Rossby-Number Similarity in the Atmospheric Boundary Layer Over a Slightly Inclined Terrain

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ABSTRACT

The structure of the steady-state flow, homogeneous along an inclined, flat, underlying surface, is studied. On the basis of the atmospheric boundary layer equations the resistance laws of geostrophic drag and heat transfer are obtained. The general form of the resistance law universal functions is found. The slope influence is shown in the figures, which were obtained by numerical solution of the resistance laws.

1. Introduction

During the last two decades, in the field of boundary layer meteorology, there has been great interest in the geostrophic drag and heat transfer laws. This is because of practical demand for a simple method to combine small-scale boundary-layer characteristics (turbulent vertical fluxes) with large-scale parameters in numerical weather forecasting models. The resistance laws were first derived by Kazanskii and Monin (1961) for the barotropic and adiabatic case. It has been extended to the diabatic, barotropic case by Monin and Zilitinkevich (1967) and simultaneously by Blackadar (1967). Yordanov and Wippermann (1972) worked out the resistance law in the case of baroclinicity. Further improvement was connected with the inclusion of non-stationary (Zilitinkevich and Deardorff, 1974; Laikhtman and Yordanov, 1979; Yordanov, 1980). Recently Gutman and Melgarejo (1981) took into consideration terrain inclination effects.

The structure of the atmospheric boundary layer is considerably modified by the influence of terrain inclination. Even a 10° slope generates a buoyancy force directed along the slope comparable in magnitude to the Coriolis force (Lykosov and Gutman, 1972) and influences the flow in the boundary layer (Brost and Wyngaard, 1978). The "slope effect" is connected with the horizontal temperature and pressure gradients due to temperature changes in the vicinity of the sloped surface (two points on the same horizontal plane have different temperatures because they are at different distances from the thermally active terrain surface). Slope flow or drainage flow originating over a sloped underlying surface has been the subject of many studies. Summaries of these works and related references can be found, e.g., in Defant (1951), Thyer (1966), Gutman (1969), Manins and Sawford (1979).

Previously, the earth's surface was assumed to be horizontal in similarity theories. Taking the slope effect into consideration, we can expect that this will decrease the scatter of empirical resistance law universal functions.

Gutman and Melgarejo (1981) obtained their resistance laws by solving governing equations for an arbitrary form of the turbulent exchange coefficient. In this paper we use a slightly changed form of the governing equations and a somewhat different approach is put forth. The form of the resistance law universal functions is derived without assuming the form of the turbulent exchange coefficient, for more realistic heat flux distribution and improved boundary conditions.

2. Governing equations

We consider the stationary turbulent boundary layer over an unbounded homogeneous rough plane, inclined to the horizontal plane by a small angle. We will formulate the problem by imposing small deviations on the basic state of atmosphere. The free atmosphere motion, extended downward to the reference ground level, will be assumed to be a basic state (Gutman, 1969). We assume that the earth's surface is warmed or cooled in such a way that the difference between temperature at a point on the underlying surface and at a point at the same height above sea level in the free atmosphere is constant for all points along the sloping surface. The vertical temperature gradient in the free atmosphere is assumed to be constant. The case for which our theory is valid is shown in Fig. 1. Under this crucial assumption [also used

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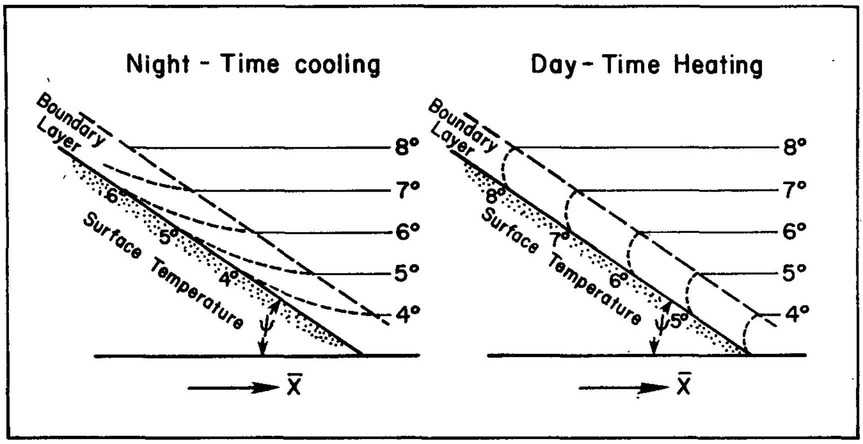

FIG. 1. Distribution of temperature isopleths in the free atmosphere (solid) and in the boundary layer (dashed) during night and day.

by other investigators (e.g., Lykosov and Gutman, 1972; Gutman and Melgarejo, 1981)] the problem can be treated as one dimensional, depending upon the normal distance from the ground, and stationary.

The motion over sloped terrain, which is governed by the dynamic balance between the Coriolis force, frictional force and drainage force (component of the Archimedian force parallel to the slope), and also by the thermal balance between turbulent, drainage and radiative heat fluxes, can be described as (Gutman, Melgarejo, 1981)

$$\left.\begin{aligned} \frac{d}{dz}\left(k\frac{du'}{dz}\right) + sfv' - \Theta'\beta\psi = 0 \\ \frac{d}{dz}\left(k\frac{dv'}{dz}\right) - sfu' = 0 \\ \frac{d}{dz}\left(\alpha_H k\frac{d\Theta'}{dz}\right) + u'\gamma\Psi = \frac{dR'}{dz} \end{aligned}\right\}, \qquad (1)$$

where

$$\left.\begin{aligned} u' &= u - G\cos\chi \\ v' &= v - G\sin\chi \\ \Theta' &= \Theta - \Theta_F \\ R' &= R - R_F \\ \Theta_F(\bar{z}) &= \Theta_F(0) + \gamma\bar{z} \end{aligned}\right\}, \qquad (2)$$

and

u, v	components of wind velocity
G	geostrophic wind velocity
χ	angle between the geostrophic wind vector and the x-axis
f	modulus of the Coriolis parameter
s	sign of the Coriolis parameter (=1 for the Northern Hemisphere and −1 for the Southern Hemisphere)
k	turbulent exchange coefficient
R, R_F	radiative heat flux in the boundary layer and in the free atmosphere; we assume that $R' = R - R_F$ is a linear function of height (see Appendix B)
α_H	inverse turbulent Prandtl number
β	buoyancy parameter [$=g/\Theta_0$]
Θ, Θ_0, Θ_F	potential temperature in the boundary layer, its mean value on the surface, and temperature in the free atmosphere
γ	vertical gradient of free atmosphere potential temperature
ψ	angle of terrain slope
z	vertical axis of Cartesian coordinates normal to the earth's surface, the origin of which is taken at an arbitrary point on the underlying surface; x-axis is oriented downward along the maximum slope
$\bar{z}$	the vertical axis.

