Baroclinic Instability and Frontogenesis with Ekman Boundary Layer Dynamics
Incorporating the Geostrophic Momentum Approximation

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ABSTRACT

The baroclinic instability of a two-dimensional uniform potential vorticity flow above a relatively thin viscous boundary layer is examined. The disturbance field is constrained by the geostrophic momentum approximation, and boundary layer dynamics are incorporated by prescribing the vertical velocity, derived by Wu and Blumen, at the bottom boundary of the inviscid layer. Characteristics of the instability and frontogenetical properties of the model are delineated by comparison with the results obtained using Ekman boundary layer dynamics to prescribe the vertical velocity at the boundary.

It is established that the unstable growth rates, phase speeds and qualitative aspects of the frontogenetical process are not significantly different from results obtained using Ekman boundary layer dynamics. However, significant modifications to the vertical velocity field at the lower boundary occur when the amplitude of the relative vorticity at the lower boundary attains a value equal to about $f$, the Coriolis parameter. In comparison with the vertical velocity field associated with Ekman layer dynamics, 1) the upward motion is smaller in cyclonic regions and larger in anticyclonic regions and 2) a broader band of relatively high values of upward motion exists. These features are interpreted in terms of the physical properties of the modified boundary layer dynamics.

1. Introduction

Charney and Eliassen (1949) demonstrated that the dynamical coupling between a relatively thin frictional boundary layer and an inviscid interior flow may be incorporated into a boundary condition on the interior velocity: the so-called Ekman suction velocity. This relatively simple parameterization of Ekman boundary layer processes, in terms of the relative vorticity of the interior flow, may be used in conjunction with quasi-geostrophic flow dynamics in a formally consistent manner. Recently Wu and Blumen (1982, hereafter WB) introduced inertial terms into the boundary layer equations, and derived a modification to the lower boundary condition that is appropriate for use with interior flow dynamics incorporating the geostrophic momentum approximation. The modified suction velocity may also be expressed solely in terms of interior flow variables, although new physical processes are incorporated into the boundary layer dynamics.

The effect on the dynamics of the interior flow by the use of the modified boundary condition will be presented, and compared to results obtained by use of the traditional Ekman boundary layer parameterization. The present development represents an extension of Blumen’s (1980, Section 4 and Appendix) analysis of baroclinic instability and frontogenesis associated with a model of uniform potential vorticity flow. The first step, which simplifies the subsequent analysis, is the transformation of the modified boundary condition from physical space to geostrophic coordinate space, which is carried out in Section 2. Then, in Section 3, a relatively simple homogeneous fluid model is employed to establish the characteristic decay rate of the inviscid flow arising from the use of the modified suction velocity. The stability problem for uniform potential vorticity flow of a stably stratified fluid is setup in Section 4, and some analytical results for the phase speed and growth rate of an unstable wave disturbance are presented. A more complete representation of the stability properties of the flow appears in Section 5, and some final remarks are given in Section 6.

2. Vertical velocity in geostrophic coordinates

The analysis is restricted to a vertical $x$, $z$ plane where ($x$, $z$) represent horizontal and vertical coor-
dinates. However, three-dimensional \((u, v, w)\) motions are considered, where \((u, w)\) are directed along the \((x, z)\) axes, \(v\) is normal to the \(x, z\) plane, and the Coriolis parameter \(f\) is constant. These simplifications permit relatively easy access to both analytical and numerical solutions, essentially because the two-dimensional constraint restricts the nonlinearity to the modified lower boundary condition in geostrophic coordinate space.

The nondimensional variables and constants that appear in the analysis have been defined by WB and by Blumen (1980). The following are frequently employed:

- \((u_T, W_T)\), horizontal and vertical velocity components at the top of the boundary layer.
- \((u_g, v_g)\), geostrophic velocity components.
- \(\zeta_G = 1 + \operatorname{Ro} \partial v_g/\partial x\), the vertical component of the absolute vorticity associated with geostrophic flow.
- \(\operatorname{Ro} < 0.3\), the Rossby number.
- \(E = 2K_v f H^2 < \operatorname{Ro}\), the Ekman number, where \(K_v\) is the eddy viscosity coefficient and \(H < 10^4\) m is the characteristic fluid depth.
- \(\delta_E = (2K_v/f)^{1/2} \sim 10^4\) m is the characteristic Ekman layer thickness.
- \((X, Z, T)\) represent geostrophic coordinate space variables and time, defined as

\[
X = x + \operatorname{Ro} u_g, \quad Z = z, \quad T = t.
\]

The transformation from physical space to geostrophic coordinate space has been provided by Hoskins (1975, Section 4).

