Effects of Baroclinicity on Resistance Laws for the Atmospheric Boundary Layer Over a Slightly Inclined Terrain

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(Manuscript received 12 July 1982, in final form 29 October 1982)

ABSTRACT

The similarity theory of an atmospheric boundary layer over a slightly inclined terrain, discussed in an earlier paper (Sorbjan, 1983) is extended to the case of geostrophic wind varying with height. The forms of resistance laws and universal functions are obtained in the cases when the Ekman height or the actual boundary layer height are used as the boundary layer height scales.

1. Introduction

Gutman and Melgarejo (1981) obtained the resistance laws for an atmospheric boundary layer (ABL) over a slightly inclined terrain. Their paper was a further extension of ABL similarity theory, first formulated by Kazanskii and Monin in 1961, and improved by different investigators during the last 20 years.

The results of Gutman and Melgarejo (1981) were generalized by Sorbjan (1983; hereafter referred to as S83). However, both of these papers assumed stationarity with baroclinicity related only to sloping terrain thermal activity. The terms representing vertical changes of the geostrophic wind and temperature advection were excluded.

The aim of this paper is to extend the results obtained in S83 by taking into consideration the assumptions that the geostrophic wind varies with height and the actual boundary layer height is used as the height scale.

2. Governing equations

We consider the airflow in the planetary boundary layer over an unbounded homogeneous rough plane, inclined to the horizontal plane by a small angle $\psi \approx 10^{-4}$–10^{-3}. We assume that horizontal and vertical scales of the problem are respectively on the order of 10^2–10^3 km and 10^2–10^3 m. We assume also that the potential temperature horizontal gradient is much smaller than $\gamma\psi$. Under these assumptions the problem can be described by the following equations (see Appendix B):

$$\left.\begin{aligned}
&\frac{d}{dz}\left(k\frac{du}{dz}\right) + sf(v - V_g) - \Theta'\beta\psi = 0\\
&\frac{d}{dz}\left(k\frac{dv}{dz}\right) - sf(u - U_g) = 0\\
&\frac{d}{dz}\left(\alpha_H k\frac{d\Theta'}{dz}\right) + (u - U_g)\gamma\psi = \frac{dR'}{dz}\\
&\frac{dU_g}{dz} \equiv \frac{f}{\kappa^2}\eta_x = S_x = \text{constant}\\
&\frac{dV_g}{dz} \equiv \frac{f}{\kappa^2}\eta_y = S_y = \text{constant}\\
&R'(z) = R'(0)\left(1 - \frac{z}{z_T}\right)
\end{aligned}\right\} \quad (1)$$

All variables are listed in Appendix A.

For the lower and upper edges of the boundary layer we can write:

$$\left.\begin{aligned}
&\text{For } z = z_0:\\
&\quad u = 0, \quad v = 0,\\
&\quad \Theta' = \Theta_0' = \text{constant} = \begin{cases} >0 & \text{during day}\\ <0 & \text{during night}\end{cases}\\
&R' = R'(0)\\
&U_g = U_{g0} = G\cos\chi\\
&V_g = V_{g0} = G\sin\chi
\end{aligned}\right\} \quad (2)$$

(we assume that surface geostrophic wind changes along the slope can be neglected due to the small terrain inclination)

$$k\frac{du}{dz} = u_*^2\cos\delta,\quad k\frac{dv}{dz} = u_*^2\sin\delta,\quad \alpha_H k\frac{d\Theta'}{dz} = \kappa u_* T_*.$$

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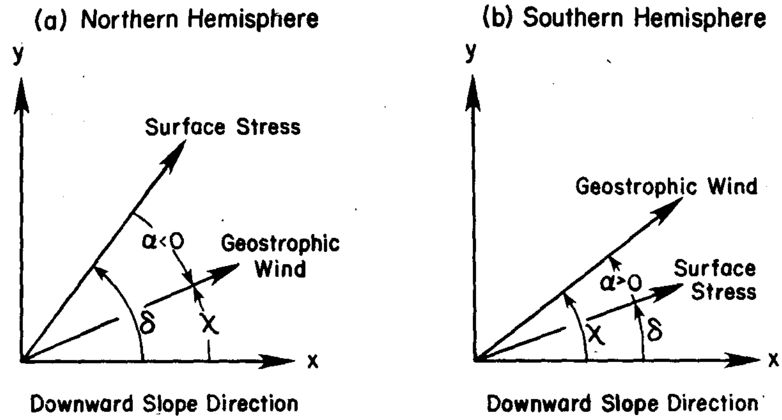

FIG. 1. Orientation of the angles α, δ, χ. Note that $\alpha = \chi - \delta$ always, independent of the hemisphere.