It should be noted that due to the smallness of Ψ, the values u, v, G are assumed to be the same as for horizontally oriented coordinates.

The form of the governing equations (1) differs from those used by Gutman and Melgarejo (1981) only by a radiative term inserted into the third equation. In the limit of zero slope this term provides for a linear decrease in the turbulent heat flux with height.

For the bottom and the top of the boundary layer we can write:

$$\left.\begin{aligned} &\text{For } z = z_0: \\ &u' = -G\cos\chi \\ &v' = -G\sin\chi \\ &\Theta' = \Theta_0' = \text{constant} \\ &\qquad \sim\begin{cases} >0 & \text{during day} \\ <0 & \text{during night} \end{cases} \quad \text{(Fig. 2)} \\ &R' = R'(0) \end{aligned}\right\}, \qquad (3)$$

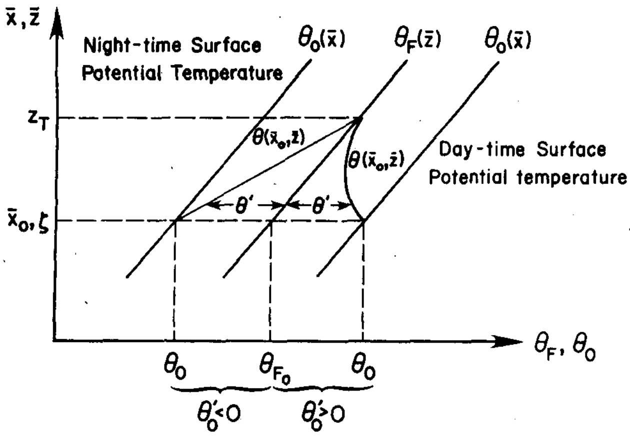

FIG. 2. Free atmosphere and boundary layer potential temperature distributions for day ($\Theta_0' > 0$) and night ($\Theta_0' < 0$).

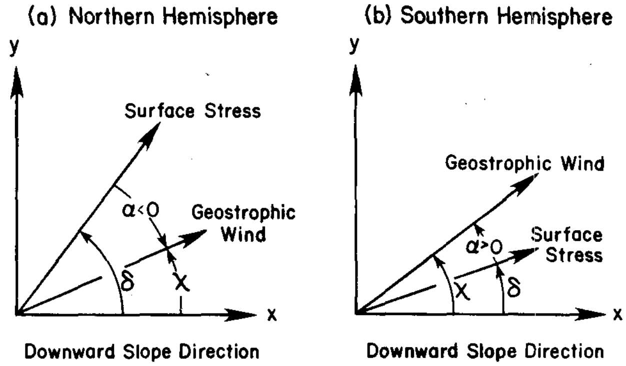

FIG. 3. Orientation of the angles α, δ, χ. Note that always $\alpha = \chi - \delta$, independent of the hemisphere.

$$\left.\begin{aligned} k\frac{du'}{dz} &= u_*^2 \cos\delta \\ k\frac{dv'}{dz} &= u_*^2 \sin\delta \\ \alpha_H k \frac{d\Theta'}{dz} &= \kappa u_* T_* \end{aligned}\right\} . \qquad (4)$$

For $z = z_T$:

$$\left.\begin{aligned} u' &= 0 \\ v' &= 0 \\ \Theta' &= 0 \\ R' &= 0 \end{aligned}\right\} , \qquad (5)$$

$$\left.\begin{aligned} k\frac{du'}{dz} &= 0 \\ k\frac{dv'}{dz} &= 0 \\ k\frac{d\Theta'}{dz} &= 0 \end{aligned}\right\} , \qquad (6)$$

where z_0 is the roughness length, z_T the boundary layer height, u_* the friction velocity, T_* the friction temperature $[= -Q_0/(\kappa c_p \rho u_*)]$, Q_0 a surface value of the heat flux, κ the Kármán constant, c_p the specific heat of the air at constant pressure, ρ the air density, and δ the angle between the surface stress and the x-axis (see Fig. 3). Unlike Gutman and Melgarejo (1981) we assume that the boundary layer height z_T is finite.

We convert the set of the governing equations into a new nondimensional form which enables us to formulate the Rossby-number similarity theory of the atmospheric boundary layer over a slightly inclined terrain. After some simple transformations set (1) can be written in the form

$$\left.\begin{aligned} &\frac{d^2}{dz^2}\left(k\frac{du'}{dz}\right) + sf\frac{d}{dz}\left(v' - s\frac{\beta\psi}{f}\Theta'\right) = 0 \\ &\frac{d^2}{dz^2}\left[k\frac{d}{dz}\left(v' - s\frac{\beta\psi}{f}\Theta'\right)\right] - sf\left(1 + \frac{\beta\gamma\psi^2}{\alpha_H f^2}\right)\frac{du'}{dz} = 0 \\ &\frac{d}{dz}\left(k\frac{d\Theta'}{dz}\right) + s\frac{\gamma\psi}{\alpha_H f}\frac{d}{dz}\left(k\frac{dv'}{dz}\right) = \frac{1}{\alpha_H}\frac{dR'}{dz} \end{aligned}\right\} , \qquad (7)$$

where the assumptions that α_H is constant with height and that R' is a linear function of height were used.

We now define new variables

$$\left.\begin{aligned} x &= k\frac{du'}{dz} \\ y &= k\frac{dv'}{dz} \\ h &= k\frac{d\Theta'}{dz} \end{aligned}\right\} . \qquad (8)$$

It will be convenient to introduce the following nondimensional variables:

$$\left.\begin{aligned} &X = \frac{x}{u_*^2}, \quad Y = \frac{y}{u_*^2}, \quad H = \frac{h}{\kappa u_* T_*} \\ &Z = \frac{z}{L_S}, \quad K = a\frac{k}{L_S^2 f} \\ &R_n' = \frac{R'}{\kappa \alpha_H u_* T_*}, \quad \mu_0 = \frac{L_E}{L_*}, \quad \mu_s = a\mu_0 \end{aligned}\right\} , \qquad (9)$$

where

$$L_* = \frac{u_*^2}{\kappa^2 \beta T_*}, \quad L_E = \frac{\kappa u_*}{f},$$

$$L_S = aL_E, \qquad a^2 = \left(1 + \frac{\beta\gamma\psi^2}{\alpha_H f^2}\right)^{-1}.$$