The expression for \(W_T\) consists of an essentially inviscid contribution, that arises from an ageostrophic divergence at the top of the boundary layer, and a viscous contribution. The former is developed in Appendix A, while the latter is provided by (34) in WB. The complete expression is

\[
W_T = -\frac{1}{2} \partial u_T / \partial x + \frac{1}{2} \left\{ (1 + \operatorname{Ro} \partial v_g / \partial x)^{-1/2} \partial v_g / \partial x \right\} \\
+ (u_g - \operatorname{Ro} \partial v_g / \partial T) (1 + \operatorname{Ro} \partial v_g / \partial x)^{-1} \\
\times \delta (1 + \operatorname{Ro} \partial v_g / \partial x)^{-1/4} / \partial x \\
+ (v_g + \operatorname{Ro} \partial u_g / \partial T) \delta (1 + \operatorname{Ro} \partial v_g / \partial x)^{-3/4} / \partial x. \tag{2}
\]

The inviscid contribution is given by the first term in (2), where

\[
u_T = (u_g - \operatorname{Ro} \partial v_g / \partial T) (1 + \operatorname{Ro} \partial v_g / \partial x)^{-1} \tag{3}
\]

is given by (26a) in WB. The Ekman suction velocity is recovered from (2) by setting \(\operatorname{Ro} = 0\) and noting that \(\partial u_g / \partial x = 0\), since \(\partial v_g / \partial y = 0\). The additional viscous contributions to \(W_T (\operatorname{Ro} \neq 0)\) have been discussed by WB (Section 4). In particular, the expression in braces in (2) is equivalent to (45) in WB.

The transformation into geostrophic coordinate space is carried out by means of

\[
\frac{\partial}{\partial x} = \left(1 + \operatorname{Ro} \frac{\partial v_g}{\partial x} \right) \frac{\partial}{\partial X} = \left(1 - \operatorname{Ro} \frac{\partial v_g}{\partial X} \right) \frac{\partial}{\partial X}, \\
\frac{\partial}{\partial t} = \left(1 - \operatorname{Ro} \frac{\partial v_g}{\partial X} \right) \frac{\partial}{\partial T}, \\
\frac{\partial}{\partial T} = \frac{\partial}{\partial T}, \tag{4}
\]

where \((X, T)\) are defined by (1). Moreover, the interior vertical velocity \(W\) at the top of the boundary layer is scaled by \(\operatorname{Ro} U_H^{-1}\) while the boundary layer vertical velocity \(W_T\) is scaled by \(\delta_E U_L^{-1}\), where \(U_L\) denote characteristic horizontal velocity and length scales of the interior flow. The matching of these velocities at \(Z = 0\) requires that

\[
\operatorname{Ro} W = E^{1/2} W_T. \tag{5}
\]

By making use of (2)–(5), the appropriate expression for the interior vertical velocity in geostrophic coordinate space may be expressed as

\[
W = \frac{1}{2} E^{1/2} \operatorname{Ro}^{-1} (1 - \operatorname{Ro} \partial v_g / \partial X)^{-1} \\
\times (1 - \operatorname{Ro} \partial v_g / \partial X)^{3/2} \partial v_g / \partial X \\
+ (1 - \operatorname{Ro} \partial v_g / \partial X)^{1/4} (v_g + \operatorname{Ro} \partial u_g / \partial T) \\
\times \partial^2 v_g / \partial X^2 - \frac{1}{4} \delta (1 - \operatorname{Ro} \partial v_g / \partial X)^{-3/4} \\
\times [u_g - \operatorname{Ro} (\partial v_g / \partial T + u_0 \partial v_g / \partial X)] \partial^2 v_g / \partial X^2, \tag{6}
\]

\(Z = 0\).

This expression provides the basis for the following analysis.

3. Viscous decay rate

The viscous decay rate, associated with the modified boundary layer dynamics, will be compared to the characteristic decay rate \((\delta_E / 2^{1/2} H)^{-1}\) that characterizes Ekman layer dynamics. For simplicity, a homogeneous fluid model that conserves potential vorticity will be employed. The basic equation, which employs the geostrophic momentum approximation, is

\[
(1 + \operatorname{Ro} \partial v_g / \partial X)^{-1} (\delta / \delta t + u_0 / \delta X) (1 + \operatorname{Ro} \partial v_g / \partial X) \\
= -(1 + \operatorname{Ro} h)^{-1} (\delta / \delta t + u_0 / \delta X) \operatorname{Ro} h, \tag{7}
\]

where \(h(x, t)\) represents the variable height of the lower boundary surface, the upper boundary surface is assumed to be level, the velocity is independent of height and \(u = u_T\) is defined by (3).