For $z = z_T$:

$$\frac{du}{dz} = \frac{f}{\kappa^2}\eta_x, \quad \frac{dv}{dz} = \frac{f}{\kappa^2}\eta_y, \quad \frac{d\Theta'}{dz} = 0, \quad R' = 0. \tag{3}$$

The orientation of angles χ and δ are shown in Fig. 1.

Under the assumption that $\eta_x = \eta_y = 0$ the problem formulation is the same as in (S83). Setting $\eta_x \neq 0$, $\eta_y \neq 0$ we consider the large-scale baroclinicity. Small-scale baroclinic effects connected with the sloping terrain local influence are expressed in (1) by terms which are proportional to ψ.

3. Ekman height scaling

We convert the set of governing equations into a new nondimensional form which enables us to formulate the Rossby-number similarity theory of the atmospheric boundary layer over a slightly inclined terrain.

It will be convenient to define new variables

$$\left.\begin{aligned} x &= k\frac{du}{dz} \\ y &= k\frac{dv}{dz} \\ h &= k\frac{d\Theta'}{dz} \end{aligned}\right\} . \tag{4}$$

We now consider first the steady-state boundary layer and introduce the nondimensional variables

$$\left.\begin{aligned} &X = \frac{x}{u_*^2}, \quad Y = \frac{y}{u_*^2}, \quad H = \frac{h}{\kappa u_* T_*} \\ &Z = \frac{z}{L_s}, \quad K = a\frac{k}{L_s^2 f}, \quad R_n' = \frac{R'}{\alpha_H \kappa u_* T_*} \\ &\mu_0 = \frac{L_E}{L_*}, \quad \mu_s = a\mu_0 \end{aligned}\right\}, \tag{5}$$

where

$$L_* = \frac{u_*^2}{\kappa^2 \beta T_*}, \quad L_E = \frac{\kappa u_*}{f},$$

$$L_s = aL_E, \qquad a^2 = \left(1 + \frac{\beta\gamma\psi^2}{\alpha_H f}\right)^{-1}.$$

From Eqs. (1), (4) and (5) it follows that

$$\left.\begin{aligned} X &= K\frac{d}{dZ}\left(\kappa\frac{u}{u_*}\right) \\ Y &= K\frac{d}{dZ}\left(\kappa\frac{v}{u_*}\right) \\ H &= K\frac{d}{dZ}\left(\frac{\Theta'}{T_*}\right) \end{aligned}\right\}, \tag{6}$$

$$\left.\begin{aligned} \frac{dX}{dZ} &= \left[\eta\frac{\Theta'}{T_*} - as\kappa\frac{v - V_{g0}}{u_*} + a^2 s\eta_y Z\right] \equiv Q \\ \frac{dY}{dZ} &= sa\kappa\frac{u - U_{g0}}{u_*} - sa^2\eta_x Z \equiv P \\ \frac{dH}{dZ} &= -\frac{(1 - a^2)}{\eta}\kappa\frac{(u - U_{g0})}{u_*} \\ &\quad + a\frac{(1 - a^2)}{\eta}\eta_x Z + \frac{dR_n'}{dZ} \end{aligned}\right\}, \tag{7}$$

where $\eta = \mu_s\psi/\kappa^2$. From (7) we can also obtain

$$\left.\begin{aligned} &\frac{d^2X}{dZ^2} + \frac{s(aY - s\eta H)}{K} = a^2 s\eta_y \\ &\frac{d^2}{dZ^2}(aY - s\eta H) - s\frac{X}{K} = -as\eta_x \\ &-\frac{dH}{dZ} = s\frac{(1 - a^2)}{a\eta}\frac{dY}{dZ} - \frac{dR_n'}{dZ} \end{aligned}\right\} . \tag{8}$$

The third equation in (8) can also be written in the form

$$\frac{d}{dZ}K\frac{d}{dZ}\left[\frac{\Theta'}{T_*}+\frac{s(1-a^2)}{a\eta}\kappa\frac{v-V_{g0}}{u_*}\right]=\frac{dR'_n}{dZ}. \quad (9)$$

Lower boundary conditions can be expressed in the form:

$$\left.\begin{aligned}&\text{For } Z=Z_0:\\ &X=X_0=\cos\delta,\quad Y=Y_0=\sin\delta,\\ &H=H_0=\frac{1}{\alpha_H^0},\quad R'_n=R'_n(0)\\ &P=P_0=-as\kappa\frac{\cos\chi}{c_g}\\ &Q=Q_0=\eta\frac{\Theta'_0}{T_*}+as\kappa\frac{\sin\chi}{c_g}\end{aligned}\right\}. \quad (10)$$

4. Universal defect profiles

Now we make the hypothesis that functions dX/dZ, dY/dZ and dH/dZ are universal in the sense of Rossby-number similarity, i.e., for $Z \gg Z_0$ they do not depend on the roughness length. The physical basis of this hypothesis for the horizontal case is discussed in Wippermann and Yordanov (1972).