Inserting (9) into (7), (8) and (1) we obtain the nondimensional equations:

$$\left.\begin{aligned} &\frac{d^2X}{dZ^2} + s\frac{(aY - s\eta H)}{K} = 0 \\ &\frac{d^2}{dZ^2}(aY - s\eta H) - s\frac{X}{K} = 0 \\ &-\frac{dH}{dZ} = s\frac{1-a^2}{a\eta}\frac{dY}{dZ} - \frac{dR'_n}{dZ} \end{aligned}\right\}, \tag{10a}$$

$$\left.\begin{aligned} X &= K\frac{d}{dZ}\left(\kappa\frac{u'}{u_*}\right) \\ Y &= K\frac{d}{dZ}\left(\kappa\frac{v'}{u_*}\right) \\ H &= K\frac{d}{dZ}\left(\frac{\Theta'}{T_*}\right) \end{aligned}\right\}, \tag{10b}$$

$$\left.\begin{aligned} \frac{dX}{dZ} &= \left(\eta\frac{\Theta'}{T_*} - as\kappa\frac{v'}{u_*}\right) \\ \frac{dY}{dZ} &= sa\kappa\frac{u'}{u_*} \end{aligned}\right\}, \tag{10c}$$

where $\eta = \mu_s\psi/\kappa^2$. We assume that K is a function of Z and μ_s. Boundary conditions (3)–(6) take the form

For $Z = Z_0$:

$$\left.\begin{aligned} &\kappa\frac{u'}{u_*} = -\kappa\frac{\cos\chi}{c_g}, \quad X = \cos\delta \\ &\kappa\frac{v'}{u_*} = -\kappa\frac{\sin\chi}{c_g}, \quad Y = \sin\delta \\ &\frac{\Theta'}{T_*} = \frac{\Theta'_0}{T_*}, \quad H = \frac{1}{\alpha_H^0} \\ &R'_n = R'_n(0) \end{aligned}\right\}. \tag{11}$$

For $Z = Z_T$

$$\left.\begin{aligned} &\kappa\frac{u'}{u_*} = 0, \quad \kappa\frac{v'}{u_*} = 0, \quad \frac{\Theta'}{T_*} = 0, \quad R'_n = 0 \\ &X = 0, \quad Y = 0, \quad H = 0 \end{aligned}\right\}, \tag{12}$$

where c_g is the geostrophic drag coefficient $[=u_*/G]$ and α_H^0 = value of α_H for neutral conditions. The third equation in (10a) can be also written in the form

$$\frac{d}{dZ}\left[K\frac{d}{dZ}\left(\frac{\Theta'}{T_*} + s\frac{1-a^2)}{a\eta}\kappa\frac{v'}{u_*}\right)\right] = \frac{dR'_n}{dZ}. \tag{10d}$$

The solutions X, Y, H of the set (10a) can be expressed in the form (see Appendix B)

$$\left.\begin{aligned} X &\approx r_1\cos\delta - r_2 s(a\sin\delta - s\eta_0) \\ Y &\approx (1-a^2)\sin\delta \\ &\quad + sa\eta_0 + as[r_1 s(a\sin\delta - s\eta_0) + r_2\cos\delta] \\ &\quad - as\eta R'_n(0)\frac{Z}{Z_T} \\ H &\approx \frac{1}{\alpha_H^0} - R'_n(0)\frac{Z}{Z_T} \\ &\quad + s\frac{(a^2-1)}{a\eta}\Big\{-a^2\sin\delta + as\eta_0 \\ &\quad + as[r_1 s(a\sin\delta - s\eta_0) + r_2\cos\delta] \\ &\quad - as\eta R'_n(0)\frac{Z}{Z_T}\Big\} \end{aligned}\right\}, \tag{13}$$

where r_1, r_2 are functions of Z and μ_s, as defined in Appendix B, and

$$R'_n(0) = \frac{1}{\alpha_H^0} - s\frac{(a^2-1)}{a\eta}\sin\delta.$$

For $Z = 0$, $r_1 = 1$, $r_2 = 0$; for $Z = Z_T$, $r_1 = r_2 = 0$.

3. Universal defect profiles

We can accept as did Wippermann and Yordanov (1972) that for sufficiently large values of Z, quantities X, Y, H are functions of Z, μ_s, a, η, δ and independent of Z_0, and are universal in the sense of Rossby-number similarity. From that and (10c), (10d), we find that the following quantities are also universal:

$$\left.\begin{aligned} &\kappa\frac{u - G\cos\chi}{u_*} = F_u(Z, \mu_s, a, \eta, \delta) \\ &as\kappa\frac{v - G\sin\chi}{u_*} - \eta\frac{\Theta - \Theta_F}{T_*} \\ &\qquad = F_v(Z, \mu_s, a, \eta, \delta) \\ &\frac{\Theta - \Theta_F}{T_*} + s\frac{(1-a^2)}{a\eta}\kappa\frac{v - G\sin\chi}{u_*} \\ &\qquad = F_\Theta(Z, \mu_s, a, \eta, \delta) \end{aligned}\right\}, \tag{14}$$

where F_u, F_v, and F_Θ are nondimensional universal functions.

For the horizontal case we have: $a = 1$, $(1 - a^2)/(a\eta) = 0$, $L_s = L_E$, $\mu_s = \mu_0$ and Eq. (12) takes the classic form of wind and temperature defect profiles.

Since we assumed a small slope angle, the surface layer structure is not strongly effected by terrain inclination and close to the ground we have

$$\left.\begin{aligned} \kappa\frac{u}{u_*} &= \cos\delta\left[\ln\frac{Z}{Z_0}+\psi\left(\frac{z}{L_*}\right)\right] \\ \kappa\frac{v}{u_*} &= \sin\delta\left[\ln\frac{Z}{Z_0}+\psi\left(\frac{z}{L_*}\right)\right] \\ \frac{\Theta-\Theta_F}{T_*} &= \frac{1}{\alpha_H^0}\left[\ln\frac{Z}{Z_0}+\varphi\left(\frac{z}{L_*}\right)\right]+\frac{\Theta_0-\Theta_F}{T_*} \end{aligned}\right\}, \quad (15)$$

If we assume the existence of a layer close to the surface where (14) and (15) are fulfilled simultaneously, then

$$\left.\begin{aligned} &\cos\delta\left[\ln\frac{Z}{Z_0}+\psi\left(\frac{z}{L_*}\right)\right]-\kappa\frac{G\cos\chi}{u_*} = F_u(Z,\mu_s,a,\eta,\delta) \\ &as\sin\delta\left[\ln\frac{Z}{Z_0}+\psi\left(\frac{z}{L_*}\right)\right]-as\kappa\frac{G\sin\chi}{u_*} - \eta_0\left[\ln\frac{Z}{Z_0}+\varphi\left(\frac{z}{L_*}\right)\right]-\eta\frac{\Theta_0'}{T_*} = F_v(Z,\mu_s,a,\eta,\delta) \\ &\frac{1}{\alpha_H^0}\left[\ln\frac{Z}{Z_0}+\varphi\left(\frac{z}{L_*}\right)\right]+\frac{\Theta_0'}{T_*} + s\frac{(1-a^2)}{a\eta}\sin\delta\left[\ln\frac{Z}{Z_0}+\psi\left(\frac{z}{L_*}\right)\right] - s\frac{(1-a^2)}{a\eta}\kappa\frac{G\sin\chi}{u_*} = F_\Theta(Z,\mu_s,a,\eta,\delta) \end{aligned}\right\}, \quad (16)$$

where $\eta_0 = \eta/\alpha_H^0$.