Introduction of geostrophic coordinates casts (7) into

\[
\left( \frac{\partial}{\partial T} + u_g \frac{\partial}{\partial X} \right) \frac{\partial v_g}{\partial X} = \frac{(1 - \operatorname{Ro} \partial v_g / \partial X)}{1 + \operatorname{Ro}} W, \tag{8}
\]

where \(W = (\partial / \partial T + u_g \partial / \partial X) h(X, T)\) is given by (6).
Further, only first-order terms in $R_0$ will be retained and the relatively small depth variations will be neglected. These assumptions provide analytical simplicity without significantly affecting the physical aspects of the solution that will be presented. Then substitution of (6) into (8), noting (5), yields

\[
\left( \frac{\partial}{\partial T} + u_g \frac{\partial}{\partial X} \right) \left( 1 + \frac{1}{2} E^{1/2} \right) \frac{\partial v_g}{\partial X} + \frac{\partial \gamma v_g}{\partial X} + \frac{\partial^2 v_g}{\partial X^2} = 0, \quad (9)
\]

where $\gamma = E^{1/2}/2 R_0$ is the reciprocal of the nondimensional decay time associated with Ekman boundary layer dissipation. It is possible to eliminate the constant translation speed $u_g$ from (9) by the appropriate scaling of the $(X, T)$ coordinates, followed by the change to a coordinate system moving with speed $u_g$. Instead, $u_g = 0$ will be employed to provide essentially the same results.

Integration of (9) with respect to $X$, and rearrangement of terms, yields

\[
\frac{\partial}{\partial T} \left( 1 + \frac{1}{2} E^{1/2} \right) v_g + \gamma v_g - \frac{3}{4} E^{1/2} u_g \frac{\partial v_g}{\partial X} = 0, \quad (10)
\]

where the constant of integration is set equal to zero under the assumption that $v_g$ vanishes at $X = -\infty$. Next, Eq. (10) may be cast into the form

\[
\frac{\partial V}{\partial \tau} + V \frac{\partial V}{\partial X} = 0, \quad (11)
\]

where $V = \frac{3}{4} R_0 \psi^{-1}$, $\tau = R_0 \exp(-\gamma T^*)$ and $T^* = (1 + E^{1/2})^{-1}$.

The Fubini-Ghiro (1935) solution of (11) will be employed [see Blumen, 1980; Eq. (7)]. It is given by

\[
V = -2 \sum_{n=1}^{\infty} (n\tau)^{-1} J_n(n\tau) \sin X, \quad (12)
\]

where $J_n$ is the Bessel function of the first kind. Since $\tau < 1$, $V$ does not become multivalued. The solution for $v_g$ is

\[
v_g = -\frac{8}{3} R_0^{-1} \sum_{n=1}^{\infty} n^{-1} J_n(n R_0^{-1/2} \gamma^{1/2} \psi) \sin X. \quad (13)
\]

An approximate value of the decay rate may be determined by noting that the maximum value of the argument of $J_n$ is $R_0 = 0.3$, and that $J_1(0.3) = 0.1483$ is about an order of magnitude larger than the second term in the series (13). Consequently, an approximate solution is

\[
v_g \approx -\frac{8}{3} R_0^{-1} J_1(R_0^{-1/2} \gamma^{1/2} \psi) \sin X \approx -\frac{8}{3} R_0^{-1} \left( \frac{1}{2} R_0^{1/2} \gamma^{1/2} \psi \right) \sin X = -\frac{8}{3} \gamma^{1/2} \psi \sin X, \quad (14)
\]

where the expression for $J_1$ has been provided by Olver (1964). The decay rate in geostrophic coordinate space is

\[
\gamma(1 + E^{1/2}/2)^{-1} \approx \gamma(1 - E^{1/2}/2),
\]

since $T^* = T(1 + E^{1/2}/2)^{-1}$. Transformation back to physical space may be accomplished by means of

\[
\sin X = \sin(x + R_0 \psi) \approx \sin x + R_0 \psi \cos x. \quad (15)
\]

Insertion of (15) into (14) yields

\[
v_g \approx \frac{-4}{3} \gamma^{1/2} \psi \sin x \quad (16)
\]

where $\gamma^* = \gamma(1 + E^{1/2}/2)^{-1}$. Although $\gamma^* < \gamma$, the dependence of dissipation on the phase of the wave is the most significant effect introduced by the modified boundary layer dynamics. Comparisons between the decay rates, determined from (16), and the Ekman decay rate ($R_0 = 0$) are displayed in Fig. 1.

4. Baroclinic instability and frontogenesis

The present analysis of baroclinic instability and frontogenesis represents a straightforward extension of the analysis presented by Blumen (1980; Section 4 and Appendix). First, some analytical results will be obtained. Then, in the next section, a more complete numerical evaluation of the stability characteristics will be presented.

