Note on the Scattering of Radiation by Moderately Nonspherical Particles

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15 March 1982

ABSTRACT

An expression for the surface area of a nonspherical particle described by the equation

$$r = r_0[1 \pm \epsilon T_n(\cos\theta)]$$

is derived, and radii of various equivalent spheres are calculated.

In a recent paper by Mugnai and Wiscombe (1980), light scattering by nonspherical particles was investigated using the Extended Boundary Condition Method (Waterman, 1971; Barber and Yeh, 1975). Rotationally symmetric nonspherical particles were parameterized by

0022-4928/82/081886-03$04.75

$$r = r_0[1 \pm \epsilon T_n(\cos\theta)], \tag{1}$$

where r, θ and ϕ are the usual spherical coordinates (because of rotational symmetry the azimuth ϕ does not appear explicitly), $T_n(\cos\theta)$ is the nth order Chebyshev polynomial and r_0 and ϵ are parameters describing the "radius" and "nonsphericity" of a considered particle.

Mugnai and Wiscombe (1980) compared scattering characteristics of nonspherical particles described by Eq. (1) with scattering characteristics of "equivalent" spheres. Considered equivalent spheres were spheres of equal volume, equal projected area and equal surface area. Here, for example, equal surface area sphere refers to a sphere with surface equal to the surface of a considered nonspherical particle. The projected area A of a convex randomly oriented nonspherical particle is related to its surface area S by the relation $A = S/4$.

To compare the scattering characteristics of nonspherical particles with scattering characteristics of equivalent spheres, we obviously need to calculate volumes and surface areas of nonspherical particles given by Eq. (1). Mugnai and Wiscombe (1980) give the following expressions for the volume V_n and the surface area S_n of a T_n particle [a T_n particle refers to a particle whose surface is described by Eq. (1)]:

$$V_n(\epsilon, r_0) = \frac{4}{3}\pi r_0^3 \begin{cases} \left[1 + \frac{3}{2}\epsilon^2 \frac{4n^2-2}{4n^2-1}\right] & \text{for } n \text{ odd} \\ \left[1 + \frac{3}{2}\epsilon^2 \frac{4n^2-2}{4n^2-1} - \frac{3\epsilon\left(1+\frac{\epsilon^2}{4}\right)}{n^2-1} - \frac{\epsilon^3}{4(9n^2-1)}\right] & \text{for } n \text{ even} \end{cases} \tag{2}$$

and

$$S_n(\epsilon, r_0) = 4\pi r_0^2 \begin{cases} \left[1 - \frac{2\epsilon}{n^2-1} + \frac{\epsilon^2}{2}\frac{4n^2-2}{4n^2-1}\right] & \text{for } n \text{ even} \\ \left[1 + \frac{\epsilon^2}{2}\frac{4n^2-2}{4n^2-1}\right] & \text{for } n \text{ odd.} \end{cases} \tag{3}$$

While the expression for the volume V_n is correct, the surface area S_n, as given by (3), is wrong. The best way to understand the erroneous nature of (3) for the surface area S_n of a T_n particle, is to compare the surface area S_n as given by this equation with the surface area S of a sphere which has the same volume V as the considered nonspherical particle T_n. One obtains $S_n/S < 1$, which is obviously incorrect (Vouk, 1948; Chýlek, 1977), since the sphere has the smallest surface area among all particles with given fixed volume V.

The correct expression for the surface area S of an arbitrary nonspherical particle given by a set of parametric equations

$$\left.\begin{aligned} x &= X(\theta, \phi) \\ y &= Y(\theta, \phi) \\ z &= Z(\theta, \phi) \end{aligned}\right\} \tag{4}$$

is (Courant, 1961)

$$S = \int_0^{2\pi}\int_0^{\pi} (EG - F^2)^{1/2} d\theta d\phi, \tag{5}$$

where

$$\left.\begin{aligned} E &= X_\phi^2 + Y_\phi^2 + Z_\phi^2 \\ F &= X_\phi X_\theta + Y_\phi Y_\theta + Z_\phi Z_\theta \\ G &= X_\theta^2 + Y_\theta^2 + Z_\theta^2 \end{aligned}\right\}, \tag{6}$$

with the subscript ϕ or θ denoting a partial derivative of an appropriate function with respect to ϕ or θ.