Taking the terms depending on Z_0 on the left side and the terms depending on Z on the right side, we can conclude that expressions on the left side are universal functions of μ_s, δ, η, a only:

$$\left.\begin{aligned} &-\cos\delta\ln(\kappa Z_0)-\kappa\frac{\cos\chi}{c_g} = A_s \\ &s(a\sin\delta - s\eta_0)\ln(\kappa Z_0)+as\kappa\frac{\sin\chi}{c_g} + \eta\frac{\Theta_0'}{T_*} = -B_s \\ &\alpha_H^0\frac{\Theta_0'}{T_*}-\left[1+s\frac{(1-a^2)}{a\eta_0}\sin\delta\right]\ln(\kappa Z_0) - s\kappa\frac{(1-a^2)}{a\eta_0}\frac{\sin\chi}{c_g} = C_s \end{aligned}\right\}, \quad (17a)$$

We have thus obtained the set of equations which in the limit of zero slope takes the form of atmospheric boundary layer resistance laws. Similarity functions A_s, B_s, C_s for sloping terrain are identical with universal functions A, B, C for horizontal terrain, when $\Psi = 0$, $\delta = 0$.

With the help of (10c) the first two equations in (17a) can also be written in the helpful form

$$\left.\begin{aligned} &\frac{1}{sa}\frac{dY}{dZ}\bigg|_{Z_0} - \cos\delta\ln(\kappa Z_0) = A_s \\ &\frac{dX}{dZ}\bigg|_{Z_0} + s(a\sin\delta - s\eta_0)\ln(\kappa Z_0) = -B_s \end{aligned}\right\}. \quad (17b)$$

In the next sections we will discuss the form of the universal functions A_s, B_s, C_s.

4. The form of functions A_s, B_s

Our aim is to define the form of similarity functions A_s, B_s. For this purpose we insert the derivatives of (13) into the resistance law (17b). From (13) we have

$$\left.\begin{aligned} &\frac{dX}{dZ}(\mu_s, Z, \cdots) = \frac{dr_1}{dZ}\cos\delta - \frac{dr_2}{dZ}s(a\sin\delta - s\eta_0) \\ &\frac{dY}{dZ}(\mu_s, Z, \cdots) = as\left[\frac{dr_1}{dZ}s(a\sin\delta - s\eta_0) + \frac{dr_2}{dZ}\cos\delta\right] - as\eta\frac{R_n'(0)}{Z_T} \end{aligned}\right\}. \quad (18)$$

We should emphasize that derivatives in (18) are dependent on μ_s because r_1, r_2 are functions of μ_s and do not depend on slope.

In the case of horizontal terrain ($\eta_0 = 0$), $\eta = 0$, $a = 1$, $\mu_s = \mu_0$ and r_1, r_2 are functions of Z and μ_0. From this time and for $\delta = 0$ Eq. (18) takes the form

$$\left.\begin{aligned} \left[\frac{dX}{dZ}(\mu_0, Z)\right]_h &= \frac{dr_1}{dZ} \\ \left[\frac{dY}{dZ}(\mu_0, Z)\right]_h &= s\frac{dr_2}{dZ} \end{aligned}\right\}, \quad (19)$$

where subscript h denotes the horizontal case.

With the help of (19), Eq. (18) can be rewritten in the form

$$\left.\begin{aligned}\frac{dX}{dZ}(\mu_s, Z, \cdots) &= \cos\delta\left[\frac{dX}{dZ}(\mu_s, Z)\right]_h \\ &\quad - (a\sin\delta - s\eta_0)\left[\frac{dY}{dZ}(\mu_s, Z)\right]_h \\ \frac{dY}{dZ}(\mu_s, Z, \cdots) &= a\cos\delta\left[\frac{dY}{dZ}(\mu_s Z)\right]_h \\ &\quad + as^2(a\sin\delta - s\eta_0)\left[\frac{dX}{dZ}(\mu_s, Z)\right]_h \\ &\quad - as\eta\frac{R_n'(0)}{Z_T}\end{aligned}\right\}, \quad (20)$$

where subscripted terms have the same form as in (19) but are functions of argument μ_s instead of μ_0.

Taking into consideration that (17a) should conform with (10c), and for $\delta = 0$, $\eta - \eta_0 = 0$ (see Fig. 3),

$$\left.\begin{aligned}-B(\mu_0) &= s\kappa\frac{\sin\alpha}{c_g} = \left[\frac{dX}{dZ}\bigg|_{Z_0}\right]_h \\ A(\mu_0) + \ln(\kappa Z_0) &= -\kappa\frac{\cos\alpha}{c_g} = \frac{1}{s}\left[\frac{dY}{dZ}\bigg|_{Z_0}\right]_h\end{aligned}\right\}, \quad (21)$$

where A, B are the universal functions for the horizontal case, and changing, according to the former notation, argument μ_0 to μ_s in (21), we can rewrite (20) as

$$\left.\begin{aligned}\frac{dX}{dZ}\bigg|_{Z_0} &= -\cos\delta B(\mu_s) \\ &\quad - s(a\sin\delta - s\eta_0)[A(\mu_s) + \ln(\kappa Z_0)] \\ \frac{1}{sa}\frac{dY}{dZ}\bigg|_{Z_0} &= \cos\delta[A(\mu_s) + \ln(\kappa Z_0)] \\ &\quad - s(a\sin\delta - s\eta_0)B(\mu_s) - as\eta\frac{R_n'(0)}{Z_T}\end{aligned}\right\}. \quad (22)$$

It should be noted that functions A, B in the above formulas have the same form of dependence on the argument as in the horizontal case. Inserting the obtained results into (17a) with the help of (A14) yields

$$\left.\begin{aligned}-B_s &= -\cos\delta B(\mu_s) - s(a\sin\delta - s\eta_0)A(\mu_s) \\ A_s &= \cos\delta A(\mu_s) - s(a\sin\delta - s\eta_0)B(\mu_s) \\ &\quad - as\eta_0 + (a^2 - 1)\sin\delta\end{aligned}\right\}, \quad (23)$$

where we assumed that $Z_T = 1$.