The interior flow satisfies the condition of uniform potential vorticity, expressed by

\[
(\partial^2/\partial X^2 + \partial^2/\partial Z^2) \Phi = 0, \quad (17)
\]

![Fig. 1. Relative rate of decay $v_g^{-1}\partial v_g/\partial t$ for indicated phases of the wave. The dashed line represents the Ekman decay rate. The parameter values are $\gamma = 0.3$, $R_0 = 0.3$, $E^{1/2}/2 = 0.09$ and $t = 1$ corresponds to approximately 10 h.](image)
where $\Phi$ is the geostrophic streamfunction. The geostrophic velocity $v_g$ and potential temperature $\theta$ are represented by

$$v_g = \partial \Phi / \partial X, \quad \theta = \partial \Phi / \partial Z.$$  \hspace{1cm} (18)

The $X$-component of the geostrophic velocity is given by

$$u_g = \ddot{u} = Z.$$  \hspace{1cm} (19)

The upper boundary condition ($W = 0$) may be expressed in terms of the conservation of potential temperature as

$$\left( \frac{\partial}{\partial T} + \dot{u} \frac{\partial}{\partial X} \right) \frac{\partial \Phi}{\partial Z} - \frac{\partial \Phi}{\partial X} = 0, \quad Z = 1.$$  \hspace{1cm} (20)

The lower boundary condition takes account of a modified suction velocity at the top of the boundary layer, and it is given by

$$\left( \frac{\partial}{\partial T} + \ddot{u} \frac{\partial}{\partial X} \right) \frac{\partial \Phi}{\partial Z} - \frac{\partial \Phi}{\partial X} + W \left( 1 - \text{Ro} \frac{\partial^2 \Phi}{\partial X^2} \right) = 0, \quad Z = 0,$$  \hspace{1cm} (21)

where (17) has been used to replace $\partial^2 \Phi / \partial Z^2$ by $-\partial^2 \Phi / \partial X^2$ in the expression for the conservation of potential temperature. The vertical velocity in (21) is provided by (6) with $u_g(0) = 0$ and $v_x$ expressed by (18). The problem examined by Blumen (1980) is represented by (17)–(21), but

$$\left( 1 - \text{Ro} \frac{\partial^2 \Phi}{\partial X^2} \right) W = \gamma \frac{\partial^2 \Phi}{\partial X^2},$$  \hspace{1cm} (22)

where $\gamma = E^{1/2} / 2 \text{Ro}$.\textsuperscript{2} The present problem represents the counterpart of the two-dimensional Eady (1949) baroclinic instability problem with the inclusion of boundary layer dissipation. The frontogenetical aspects become apparent when the geostrophic coordinate solution is transformed back to physical space. Various features of this transformation have been discussed by Hoskins (1975) and Blumen (1981).

Comparisons will be made between solutions that contain the appropriate suction velocity (6) and the approximation, for this problem, given by (22). First, some analytical properties of the solutions will be established.

Again, to simplify the analysis, only first-order terms will be retained in (6). Then, making use of (18) and (19), condition (21) reduces to

$$\frac{\partial}{\partial T} \left( \frac{\partial \Phi}{\partial Z} + \frac{1}{2} E^{1/2} \frac{\partial^2 \Phi}{\partial X^2} \right) - \frac{\partial \Phi}{\partial X} + \gamma \frac{\partial^2 \Phi}{\partial X^2} \right) = 0, \quad Z = 0.$$  \hspace{1cm} (23)

$$= \frac{3}{4} E^{1/2} \frac{\partial}{\partial X} \left( \frac{\partial \Phi}{\partial X} \frac{\partial^2 \Phi}{\partial X^2} \right), \quad Z = 0.$$  \hspace{1cm} (23)

The numerical integrations presented in the following section establish that the principal contribution to the unstable solution is provided by linear Ekman layer dynamics. The order $E^{1/2}$ corrections affect both the phase speed $c$ and growth rate $\sigma$ of the dominant wave solution. Hoskins (1976, Section 7) has shown that a good analytical approximation to the nonlinear inviscid Eady problem may be provided by expanding $\Phi$ in a power series

$$\Phi = a \Phi_1 + a^2 \Phi_2 + \cdots,$$  \hspace{1cm} (24)

where $a \leq 1$ denotes the initial wave-amplitude. This approach is also suitable for the present purpose of providing analytical expressions for the modifications to both $c$ and $\sigma$ that may be used to interpret the numerical results.