From (7a,b) and (9) we can write

$$\left.\begin{aligned}&\kappa\frac{u-G\cos\chi}{u_*}=F_u(Z,\mu_s,a,\eta,\delta,\lambda_x,\lambda_y)\\ &as\kappa\frac{v-G\sin\chi}{u_*}-\eta\frac{\Theta-\Theta_F}{T_*}\\ &\qquad=F_v(Z,\mu_s,a,\eta,\delta,\lambda_x,\lambda_y)\\ &\frac{\Theta-\Theta_F}{T_*}+s\frac{(1-a^2)}{a\eta}\kappa\frac{v-G\sin\chi}{u_*}\\ &\qquad=F_\vartheta(Z,\mu_s,a,\eta,\delta,\lambda_x,\lambda_y)\end{aligned}\right\}, \quad (11)$$

where the internal baroclinic parameters λ_x, λ_y and external baroclinic parameters η_x, η_y are related by

$$\left.\begin{aligned}\lambda_x&=\eta_x\cos\delta+\eta_y\sin|\delta|\\ \lambda_y&=-\eta_x\sin|\delta|+\eta_y\cos\delta\end{aligned}\right\}. \quad (12)$$

In the surface layer we have

$$\left.\begin{aligned}&\kappa\frac{u}{u_*}=\cos\delta\left[\ln\left(\frac{z}{z_0}\right)+\psi\left(\frac{z}{L_*}\right)\right]\\ &\kappa\frac{v}{u_*}=\sin\delta\left[\ln\left(\frac{z}{z_0}\right)+\psi\left(\frac{z}{L_*}\right)\right]\\ &\kappa\frac{\Theta'}{T_*}=\frac{1}{\alpha_H^0}\left[\ln\left(\frac{z}{z_0}\right)+\varphi\left(\frac{z}{L_*}\right)\right]+\frac{\Theta'_0}{T_*}\end{aligned}\right\}, \quad (13)$$

where ψ and φ are the universal surface-layer functions.

Matching the inner (13) and outer (11) profiles and using classic procedure (e.g., Zilitinkevich, 1970) we obtain the so-called resistance laws:

$$\left.\begin{aligned}&-\ln(\kappa Z_0)\cos\delta-\kappa\frac{\cos\chi}{c_g}\\ &\qquad=A_s(\mu_s,a,\eta,\delta,\lambda_x,\lambda_y)\\ &s(a\sin\delta-s\eta_0)\ln(\kappa Z_0)+as\kappa\frac{\sin\chi}{c_g}+\eta\frac{\Theta'_0}{T_*}\\ &\qquad=-B_s(\mu_s,a,\eta,\delta,\lambda_x,\lambda_y)\\ &\alpha_H^0\frac{\Theta'_0}{T_*}-\left[1+s\frac{(1-a^2)}{a\eta_0}\sin\delta\right]\ln(\kappa Z_0)\\ &-s\kappa\frac{(1-a^2)}{a\eta_0}\frac{\sin\chi}{c_g}=C_s(\mu_s,a,\eta,\delta,\lambda_x,\lambda_y)\end{aligned}\right\}, \quad (14)$$

where $Z_0 = z_0/L_s$, $\eta_0 = \eta/\alpha_H^0$ and the universal functions A_s, B_s and C_s for the sloping terrain take the form of the universal functions A, B and C for horizontal terrain (i.e., when $\psi = \delta = 0$). The form (14) is the same as that obtained analogously in (S82).

5. Form of the universal functions

The resistance laws (14) can be rewritten in the more convenient form

$$\left.\begin{aligned}&-X_0\ln(\kappa Z_0)+\frac{1}{as}P_0=A_s\\ &s(aY_0-s\eta_0)\ln(\kappa Z_0)+Q_0=-B_s\\ &-\left[1+\frac{s(1-a^2)}{a\eta_0}Y_0\right]\ln(\kappa Z_0)\\ &\qquad+\frac{\alpha_H^0}{a^2}\frac{\Theta'_0}{T_*}-\frac{1-a^2}{a^2\eta_0}Q_0=C_s\end{aligned}\right\}. \quad (15)$$

If we rewrite (15) for the barotropic case ($\lambda_x = \lambda_y = 0$) and subtract from the baroclinic form of the equations ($\lambda_x \neq 0$, $\lambda_y \neq 0$) we obtain

$$\left.\begin{aligned}&\delta A_s\equiv A_s(\mu_s,a,\eta,\delta,\lambda_x,\lambda_y)-A_s(\mu_s,a,\eta,\delta,0,0)=\frac{1}{as}\delta P_0-\ln(\kappa Z_0)\delta X_0\\ &\delta B_s\equiv B_s(\mu_s,a,\eta,\delta,\lambda_x,\lambda_y)-B_s(\mu_s,a,\eta,\delta,0,0)=-\delta Q_0-as\ln(\kappa Z_0)\delta Y_0\\ &\delta C_s\equiv C_s(\mu_s,a,\eta,\delta,\lambda_x,\lambda_y)-C_s(\mu_s,a,\eta,\delta,0,0)=\frac{\alpha_H^0}{a^2}\delta\left(\frac{\Theta'_0}{T_*}\right)-\frac{1-a^2}{a^2\eta_0}\delta Q_0-s\frac{1-a^2}{a\eta_0}\ln(\kappa Z_0)\delta Y_0\end{aligned}\right\}. \quad (16)$$

where δP_0, δQ_0, δX_0, δY_0, $\delta(\Theta_0'/T_*)$ are defined in an analogous way to δA_s, δB_s and δC_s, as differences between baroclinic and barotropic values.