The above form of the similarity functions A_s, B_s is different from that obtained by Gutman, and Melgarejo (1981) by the last terms of the second equation in the set (23). This is the result of the different form of the heat flux equation in the set (1). Finally, the resistance law of the geostrophic drag takes the form

$$\left.\begin{aligned}\eta\frac{\Theta_0'}{T_*} + as\frac{\kappa\sin\chi}{c_g} &= -\cos\delta B(\mu_s) \\ &\quad - s(a\sin\delta - s\eta_0)[A(\mu_s) - \ln(a\mathrm{Ro}c_g)] \\ -\frac{\kappa\cos\chi}{c_g} &= \cos\delta[A(\mu_s) - \ln(a\mathrm{Ro}c_g)] \\ &\quad - s(a\sin\delta - s\eta_0)B(\mu_s) \\ &\quad - as\eta_0 + (a^2 - 1)\sin\delta\end{aligned}\right\}, \quad (24)$$

where the identity $\kappa Z_0 = (a\mathrm{Ro}c_g)^{-1}$ was used, and $\mathrm{Ro} = G/(f^{-1}z_0)$ is the Rossby number.

5. The form of function C_s

We now define the form of function C_s. Integrating the third equation of (10a) with respect to Z we obtain (see (A15))

$$K\frac{d}{dZ}\left(\frac{\Theta'}{T_*}\right) = H = \frac{1}{\alpha_H^0} + s\frac{(a^2 - 1)}{a\eta}(Y - \sin\delta) + \left[\frac{s(a^2 - 1)}{a\eta}\sin\delta - \frac{1}{\alpha_H^0}\right]\frac{Z}{Z_T}. \quad (25)$$

After another integration we have

$$\frac{\Theta'}{T_*}\bigg|_{Z_T} - \frac{\Theta'}{T_*}\bigg|_{Z_0} = s\frac{(a^2 - 1)}{a\eta}\int_{Z_0}^{Z_T}\frac{Y}{K}dZ + \left[\frac{1}{\alpha_H^0} - s\frac{(a^2 - 1)}{a\eta}\sin\delta\right]\int_{Z_0}^{Z_T}\left[1 - \frac{Z}{Z_T}\right]\frac{dZ}{K}. \quad (26)$$

Taking into consideration the conditions (3) and (5) yields

$$\frac{\Theta_0'}{T_*} = s\frac{(1 - a^2)}{a\eta}\int_{Z_0}^{Z_T}\frac{Y}{K}dZ - \left[\frac{1}{\alpha_H^0} - s\frac{(a^2 - 1)}{a\eta}\sin\delta\right]\int_{Z_0}^{Z_T}\left[1 - \frac{Z}{Z_T}\right]\frac{dZ}{K}. \quad (27)$$

Integration of the identity

$$Y = K\frac{d}{dZ}\left(\kappa\frac{v'}{u_*}\right)$$

with conditions (11)–(12) gives

$$\int_{Z_0}^{Z_T}\frac{Y}{K}dZ = -\kappa\frac{v'}{u_*}\bigg|_{Z_0} = \kappa\frac{\sin\chi}{c_g}. \quad (28)$$

Inserting Eqs. (27) and (28) into (17) we obtain

$$C_s = -\left[1 + s\frac{(1 - a^2)}{a\eta_0}\sin\delta\right] \times \left\{\int_{Z_0}^{Z_T}\left[1 - \frac{Z}{Z_T}\right]\frac{dZ}{K(\mu_s, Z)} + \ln(\kappa Z_0)\right\}. \quad (29)$$

In the case of horizontal terrain (27) takes the form

$$\frac{\Theta_0'}{T_*} = -\frac{1}{\alpha_H^0}\int_{Z_0}^{Z_T}\left[1-\frac{Z}{Z_T}\right]\frac{dZ}{K(\mu_s, Z)}. \quad (30)$$

From (30) and from the definition of the universal function $C(\mu_0)$ for the horizontal case we have

$$C(\mu_0) \equiv \alpha_H \frac{\Theta_0'}{T_*} - \ln(\kappa Z_0)$$

$$= -\int_{Z_0}^{Z_T}\left[1-\frac{Z}{Z_T}\right]\frac{dZ}{K(\mu_0, Z)} - \ln(\kappa Z_0). \quad (31)$$

With the help of (31), we can rewrite (29) as

$$C_s = \left[1 + s\frac{(1-a^2)}{a\eta_0}\sin\delta\right]C(\mu_s). \quad (32)$$

Finally, the resistance law of heat transfer has the form

$$\alpha_H^0 \frac{\Theta_0'}{T_*} - \kappa s \frac{(1-a^2)}{a\eta_0}\frac{\sin\chi}{c_g}$$

$$= \left[1 + s\frac{(1-a^2)}{a\eta_0}\sin\delta\right]\left[C(\mu_s) - \ln(a\mathrm{Ro}c_g)\right]. \quad (33)$$

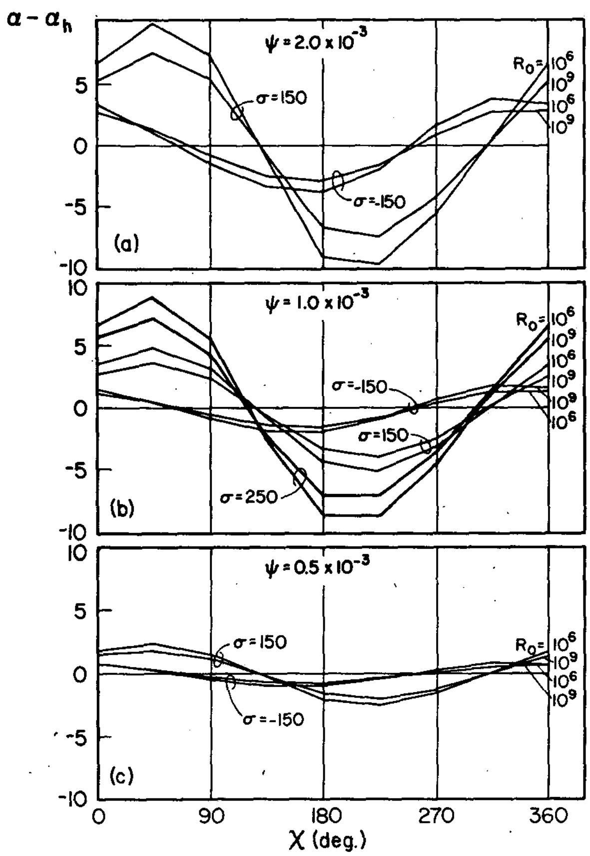

FIG. 4. Changes of cross-isobar angle α due to terrain slope (α_h cross-isobar angle in horizontal case): (a) $\psi = 2.0 \times 10^{-3}$, (b) $\psi = 1.0 \times 10^{-3}$, (c) $\psi = 0.5 \times 10^{-3}$.

