. His interpretation focuses on the details of the disturbance structure, while our approach is concerned with the domain-averaged properties of the disturbance appearing in the energy transformations.
reviewer suggested using the potential vorticity constraint (9) to establish (14). This research was funded by the National Science Foundation through Grant ATM-7802699 and the Naval Environmental Prediction Research Facility through Contract N00014-78-C-0775. The first author's attendance at The Pennsylvania State University during the 1978-79 academic year was sponsored by the National Science Foundation through the final year of a three-year graduate fellowship. Publication of this note was supported by the Severe Storms Branch of the Goddard Laboratory for Atmospheric Sciences, NASA/Goddard Space Flight Center.

REFERENCES