In the considered case of a nonspherical particle T_n described by (1), we have

$$\left.\begin{aligned} x &= r_0[1 \pm \epsilon T_n(\cos\theta)] \sin\theta \cos\phi \\ y &= r_0[1 \pm \epsilon T_n(\cos\theta)] \sin\theta \sin\phi \\ z &= r_0[1 \pm \epsilon T_n(\cos\theta)] \cos\theta \end{aligned}\right\}, \tag{7}$$

which leads to the following expression for the surface area S_n of a T_n particle:

$$S_n = 2\pi r_0^2 \int_0^{\pi} (1 + \epsilon T_n) \times [(1 + \epsilon T_n)^2 + \epsilon^2 T_{n\theta}^2]^{1/2} \sin\theta d\theta, \tag{8}$$

where $T_{n\theta}$ represents a derivative of T_n with respect to θ.

Assuming $\epsilon < 1$ and keeping only the terms up to ϵ^6 in Eq. (8), we can write

$$S_n(\epsilon, r_0) = 2\pi r_0^2 \int_0^{\pi} (1 + \epsilon T_n)^2 \left[1 + \frac{\epsilon^2}{2}\left(\frac{T_{n\theta}}{1+\epsilon T_n}\right)^2 - \frac{\epsilon^4}{8}\left(\frac{T_{n\theta}}{1+\epsilon T_n}\right)^4 + \frac{\epsilon^6}{16}\left(\frac{T_{n\theta}}{1+\epsilon T_n}\right)^6\right] \sin\theta d\theta. \tag{9}$$

Finally, after the integration, we obtain

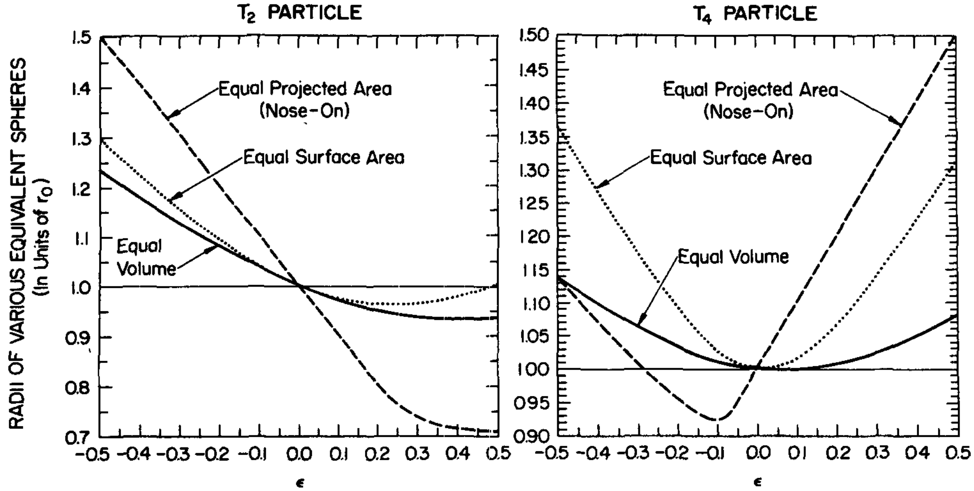

FIG. 1. Replacement for Fig. 2. of Mugnai and Wiscombe (1980), comparing various equivalent sphere radii for nonspherical particles.

$$S_n(\epsilon, r_0) = 4\pi r_0^2 \begin{cases} \left[1 + \dfrac{\epsilon^2(n^4+2n^2-1)}{4n^2-1} - \dfrac{3\epsilon^4 n^4}{64}\left(1 + \dfrac{20n^2-1}{(16n^2-1)(4n^2-1)}\right)\right] & \text{for } n \text{ odd} \\ \left[1 - \dfrac{2\epsilon}{n^2-1} + \dfrac{\epsilon^2(n^4+2n^2-1)}{4n^2-1} - \dfrac{3\epsilon^4 n^8}{(16n^2-1)(4n^2-1)} \right. & \\ \qquad \left. - 6\epsilon^5 n^8 \dfrac{1}{(n^2-1)(9n^2-1)(25n^2-1)}\right] & \text{for } n \text{ even} \end{cases}$$

where we have retained only terms up to ϵ^5.

Thus, although the main conclusions of Mugnai and Wiscombe (1980) remain valid (especially those concerning the equal volume equivalent sphere), a minor revision of Fig. 2 of their paper is required. Fig. 1 of this note gives the corrected version of Mugnai and Wiscombe's Fig. 2 (only the curve labeled "Equal Surface Area" is different).

Acknowledgments. This research was supported in part by the U.S. Army Research Office Grant DAAG2980C0108 and by the National Science Foundation Grant ATM 8007443.

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