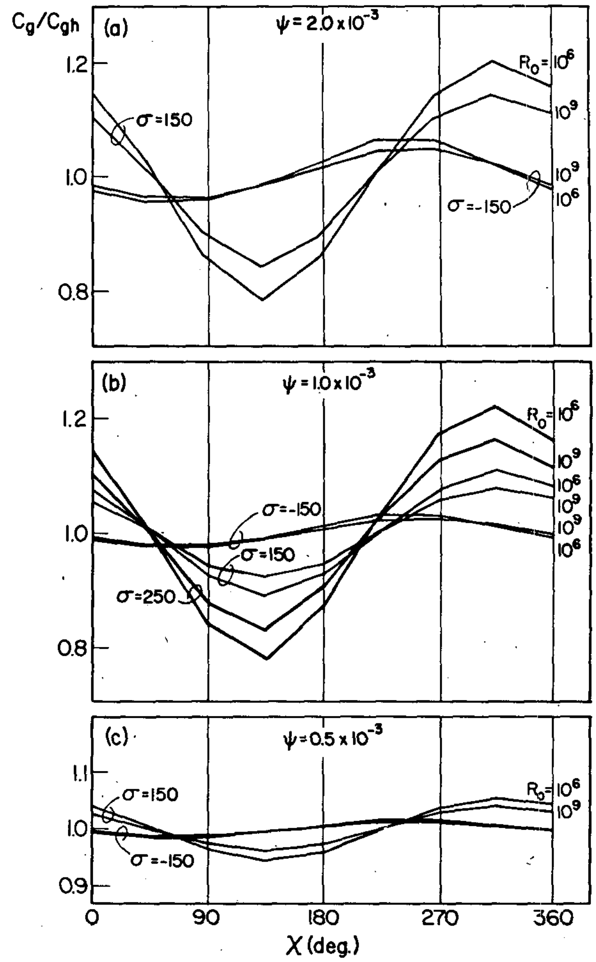

FIG. 5. Changes of geostrophic drag coefficient c_g due to terrain slope (c_{gh} geostrophic drag coefficient in horizontal case): (a) $\psi = 2.0 \times 10^{-3}$, (b) $\psi = 1.0 \times 10^{-3}$, (c) $\psi = 0.5 \times 10^{-3}$.

It should be noted that values of Θ_0' and μ_s are connected. This connection can be expressed by introducing the external stratification parameter σ defined as

$$\sigma = \frac{\beta(\Theta_{F_0} - \Theta_0)}{fG} = -\frac{\Theta_0'}{T_*}\frac{c_g}{\kappa^3}\frac{\mu_s}{a}. \quad (34)$$

6. Solution of the resistance law equations

The resistance law equations establish a set of three equations with three unknown functions c_g, δ, μ_s, depending on five governing parameters σ, Ro, a, χ, ψ. Functions $A(\mu_s)$, $B(\mu_s)$, $C(\mu_s)$ are assumed to be known. Values of $c_h = T_*/\Theta_0'$ can be obtained from (34) when the solution is known.

The solutions of the resistance laws, obtained numerically using Newton's method for three slopes $\psi = 0.5 \times 10^{-3}$, $\psi = 1.0 \times 10^{-3}$ and $\psi = 2.0 \times 10^{-3}$, are shown in Figs. 4–7. The chosen values ψ corre-

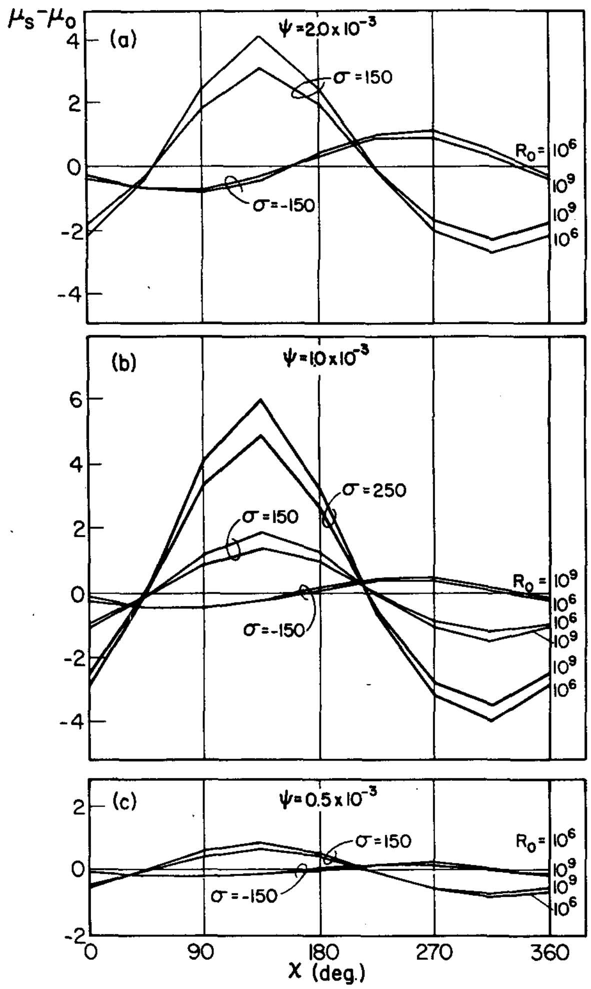

FIG. 6. Changes of internal stability parameter due to terrain slope: (a) $\psi = 2.0 \times 10^{-3}$, (b) $\psi = 1.0 \times 10^{-3}$, (c) $\psi = 0.5 \times 10^{-3}$.

spond to large-scale processes with characteristic horizontal scales of the order of 1000 km. Universal functions A, B, C were taken after Arya (1975), who derived them on the basis of Wangara data, in the range $\mu_0 > -50$:

$$\left.\begin{aligned} A(\mu_0) &= 1.01 - 0.105\,\mu_0 \\ &\quad - 0.00099\mu_0^2 + 0.00000081\mu_0^3 \\ B(\mu_0) &= 5.14 + 0.142\mu_0 \\ &\quad + 0.00177\mu_0^2 + 0.0000032\mu_0^3 \\ C(\mu_0) &= 1.86 - 0.377\mu_0 \\ &\quad - 0.00539\mu_0^2 + 0.0000572\mu_0^3 \end{aligned}\right\} . \quad (35)$$

Solutions are presented as the differences (or ratios) of the quantities over the slope terrain: α, $c_g = u_*/G$, μ_s, $c_h = T_*/\Theta_0'$ and analogous quantities over the horizontal terrain: α_h, c_{gh}, μ_0, c_{hh}, for two Rossby numbers Ro = 10^6 and Ro = 10^9, and for $a = 1$. Parameter $s = 1$, so the results are valid for the Northern Hemisphere.