The first-order problem for the determination of

$$\Phi = \text{Re} \{A, \sinh kZ + B \cosh kZ \} e^{i(kX - \sigma T)},$$  \hspace{1cm} (25)

which satisfies (17), (20) and (23), is an eigenvalue problem for the complex phase speed $c = c_r + i c_i$ as a function of wavenumber $k$. The only difference between this system of equations and the system presented by Blumen (1980, Appendix) is the presence of a small $O(E^{1/2})$ linear term in (23). The expression for the unstable growth rate is provided in Appendix B and a very accurate expression for $E^{1/2} / 2 \ll 1$ is given by

$$kc_i = \sigma$$  \hspace{1cm} (26)

$$\approx \sigma_E \left[ 1 - \frac{1}{2} E^{1/2} k \left( \tanh \frac{k}{2} + \coth \frac{k}{2} \right) \right],$$  \hspace{1cm} (26)

where $\sigma_E$ denotes the growth rate for the Eady problem with an Ekman suction velocity at the lower boundary, given by (22). The expression for $\sigma_E$ has been derived by Hide (1969), for example. The principal feature of (26) is an $O(E^{1/2})$ reduction in the growth rate that may be attributed to the *inviscid* ageostrophic divergence contained in (2), which gives rise to the $O(E^{1/2})$ vorticity tendency appearing in Eq. (23).\textsuperscript{3} This feature complements the dissipation associated with the Ekman suction velocity or equivalently the relative vorticity, $\gamma \theta^2 \Phi / \partial X^2$, in (23).

Similarly, the phase velocity $c_E$, associated with the Eady-Ekman model, is also reduced by the presence of the $O(E^{1/2})$ vorticity tendency (Table 2). The expression for $c_E$ is given by (B2) in Appendix B.

The second-order solution in (24) may be represented as

$$\Phi = \text{Re} \{A_2 \sinh 2kZ + B_2 \cosh 2kZ \} e^{i(kX - \sigma_T)},$$  \hspace{1cm} (27)

\textsuperscript{3} However, Professor R. T. Williams has pointed out that (26) includes an error of order $E^{1/2}$ because the inviscid boundary condition is applied at $Z = 0$ rather than at the top of the boundary layer. Although this error is small compared to $\sigma_E$, it is the same order as the correction term in (26).
Further analysis will not be undertaken, but two aspects of the solution (24) may be noted by comparing (27) to (25). First, the relative growth rate of the second-order solution is \(2\sigma\), which tends to counteract the reduced growth rate of the first-order solution given by (26). In fact, the numerical results that are presented in the following section show that a net effect of the \(O(\varepsilon^{1/2})\) terms that appear in the modified boundary condition (6) is to slightly increase the relative growth rate of the unstable disturbance. The second aspect relates to the phase speed, which is the same for both the first- and second-order solutions.

5. Numerical results

The nonlinear stability problem, defined by Eqs. (6) and (17)–(21), has been solved by a numerical approach for the following three cases: 1) \(\gamma = 0.15\) (\(\varepsilon^{1/2} = 0.09\)), 2) \(\gamma = 0.30\) (\(\varepsilon^{1/2} = 0.18\)), and 3) \(\gamma = 0.45\) (\(\varepsilon^{1/2} = 0.27\)), where \(\varepsilon \approx 0.3\) in each case. These parameters satisfy the constraint \(\varepsilon < \varepsilon_0\) imposed by the boundary layer scaling in WB. A wave of the form (25) defines the initial disturbance. The complex amplitudes \((A_1, B_1)\) and wavenumber \(k\) were determined by solving the linear stability problem, using (22) rather than (6), to find \(a(k)\). The undetermined amplitude was specified as \(a = 0.3\), but the initial amplitudes \(|\Phi(T = 0)|\) differ because the amplitudes \((A_1, B_1)\) depend on \(\gamma\). The growth rates \(\sigma(k)\) do not differ appreciably from their maxima in the vicinity of \(k = 1.5\) (wavelength \(\approx 4.2 \times 10^6\) m), which was chosen to represent the initial wave in all three cases.

Hoskins' (1976, Appendix) numerical technique was employed, using 43 equally spaced points along the \(X\)-axis \((\Delta X = 10^5\) m) and 21 equally spaced points along the \(Z\)-axis \((\Delta Z = 5 \times 10^6\) m). The time step \(\Delta T = 0.01\) corresponds to \(\sim 6\) min. Tests using 8 and 16 waves to represent the flow fields showed little difference in accuracy, and both reproduced analytical results with negligible error. The reason is that the two waves \((k, 2k)\) make the most significant contribution to each solution. This feature is evident in Fig. 2, which shows the amplitude growth for these two waves. The growth rate of the dominant wave \((k)\) is virtually the same as the linear growth rate. The initial vorticity field is characterized by \(|1 - \varepsilon^{3/2} \phi/\partial X^2| = 0.8\), and the integrations were terminated when \(|1 - \varepsilon^{1/2} \phi/\partial X^2| = 0.1\), which first occurs at the upper boundary \(Z = 1\). Finally, the transformation \((X, Z, T) \rightarrow (x, z, t)\) is accomplished by means of

\[
x = X - \varepsilon \phi \phi/\partial X, \quad z = Z, \quad t = T,
\]

at each point in the domain.