On the basis of Appendix C and for $a = 1$ [for $\beta \approx 0.3 \times 10^{-1}$ m s^{-2} K^{-1}, $\gamma \approx 10^{-2}$ K m^{-1}, $\alpha_H^0 \approx 1$, $\psi \approx 10^{-3}$, $f \approx 10^{-4}$ s^{-1}, we have $a^2 \approx 1/(1 + 10^{-5}) = 1$], we obtain

$$\left.\begin{aligned} \delta P_0 &= \eta_x b_1(\mu_0) + \eta_y b_2(\mu_0) \\ \delta Q_0 &= s(\eta_x b_2(\mu_0) - \eta_y b_1(\mu_0)) \\ \delta X_0 &= s(\eta_x b_3(Z_0, \mu_0) - \eta_y b_4(Z_0, \mu_0)) \\ \delta Y_0 &= \eta_x b_4(Z_0, \mu_0) + \eta_y b_3(Z_0, \mu_0) \\ \frac{\alpha_H^0}{a^2}\,\delta\left(\frac{\Theta_0'}{T_*}\right) &= 0 \end{aligned}\right\}, \tag{17}$$

where $b_1 = \varphi_y'(Z_0, \mu_0)$, $b_2 = \varphi_x'(Z_0, \mu_0)$ are functions of the parameter μ_0, independent of Z_0, since A_s, B_s, C_s are universal functions. The functions $b_3 = \varphi_x(Z_0, \mu_0)$, $b_4 = \varphi_y(Z_0, \mu_0)$ depend on Z_0 but their product with $\ln(Z_0)$ is independent of Z_0 from the assumption of Rossby-number similarity. Substituting (17) into (16) we finally obtain

$$\left.\begin{aligned} \delta A_s &= c_1\eta_x + c_2\eta_y \\ \delta B_s &= c_3\eta_x + c_4\eta_y \\ \delta C_s &= 0 \end{aligned}\right\}, \tag{18}$$

where functions c_1–c_4, which depend on stability and the sign of the Coriolis parameter but do not depend on slope characteristics, can be expressed by functions b_1–b_4. External parameters of baroclinicity η_x, η_y can be converted to internal parameters of baroclinicity λ_x, λ_y with the help of (12). Notice that in the horizontal case Eqs. (18) take the form obtained by Yordanov (1973). The general form of the functions A_s, B_s, $C_s(\mu_s, a, \eta, \delta, 0, 0)$ was derived in S83.

6. Actual ABL height scaling

It was shown in a number of papers that the Ekman height L_E is an unsatisfactory parameter characterizing the ABL, especially when the unsteady or convective or tropical boundary layer is considered (e.g., Arya and Wyngaard, 1975; Yamada, 1976). It was suggested first by Deardorff (1972) through three-dimensional numerical ABL simulation that the Ekman height should be replaced by the actual ABL height h_s. Over land in the daytime h_s can be determined from temperature and humidity soundings. At night, however, there is a controversy dealing with the nocturnal ABL height definition (Arya, 1981). The values of h_s generally seem determinable from a rate equation which incorporates primarily the effects of entrainment and mean vertical velocity on the top of the boundary layer.

From these arguments it seems convenient to extend the results obtained in the earlier sections to the case when the actual ABL height h_s is used as a scaling parameter. To this aim we redefine some nondimensional variables introduced in (5) as follows:

$$Z_n = \frac{z}{h_s}, \quad K_n = a\frac{k}{h_s^2 f}, \tag{19}$$

$$\mu = \frac{h_s}{L_*}, \quad n = \frac{h_s}{L_s}, \quad M_x = \frac{h_s}{u_*}S_x, \quad M_y = \frac{h_s}{u_*}S_y. \tag{20}$$