It was found that to satisfy the condition $\mu_0 > -50$ in (35), the parameter σ should not be less than -150.

From the above figures it follows that the effect of slope increases with increasing absolute values of external stability parameter σ, with decreasing values of Rossby number Ro, and with increasing slope inclination ψ.

The differences $\alpha - \alpha_h$, $\mu_s - \mu_0$, $c_h - c_{hh}$ and the ratio c_g/c_{gh} are periodic functions of the angle χ between the fall line vector and the geostrophic wind vector. The functions obtained for unstable conditions are passed in phase on about 90° with respect to stable conditions. The position of the maxima and minima depends on the orientation of the geostrophic wind relative to the fall line vector and stability, and is the result of contrary or complementary action of the pressure gradient and drainage forces.

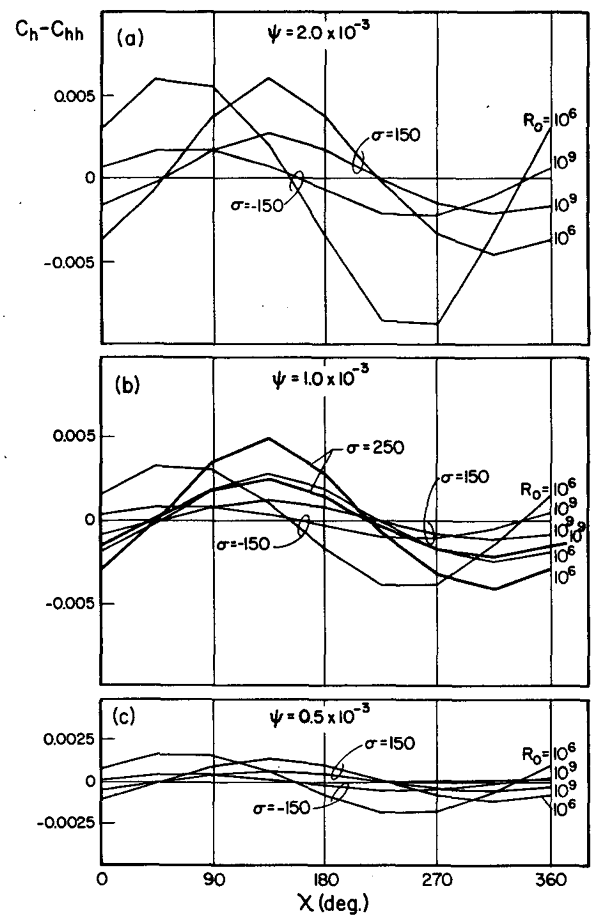

FIG. 7. Changes of Stanton-number $c_h = T_*/\Theta_0'$ due to terrain slope (c_{hh} Stanton number in horizontal case): (a) $\psi = 2.0 \times 10^{-3}$, (b) $\psi = 1.0 \times 10^{-3}$, (c) $\psi = 0.5 \times 10^{-3}$.

7. Concluding remarks

In our paper we have defined the universal profiles of the wind velocity and temperature defects (14) and have obtained the geostrophic drag and heat transfer resistance laws (17) for the boundary layer over a slightly inclined terrain. We have also found the relations (23) and (32) between sloping terrain similarity functions A_s, B_s, C_s and horizontal terrain similarity functions A, B, C. The resulting formulas are different from those obtained by Gutman and Melgarejo (1981). The difference is the result of the improved heat flux equation.

Knowing the slope corrections presented here allows one to calculate the universal functions A, B, C using data taken over sloping terrain. These formulas allow estimation of the slope influence on the boundary layer characteristics.

We have shown that small slopes (of the order 10^{-3}) can cause substantial changes of the boundary layer parameters. Our results were obtained under the assumptions that the ground surface is an unbounded inclined rough plane, and that the difference between the temperature at a point on the underlying surface and at the base of the free atmosphere directly above is constant along the sloping surface.

List of Symbols

a	parameter $\left[=\left(1+\frac{\beta\gamma\psi^2}{\alpha_H f}\right)^{-1/2}\right]$
A_s, B_s, C_s	universal functions of sloping terrain resistance laws
A, B, C	universal functions for the horizontal case
c_g	geostrophic drag coefficient at constant pressure
c_p	specific heat of air at constant pressure
c_h	Stanton number $[=T_*/\Theta_0']$
f	modulus of the Coriolis parameter
G	modulus of the geostrophic wind
h_s	terrain elevation
k	eddy viscosity
K	nondimensional eddy viscosity
L_*	Monin-Obukhov length $[=u_*^2/(\kappa^2\beta T_*)]$
L_E	Ekman length $[=\kappa u_*/f]$
L_s	Ekman length over sloping terrain $[=aL_E]$
Q_0	surface potential temperature flux
R	radiative heat flux
Ro	Rossby number $[=G/(fz_0)]$
s	sign of the Coriolis parameter
T_*	temperature scale $[=-Q_0/(\kappa c_p\rho u_*)]$
u, v	x and y components of the wind velocity
u_*	friction velocity
$\bar{x}$, $\bar{z}$	horizontal and vertical axes of Cartesian coordinates
x, z	down slope and normal axes of terrain-following Cartesian coordinates
z_0	roughness parameter
z_T	boundary layer height
α	angle between geostrophic wind and surface stress vectors
α_H	the inverse turbulent Prandtl number
β	buoyancy parameter $[=g/\Theta_0]$
γ	vertical gradient of potential temperature in the free atmosphere
δ	angle between the surface stress vector and the x axis
κ	von Kármán constant
η	parameter $[=\mu_s\psi/\kappa^2]$
η_0	parameter $[=\eta/\alpha_H^0]$
Θ_F	potential temperature of the free atmosphere
Θ_0	constant average value of the potential temperature
Θ	potential temperature in the boundary layer
μ_0	internal stability parameter over horizontal terrain $[=L_E/L_*]$
μ_s	internal stability parameter over the sloping terrain $[=L_s/L_*]$
ρ	air density
σ	external stability parameter $[=-\beta\Theta_0'/(fG)]$
χ	angle between the geostrophic wind vector and the x axis
ψ	angle of terrain slope.