The difference between the termination times for the linear and nonlinear integrations are shown for each case in Table 1. These results confirm that the \(O(\varepsilon^{1/2})\) dissipation that accompanies the use of the modified boundary condition (6), noted in (26), is counterbalanced by an increased disturbance growth when the nonlinear terms are included in the lower boundary condition (see Fig. 2). The net effect is a slightly larger growth rate than the linear growth rate.

The fields of \(w(x, z)\) are shown in Fig. 3, for the final times corresponding only to case 1, since cases 2 and 3 exhibit similar characteristic features. There are quantitative differences between the results obtained using (6) and (22) to evaluate the suction velocity. However, the most significant differences are confined to low levels, and the qualitative aspects of frontogenesis are as described by Blumen (1980) using (22) at \(z = 0\). Consequently attention will be directed to a comparison between the vertical velocity fields (6) and (22) at \(z = 0\). However, the last term in (6) was not evaluated, since the ratio of the last two terms in (6) at termination of the integrations is

![Fig. 2. Amplitude growth as a function of time for wavenumbers \(k = 1.5\) (solid) and \(2k = 3.0\) (dashed), determined from the numerical integrations for each case defined in the text.](image)

<table>
<thead>
<tr>
<th>Case</th>
<th>Nonlinear</th>
<th>Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.68</td>
<td>6.95</td>
</tr>
<tr>
<td>2</td>
<td>8.91</td>
<td>9.33</td>
</tr>
<tr>
<td>3</td>
<td>11.24</td>
<td>11.61</td>
</tr>
</tbody>
</table>
FIG. 3. Vertical velocity $w(x, z)$ for case 1 at the termination times given in Table 1: (a) Ekman boundary layer parameterization (linear), (b) modified boundary layer parameterization (nonlinear). The dashed line represents the axis of maximum cyclonic relative vorticity. The units are $10^{-2}$ m s$^{-1}$ and the distance between tick marks is $\Delta x = 400$ km.

\[
\frac{1}{4} \text{Ro}^2 \left(1 - \text{Ro} \frac{\partial^2 \Phi}{\partial x^2}\right)^{-3/4} \frac{\partial^2 \Phi}{\partial x \partial T} \approx \frac{\text{Ro}_{cr}}{3 \left(1 - \text{Ro} \frac{\partial^2 \Phi}{\partial x^2}\right)^{1/2}} \approx \frac{(0.3)(0.4)}{3(0.1)^{1/2}} \approx 0.13.
\]

There are three remaining contributions to $w$:

\[
w = w_1 + w_2 + w_3.
\]

The physical space representations, provided by (2) and (3), are

\[
w = \frac{1}{2} \text{E}^{1/2} \frac{\partial}{\partial x} \left[(1 + \text{Ro} \partial v_e/\partial x)^{-3/4} \partial v_e/\partial t\right]
\]

where $\gamma = \text{E}^{1/2}/2 \text{Ro}$.

A comparison of the vertical velocity at $z = 0$ with the Ekman suction velocity, $w_e = \gamma \partial v_e/\partial x$, appears in Fig. 4. Note that the frontogenetical characteristics of the Ekman suction velocity occur as a consequence of the transformation from geostrophic coordinate space to physical space. Further, the wave disturbances move in the positive $x$-direction at phase speed $c_e$, shown in Table 2. The differences between the analytically determined phase speeds (B2) shown and the numerically determined values are negligible. The appropriate values of $c_e$ have been subtracted out, in order to display the stationary disturbances shown in Fig. 4.

\[
\begin{array}{ccc}
\text{Case} & \text{Nonlinear} & \text{Linear} \\
1 & 0.448 & 0.462 \\
2 & 0.420 & 0.437 \\
3 & 0.406 & 0.422 \\
\end{array}
\]