The rest of the parameters are unchanged. Instead of (6)–(9) we now have

$$\left.\begin{aligned} X &= nK_n\frac{d}{dZ_n}\left(\kappa\frac{u}{u_*}\right) \\ Y &= nK_n\frac{d}{dZ_n}\left(\kappa\frac{v}{u_*}\right) \\ H &= nK_n\frac{d}{dZ_n}\left(\frac{\Theta'}{T_*}\right) \end{aligned}\right\}, \tag{21}$$

$$\left.\begin{aligned} &\frac{dX}{dZ_n} = n\left[\eta\frac{\Theta'}{T_*} - as\kappa\frac{v - V_{g0}}{u_*} + as\kappa M_x Z_n\right] \\ &\frac{dY}{dZ_n} = nsa\kappa\left[\frac{u - U_{g0}}{u_*} - M_y Z_n\right] \\ &n\frac{d}{dZ_n}K_n\frac{d}{dZ_n} \\ &\quad\times\left[\frac{\Theta'}{T_*} + \frac{s(1 - a^2)}{a\eta}\kappa\frac{v - V_{g0}}{u_*}\right] = \frac{dR_n'}{dZ_n} \end{aligned}\right\}, \tag{22}$$

$$\left.\begin{aligned} &\frac{d^2X}{dZ_n^2} + \frac{s(aY - s\eta H)}{K_n} = asn\kappa M_y \\ &\frac{d^2}{dZ_n^2}(aY - s\eta H) - s\frac{X}{K_n} = -sn\kappa M_x \\ &-\frac{dH}{dZ_n} = s\frac{(1 - a^2)}{a\eta}\frac{dY}{dZ_n} - \frac{dR_n'}{dZ_n} \end{aligned}\right\}. \tag{23}$$

On the basis of (22) and the previous arguments we can write

$$\left.\begin{aligned} &\kappa\frac{u - G\cos\chi}{u_*} \\ &\qquad = F_u(Z_n, \mu, \mu_s, a, \eta, \delta, M_x, M_y) \\ &as\kappa\frac{v - G\sin\chi}{u_*} - \eta\frac{\Theta - \Theta_F}{T_*} \\ &\qquad = F_v(Z_n, \mu, \mu_s, a, \eta, \delta, M_x, M_y) \\ &\frac{\Theta - \Theta_F}{T_*} + s\frac{(1 - a^2)}{a\eta}\kappa\frac{v - G\sin\chi}{u_*} \\ &\qquad = F_\vartheta(Z_n, \mu, \mu_s, a, \eta, \delta, M_x, M_y) \end{aligned}\right\}. \tag{24}$$

Matching (13) and (24) we obtain

$$\left.\begin{aligned} &-\ln(Z_{0_n})\cos\delta - \kappa\frac{\cos\chi}{c_g} \\ &\qquad = a_s(\mu, \mu_s, a, \eta, \delta, M_x, M_y) \\ &s(a\sin\delta - s\eta_0)\ln(Z_{0_n}) + as\kappa\frac{\sin\chi}{c_g} + \eta\frac{\Theta_0'}{T_*} \\ &\qquad = -b_s(\mu, \mu_s, a, \eta, \delta, M_x, M_y) \\ &\alpha_H^0\frac{\Theta_0'}{T_*} - \left[1 + s\frac{(1-a^2)}{a\eta_0}\sin\delta\right]\ln(Z_{0_n}) \\ &\qquad - s\kappa\frac{(1-a^2)}{a\eta_0}\frac{\sin\chi}{c_g} \\ &\qquad = c_s(\mu, \mu_s, a, \eta, \delta, M_x, M_y) \end{aligned}\right\}, \quad (25)$$

where $Z_{0_n} = z_0/h_s$. Functions a_s, b_s, c_s are identical with functions a_0, b_0, c_0 for horizontal terrain, when $\psi = 0$, $\delta = 0$. Moreover, in this case the resistance laws (25) have the same form as equations obtained by Zilitinkevich and Deardorff (1974) for barotropic conditions.

The general form of Eqs. (23) and (24) is the same as for the equations obtained with Ekman height scaling in S83. This argument enables us to adopt the results obtained in S83 and rewrite them for $a = 1$, i.e.,

$$\left.\begin{aligned} &a_s(\mu, \mu_0, \eta, \delta, 0, 0) = a_0(\mu_0, \mu)\cos\delta \\ &\qquad - s(\sin\delta - s\eta_0)b_0(\mu_0, \mu) - s\eta_0 \\ &b_s(\mu, \mu_0, \eta, \delta, 0, 0) = b_0(\mu_0, \mu)\cos\delta \\ &\qquad + s(\sin\delta - s\eta_0)a_0(\mu_s, \mu) \\ &c_s(\mu, \mu_0, \eta, \delta, 0, 0) = c_0(\mu_0, \mu) \end{aligned}\right\}, \quad (26)$$

where a_0, b_0, c_0 are the similarity functions for the horizontal case. The first two equations of (23) can be written in the form

$$\frac{d^2T}{dZ_n^2} - i\frac{T}{K_n} = -n\kappa siM, \quad (27)$$

where $M = M_x + iM_y$. Eq. (27) is analogous to Eq. (C1). Since from (C1) we obtained set (18), analogously from (27) we obtain

$$\left.\begin{aligned} \delta a_s &= n(C_1M_x + C_2M_y) \\ \delta b_s &= n(C_3M_x + C_4M_y) \\ \delta c_s &= 0 \end{aligned}\right\}, \quad (28)$$

where δa_s, δb_s, δc_s are differences between values of a_s, b_s, c_s in the baroclinic and barotropic cases, defined analogously to (16). Functions C_1–C_4 are defined in a corresponding way to functions c_1–c_4 in (18).