The primes denote the deviations of boundary layer parameters from free atmosphere parameters.

APPENDIX A

Solution of Governing Equations

For the solution of the governing equations the method called the WKBJ approximation in standard textbooks on differential equations was used.

The first two equations of (10a) are equivalent to

$$\frac{d^2T}{dZ^2}-\frac{iT}{K}=0, \tag{A1}$$

where

$$T=X+iS,$$

$$S=s(aY-s\eta H),$$

$$i=\sqrt{-1}.$$

The boundary conditions for (A1) are

$$\left.\begin{aligned} &\text{for}\quad Z = Z_0 \to 0: \\ &\qquad T = \cos\delta + is(a\sin\delta - s\eta_0) \\ &\qquad K = K_0 \\ &\text{for}\quad Z = Z_T\colon T = 0 \end{aligned}\right\}, \quad \text{(A2)}$$

where $\eta_0 = \eta/\alpha_H^0$.

We also assume that K doesn't change very rapidly with the height.

We will look for a solution in the form

$$T = \exp[i\varphi(z)]. \quad \text{(A3)}$$

Inserting (A3) into (A1) we get

$$i\varphi'' - (\varphi')^2 - \frac{i}{K} = 0, \quad \text{(A4)}$$

where primes denote derivatives with respect to Z. In first approximation we assume $\varphi'' \approx 0$, then,

$$\varphi' \approx \pm \frac{i-1}{\sqrt{2K}}, \quad \text{(A5)}$$

$$\varphi'' \approx \pm \frac{(i-1)K'}{2\sqrt{2}K^{3/2}}. \quad \text{(A6)}$$

The condition for φ'' to be small is

$$|\varphi''| \approx \frac{1}{2}\left|\frac{K'}{K^{3/2}}\right| \ll \left|\frac{1}{K}\right|.$$

Inserting (A6) into (A4) we obtain as a second approximation the results

$$\varphi' \approx \frac{iK'}{2K} + \frac{i-1}{\sqrt{2K}}, \quad \text{(A7)}$$

$$\varphi \approx -i\ln\sqrt{K} \pm (i-1)\int \frac{dZ}{\sqrt{2K}}. \quad \text{(A8)}$$

Assuming that a second approximation is sufficient, we can write the general solution of our problem in the form

$$T \approx \sqrt{K}\{c_1 \exp[(1+i)m] + c_2 \exp[-(1+i)m]\}, \quad \text{(A9)}$$

where

$$m = \int_{Z_0}^{Z} \frac{dZ}{\sqrt{2K}}$$

is a function of Z. From boundary conditions (A2) we obtain

$$\left.\begin{aligned} c_1 &= -\frac{1}{2\sqrt{K_0}}\frac{\exp[-(1+i)m_T]}{\sinh[(1+i)m_T]} \\ &\quad \times [\cos\delta + is(a\sin\delta - s\eta_0)] \\ c_2 &= \frac{1}{2\sqrt{K_0}}\frac{\exp[(1+i)m_T]}{\sinh[(1+i)m_T]} \\ &\quad \times [\cos\delta + is(a\sin\delta - s\eta_0)] \end{aligned}\right\}, \quad \text{(A10)}$$

where $m_T = m(Z_T)$. Finally we express the solution in the form

$$T \approx [\cos\delta + is(a\sin\delta - s\eta_0)]r, \quad \text{(A11)}$$

where $r = r_1 + ir_2$ is a function of Z and μ_s, which can be obtained from (A9) and (A10). It can be easily checked that $r = 1$ for $Z = 0$, and $r = 0$ for $Z = Z_T$. For constant K and zero slope the obtained solution is equivalent to the Ekman solution of the motion equations. For K changing with height the solution is accurate when $|K'| \ll 2\sqrt{K}$, i.e., when the coefficient of turbulent exchange is an arbitrarily but slowly changing function of height. The resulting solution seems to be efficient for our aims since we would rather use the general form of the equations than the values given by them.

Separating the real from the imaginary parts, we get

$$\left.\begin{aligned} X &\approx r_1\cos\delta - r_2 s(a\sin\delta - s\eta_0) \\ S &= s(aY - s\eta H) \\ &= sr_1(a\sin\delta - s\eta_0) + r_2\cos\delta \end{aligned}\right\}. \quad \text{(A12)}$$

The integral of the third equation of the set (10a) can be written in the form

$$H = \frac{1}{\alpha_H^0} + s\frac{(a^2-1)}{a\eta}(Y - \sin\delta) + R_n' - R_n'(0). \quad \text{(A13)}$$

From (12) and (A13) we obtain

$$R_n'(0) = \frac{1}{\alpha_H^0} - s\frac{(a^2-1)}{a\eta}\sin\delta. \quad \text{(A14)}$$

We assume that R_n' changes linearly with height as

$$R_n'(Z) = R_n'(0)\left(1 - \frac{Z}{Z_T}\right).$$

From this and Eqs. (A13) and (A14) we obtain that in the limit of zero slope the turbulent heat flux

$$H = R_n' = \frac{1}{\alpha_H^0}\left(1 - \frac{Z}{Z_T}\right).$$

Then, from (A12), (A13) and (A14) it follows that

$$H = \frac{1}{\alpha_H^0} + s\frac{(a^2-1)}{a\eta}(Y - \sin\delta) - R_n'(0)\frac{Z}{Z_T},$$

$$Y = asS + (1-a^2)\sin\delta + as\eta_0 - as\eta R_n'(0)\frac{Z}{Z_T}. \quad \text{(A15)}$$

Finally we have

$$\left.\begin{aligned} X &\approx r_1\cos\delta - r_2 s(a\sin\delta - s\eta_0) \\ Y &\approx (1-a^2)\sin\delta + sa\eta_0 + as[r_1 s(a\sin\delta - s\eta_0) + r_2\cos\delta] - as\eta R_n'(0)\frac{Z}{Z_T} \\ H &\approx \frac{1}{\alpha_H^0} - R_n'(0)\frac{Z}{Z_T} + s\frac{(a^2-1)}{a\eta} \\ &\quad \times\left\{-a^2\sin\delta + as\eta_0 + as[r_1 s(a\sin\delta - s\eta_0) + r_2\cos\delta] - as\eta R_n'(0)\frac{Z}{Z_T}\right\} \end{aligned}\right\}. \quad \text{(A16)}$$

Quantities X, Y, H are functions of Z, Z_0, μ_s (by r_1, r_2) and δ, a, η_0.

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