FIG. 4. Vertical velocities $w(x, 0)$ at the (a) initial time and at the (b) termination times for case 1. The dashed line represents the Ekman suction velocity. The distance between tick marks is $\Delta x = 200$ km. The arrow denotes the position of maximum cyclonic relative vorticity associated with the modified suction velocity parameterization. $\text{Ro} = 0.3$, the velocity scale $U = 30$ m s$^{-1}$, and the aspect ratio is $H/L = 10^{-2}$, so that $w_e = 1$ corresponds to a dimensional value $\text{Ro}UH/L = 9 \times 10^{-2}$ m s$^{-1}$.
Hoskins (1975) has shown that

\[ \zeta_{GM} = 1 + R \partial \nu_y / \partial x = (1 - R \partial \nu_y / \partial x)^{-1} \]

\[ = (1 - R \partial^2 \Phi / \partial x^2)^{-1}. \]  \hspace{1cm} (32)

Consequently, the initial cyclonic relative vorticity \( R \partial \nu_y / \partial x \) is 0.25 or, equivalently, \( f/4 \). The termination values used in the construction of Fig. 4 correspond to a cyclonic relative vorticity equal to \( 9f \). However, these values occur at the upper boundary. The corresponding values at the lower boundary, \( z = 0 \), appear in Table 3. The frontolynic nature of the boundary layer parameterization is evident, although it is less pronounced with the nonlinear modification.

The significant nonlinear features are:

1) The relatively smaller values of the upward motion in cyclonic regions\(^4\) and the relatively larger values of downward motion in anticyclonic regions; and

2) The relatively broader band of upward motion.

The presence of these features may be understood by examining each contribution to the total field of motion, shown in Fig. 5. First, the Ekman velocity \( w_E = \gamma \partial \nu_y / \partial x \) will always have a maximum amplitude at extrema of the relative vorticity. The comparable nonlinear term is represented by \( w_2 \) in (30b). Vertical motion in both cases is associated with the divergence of the viscous stress at the lower rigid surface, which produces a mass flux toward low pressure or, equivalently, toward centers of cyclonic relative vorticity (Pedlosky, 1979; Wu and Blumen, 1982). However, the relative vorticity in \( w_2 \) is divided by the \( \sqrt{3} \) power of the absolute vorticity. Consequently, \( w_2 < w_E \) in cyclonic regions and \( w_2 > w_E \) in anticyclonic regions for the same value of \( \partial \nu_y / \partial x \). Although the amplitudes of \( \partial \nu_y / \partial x \) are not exactly the same, as shown in Table 3, feature 1) may still be attributed to these inequalities between \( w_2 \) and \( w_E \). Feature 2) may be attributed to \( w_2 \), given by (30c), which WB relate to the advection of geostrophic relative vorticity by the depth-integrated cross-isobar flow in the boundary layer. Since the depth-integrated flow is directed to the left of \( v_E \), the net motion will be directed into cyclonic regions and out from anticyclonic regions. Consequently, the advection of relative vorticity by this flow will be either positive or zero in both situations. This process is reflected in \( w_3 \); the depth-integrated flow is opposite in sign to \( v_E \) and, consequently, \(-v_E \partial^2 \nu_y / \partial x^2 \approx (kv_E)^2 \geq 0\). Although \( w_3 = 0 \) at extrema of \( \partial \nu_y / \partial x \), the nonzero contribution broadens the field of upward motion in cyclonic regions. The net effect produced in anticyclonic regions will be considered below.

The divergence of the inviscid ageostrophic motion

\(^4\) The same feature occurs if the linear and nonlinear cases are compared at the same time.

Table 3. Amplitudes of positive relative vorticity z = 0, 1 in units of \( f \), the Coriolis parameter.

<table>
<thead>
<tr>
<th>Case</th>
<th>( z = 1 )</th>
<th>( z = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(nonlinear)</td>
<td>(linear)</td>
</tr>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.15</td>
</tr>
<tr>
<td>4</td>
<td>1.47</td>
<td>1.12</td>
</tr>
<tr>
<td>9</td>
<td>2.18</td>
<td>1.46</td>
</tr>
<tr>
<td>2</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>4</td>
<td>0.75</td>
<td>0.59</td>
</tr>
<tr>
<td>9</td>
<td>0.99</td>
<td>0.72</td>
</tr>
<tr>
<td>3</td>
<td>0.08</td>
<td>0.08</td>
</tr>
<tr>
<td>4</td>
<td>0.47</td>
<td>0.39</td>
</tr>
<tr>
<td>9</td>
<td>0.57</td>
<td>0.44</td>
</tr>
</tbody>
</table>

At the top of the boundary layer is responsible for the vertical motion associated with \( w_1 \), given by (30a). The principal contribution is expressed by the vorticity tendency. Both the unstable growth and the translation of the vorticity field contribute to this tendency. Consequently, extrema of \( w_1 \) occur in advance of extrema of \( \partial \nu_y / \partial x \). This feature tends to displace the position of the maximum upward motion. However, the net contribution from the sum of \( w_1 \) and \( w_3 \) is essentially symmetric about the position of the relative vorticity maximum, so that the total vertical velocity field, shown in Fig. 4, shows only a slight displacement.