7. Concluding remarks

The similarity theory which has been presented is based on several crucial assumptions. We assume that the difference between the potential temperature close to the ground and that at the same level in the free atmosphere is constant along a slope. We also assume that geostrophic wind changes along the slope can be neglected. This requires, that if we consider motions on the scale of 10^5–10^6 m in length, we must consider a terrain slope of 10^{-4}–10^{-3} to have ground surface altitude differences smaller than 10^2 m. Finally we assumed that the potential-temperature horizontal gradient is much smaller than $\gamma\psi$. Using these assumptions we obtained the form of resistance laws and universal functions for a baroclinic ABL which are consistent with those obtained earlier for horizontal terrain.

APPENDIX A

List of Symbols

a	parameter $\left[=\left(1+\frac{\beta\gamma\psi^2}{\alpha_{Hf}}\right)^{-1/2}\right]$
a_s, b_s, c_s	universal functions of sloping terrain resistance laws for time-dependent ABL
a_0, b_0, c_0	universal functions of time-dependent ABL for the horizontal case
A_s, B_s, C_s	universal functions of sloping terrain resistance laws
A, B, C	universal functions for the horizontal case
c_g	geostrophic drag coefficient
c_p	specific heat of air at constant pressure
f	modulus of the Coriolis parameter
G	modulus of the surface geostrophic wind
h_s	actual boundary height
k	eddy viscosity
K, K_n	nondimensional eddy viscosity
L_*	Monin-Obukhov length $[=u_*^2/(\kappa^2\beta T_*)]$
L_E	Ekman length $[=\kappa u_* f]$
L_s	Ekman length over sloping terrain $[=aL_E]$
M_x, M_y	parameters of baroclinicity $[M_x = (h_s/u_*)S_x,\ M_y = (h_s/u_*)S_y]$
n	parameter $[=\mu/\mu_s]$
Q_s	surface potential temperature flux
R, R_F, R'	radiative heat flux in the boundary layer, in the free atmosphere and the difference between these two quantities
Ro	surface Rossby number $[=G/(fz_0)]$
s	sign of the Coriolis parameter
S_x, S_y	components of the geostrophic wind vertical gradient
T_*	temperature scale $[=-Q_s/(\kappa c_p\rho u_*)]$

$\bar{u}, \bar{v}, \bar{w}$	components of wind velocity in horizontally oriented Cartesian coordinates
u, v, w	components of wind velocity in terrain-following Cartesian coordinates
u_*	friction velocity
$\bar{x}, \bar{y}, \bar{z}$	axes of horizontally oriented Cartesian coordinates
x, y, z	axes of terrain-following Cartesian coordinates
z_0	roughness parameter
z_T	boundary layer height
Z, Z_n	nondimensional height
α	angle between geostrophic wind and surface stress vectors
α_H	the inverse turbulent Prandtl number
β	buoyancy parameter $[=g/\Theta_0]$
γ	vertical gradient of potential temperature in the free atmosphere
δ	angle between the surface stress vector and the x axis
κ	von Kármán constant
η	parameter $[=\mu_s\psi/\kappa^2]$
η_0	parameter $[=\eta/\alpha_H^0]$
Θ_F	potential temperature of the free atmosphere
Θ_0	constant average value of the potential temperature
Θ	potential temperature in the boundary layer
Θ'	difference between temperature in boundary layer and free atmosphere
Θ_0'	surface value of Θ'
μ	stability parameter of time-dependent ABL $[=h_s/L_*]$
μ_0	internal stability parameter over horizontal terrain $[=L_E/L_*]$
μ_s	internal stability parameter over the sloping terrain $[=L_s/L_*]$
ρ	air density
χ	angle between the geostrophic wind vector and x axis
ψ	angle between the geostrophic wind vector and x axis
λ_x, λ_y	internal baroclinic parameters $[\lambda_s = \frac{\kappa^2}{f}\frac{\partial u_g}{\partial Z}, \eta_y = \frac{\kappa^2}{f}\frac{\partial v_g}{\partial Z}$, where u_g is parallel to surface stress vector and v_g is adequately perpendicular component of geostrophic wind]
η_x, η_y	external baroclinic parameters $\left[\eta_x = \frac{\kappa^2}{f}\frac{\partial U_g}{\partial Z}, \eta_y = \frac{\kappa^2}{f}\frac{\partial V_g}{\partial Z}\right]$
π	pressure term $\left[=c_P\Theta\left(\frac{P}{P_0}\right)^{R/c_P}, P_0 = 1000\ \text{hPa}\right]$.

APPENDIX B

Derivation of the Governing Equations

We start from the general equations of an atmospheric boundary layer, which we write in the form

$$\left.\begin{aligned} &\frac{D\bar{u}}{Dt} - sf\bar{v} = -\frac{\Theta}{\Theta_0}\frac{\partial\pi}{\partial\bar{x}} + \overline{Fu}, && \frac{D\Theta}{Dt} = \bar{F}_\Theta \\ &\frac{D\bar{v}}{Dt} + sf\bar{u} = -\frac{\Theta}{\Theta_0}\frac{\partial\pi}{\partial\bar{y}} + \overline{Fv}, && \pi = c_P\Theta_0\left(\frac{P}{P_0}\right)^{R/c_P} \\ &\frac{\partial\bar{u}}{\partial\bar{z}} + \frac{\partial\bar{v}}{\partial\bar{z}} + \frac{\partial\bar{w}}{\partial\bar{z}} = 0, && \frac{\Theta}{\Theta_0}\frac{\partial\pi}{\partial\bar{z}} = -g \end{aligned}\right\}, \tag{B1}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial\bar{x}} + \bar{v}\frac{\partial}{\partial\bar{y}} + \bar{w}\frac{\partial}{\partial\bar{z}},$$

$$\bar{F} = \frac{\partial}{\partial\bar{z}}K\frac{\partial}{\partial\bar{z}},$$

$$\bar{F}_\Theta = \frac{\partial}{\partial\bar{z}}K\frac{\partial\Theta}{\partial\bar{z}} - \frac{\partial R}{\partial\bar{z}},$$

and the bar denotes that variables are designated with respect to Cartesian coordinates oriented as shown in Fig. 2. The $\bar{x}$ axis lies in the plane of slope gradient vector. The pressure, temperature and radiative heat flux are expressed in terms of standard (free atmosphere) variables and deviations (primed), i.e.,

$$\left.\begin{aligned} \pi &= \pi_g + \pi' \\ \Theta &= \Theta_F + \Theta' \end{aligned}\right\}. \tag{B2}$$

$$R = R_F + R'$$

We assume that standard variables are defined as

$$\left.\begin{aligned} &\frac{\partial\Theta_F}{\partial t} + \bar{U}_g\frac{\partial\Theta_F}{\partial\bar{x}} + \bar{V}_g\frac{\partial\Theta_F}{\partial\bar{y}} = -\frac{\partial R_F}{\partial\bar{z}} \\ &\frac{\partial\Theta_F}{\partial z} = \gamma = \text{constant} > 0 \\ &\frac{\Theta_F}{\Theta_0}\frac{\partial\pi_g}{\partial\bar{x}} = -sf\bar{V}_g \\ &\frac{\Theta_F}{\Theta_0}\frac{\partial\pi_g}{\partial\bar{y}} = sf\bar{U}_g \\ &\frac{\Theta_F}{\Theta_0}\frac{\partial\pi_g}{\partial\bar{z}} = -g \end{aligned}\right\}, \tag{B3}$$

where $\bar{U}_g$, $\bar{V}_g$ are the wind components in the free atmosphere.

With the help of (B2), (B3) the set (B1) becomes

$$\left.\begin{aligned}
&\frac{D\bar{u}}{Dt} - sf(\bar{v} - \bar{V}_g) = -\frac{\partial \pi'}{\partial \bar{x}} + \overline{Fu}\\
&\frac{D\bar{v}}{Dt} + sf(\bar{u} - \bar{U}_g) = -\frac{\partial \pi'}{\partial \bar{y}} + \overline{Fv}\\
&\frac{\partial \bar{u}}{\partial \bar{z}} + \frac{\partial \bar{v}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0\\
&\frac{D\Theta'}{Dt} + (\bar{u} - \bar{V}_g)\frac{\partial \Theta_F}{\partial \bar{x}} + (\bar{v} - \bar{V}_g)\frac{\partial \Theta_F}{\partial \bar{y}} + \bar{w}\gamma = \bar{F}'_\Theta\\
&\frac{\partial \pi'}{\partial \bar{z}} = \beta\Theta'\\
&\text{where}\quad \bar{F}'_\Theta = \frac{\partial}{\partial \bar{z}} K\left(\frac{\partial \Theta'}{\partial \bar{z}} + \gamma\right) - \frac{\partial R'}{\partial \bar{z}}.
\end{aligned}\right\}, \qquad \text{(B4)}$$

The equations are subsequently rewritten in a coordinate system that is locally parallel to the terrain. The transformation matrix which transforms the vectors from the barred system into the new system has the form

$$\mathbf{A} = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix}, \qquad \text{(B5)}$$

where ψ is the angle of the terrain slope (Fig. 2).