6. Concluding remarks

Modifications to Ekman boundary layer dynamics by the inclusion of inertial terms, under the geostrophic momentum approximation, also modifies the suction velocity \( w \) at the top of the boundary layer. The effect of this modification on the stability properties of a simple model of uniform potential vorticity flow has been examined. On this basis, it has been established that some stability properties, such as the growth rate and phase speed of the disturbance are primarily related to physical processes associated with Ekman layer dy-
namic. However, dynamical processes, associated with both boundary layer accelerations and ageostrophic divergence at the top of the boundary layer, introduce quantitative modifications to the lower vertical motion field that become significant when the relative vorticity is comparable in magnitude to the Coriolis parameter. These features are primarily associated with the relationship between magnitude and position of the maximum velocities and the relative vorticity field.

The present results point-up the sensitivity of the low-level vertical motion field to the parameterization of boundary layer dynamics. Secondary effects associated, for example, with cloud formation and latent heat release need to be considered by the use of more complex dynamical models.

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APPENDIX A

Inviscid Contribution to the Vertical Velocity at the Top of the Boundary Layer

There is an inviscid contribution to the vertical velocity at the top of the boundary layer, as a consequence of ageostrophic divergence, that is independent of depth. The order of magnitude of this contribution \( W_I \) to the vertical velocity field is

\[
W_I \sim -\left(\partial u_I/\partial x\right) \eta
\]

(A1)

where \( \eta \approx 1 \) is the characteristic nondimensional Ekman layer depth and \( u_I \) is given by (3). The expression for the inviscid contribution provided by WB [Eq. (30)] is represented as a limit, in which the scaled vertical coordinate recedes to infinity. The relevant aspects of this limiting process in the boundary layer are discussed in detail by Pedlosky (1979, Section 4.4). However, the inviscid contribution should be essentially independent of this limiting process, although the characteristic boundary layer depth is not necessarily \( \eta \). Consequently, it is assumed that the inviscid contribution may be represented by

\[
W_I \approx -\partial u_I/\partial x[1 - \frac{1}{2}(1 + \hbox{Ro} \partial v_g/\partial x)^{-1/4}],
\]

(A2)

where \( \eta = 1 \) in WB [Eq. (30)]. The bracketed term in (A2) represents the characteristic depth of the boundary layer. A suitable approximation for the parameter range considered in Section 5 is provided by

\[
W_I \approx -\frac{1}{2} \partial u_I/\partial x[1 - \frac{1}{2}(1 - \frac{1}{4} \hbox{Ro} \partial v_g/\partial x)]
\]

\[
\approx -\frac{1}{2} \partial u_I/\partial x,
\]

(A3)

which appears in (2).

APPENDIX B

Analytical Expressions for the Growth Rate and Phase Speed

The growth rate \( \sigma \) and phase speed \( c_r \) may be determined from the O(a) eigenvalue problem defined by (17)-(20) and (23). The following approximate expressions may be derived, assuming that \( k \approx 1 \) and \( E^{1/2} \ll 1 \):

\[
\sigma \approx (\hat{a} - \frac{1}{2} \gamma^* M),
\]

(B1)

\[
c_r \approx \frac{1}{2} - \left( \frac{\hat{b}}{k} + \frac{E^{1/2}}{4} \right).
\]

(B2)

The parameters in (B1) and (B2) are

\[
\hat{a} = \left[ \frac{1}{2} x^* + \left( \frac{\gamma^* M}{2} \right) \right] + \left[ \frac{1}{2} \left( x^* + \left( \frac{\gamma^* M}{2} \right)^2 \right) \right] + (k \gamma^*) \left[ 1 - \frac{M}{2} \left( 1 + \frac{E^{1/2}}{2} \right)^2 \right]^{1/2},
\]

\[
\hat{b} = \left[ \frac{1}{2} \left( x^* + \left( \frac{\gamma^* M}{2} \right)^2 \right) \right]^{1/2},
\]

\[
M = \frac{1}{2}(\coth k/2 + \tan h k/2),
\]

\[
x^* = (\coth k/2 - k/2)(k/2 - \tan h k/2) \times (1 - \frac{1}{2} E^{1/2} M),
\]

\[
\gamma^* = \gamma(1 - \frac{1}{2} E^{1/2} M).
\]

When \( E^{1/2} \) terms are omitted but \( \gamma \) is retained, then \( (\sigma, c_r) \) reduce to the exact eigenvalues for the Eady-Ekman problem.

REFERENCES