For this transformation we have

$$\frac{D}{Dt} = \frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \qquad \text{(B6)}$$

where the variables in the new coordinate system are denoted without bars.

Taking (B4a) $\times \cos\psi$ + (B4c) $\times \sin\psi$ and (B4a) $\times (-\sin\psi)$ + (B4c) $\times \cos\psi$ and using (B6), we obtain the new system of equations

$$\left.\begin{aligned}
&\frac{du}{dt} - sf(v - V_g)\cos\psi = -\frac{\partial \pi'}{\partial x} + \beta\Theta'\sin\psi + Fu\\
&\frac{dv}{dt} + sf(u\cos\psi + w\sin\psi - U_g) = -\frac{\partial \pi'}{\partial y} + Fv\\
&\frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} + \frac{\partial w}{\partial z} = 0\\
&\frac{d\Theta'}{dt} + \left[(u - U_g)\frac{\partial \Theta_F}{\partial x} + (v - V_g)\frac{\partial \Theta_F}{\partial y}\right] + [(u - U_g)\sin\psi + w\cos\psi]\gamma = F'_\Theta
\end{aligned}\right\}, \qquad \text{(B7)}$$

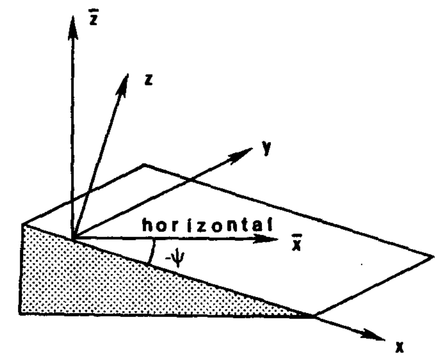

FIG. 2. Orientation of coordinate systems.

where

$$\left.\begin{aligned}
F &= \sin^2\psi\frac{\partial}{\partial x}K\frac{\partial}{\partial x} + \sin\psi\cos\psi \times \left(\frac{\partial}{\partial x}K\frac{\partial}{\partial z} + \frac{\partial}{\partial z}K\frac{\partial}{\partial x}\right) + \cos^2\psi\frac{\partial}{\partial z}K\frac{\partial}{\partial z}\\
F'_\Theta &= \sin^2\psi\frac{\partial}{\partial x}K\frac{\partial \Theta'}{\partial x} + \sin\psi\cos\psi \times \left(\frac{\partial}{\partial x}K\frac{\partial}{\partial z} + \frac{\partial}{\partial z}K\frac{\partial}{\partial x}\right)\Theta' + \cos^2\psi\frac{\partial}{\partial z}K\frac{\partial \Theta'}{\partial z}\\
&\quad + \frac{\partial K\gamma}{\partial x}\sin\psi + \frac{\partial K\gamma}{\partial z}\cos\psi - \frac{\partial R'}{\partial x}\sin\psi - \frac{\partial R'}{\partial z}\cos\psi.
\end{aligned}\right\}. \qquad \text{(B8)}$$

Now we simplify the set of the governing equations. Let us scale (B7) with the velocity scale U_0, horizontal and vertical scales L_0 and h, temperature difference scale Δ and pressure scale $\pi_0 = gh\Delta/\Theta_0$. We require that the two characteristic numbers R_1, R_2 which appear in the substantial derivative and the pressure terms of the first two equations in (B7) are small, i.e., $R_1 = U_0/(fL_0) \ll 1$ and $R_2 = gh\Delta/(\Theta_0 U_0 f L_0) \ll 1$. For $U_0 \approx 10$ m s^{-1}, $f \approx 10^{-4}$ s^{-1}, $g \approx 10$ m s^{-1}, $\Delta \approx 30$ K and $\Theta_0 = 300$ K we obtain $L_0 \gg 10^5$ m, $h \ll 10^3$ m. Under these assumptions we can omit total derivatives and pressure terms in the first two equations of (B7).

Since a horizontal scale is greater than a vertical scale, $\partial/\partial x \approx \partial/\partial y \ll \partial/\partial z$. From this and from the boundary condition that $w = 0$ at $z = 0$, and also from the continuity equation, we have $w = 0$.

Letting $\sin\psi \approx 0$ and $\cos\psi = 1$, we obtain

$$F = \frac{\partial}{\partial z}K\frac{\partial}{\partial z}, \quad F'_\Theta = \frac{\partial}{\partial z}K\left[\frac{\partial \Theta'}{\partial z} + \gamma\right] - \frac{\partial R'}{\partial z}.$$

Moreover, we omit the term $\partial(K\gamma)/\partial z$ in F'_Θ because it does not play a substantial role in the surface layer where $\gamma \ll \partial\Theta'/\partial z$ (Gutman and Melgarejo, 1981).

Further assumptions of steady state and horizontal homogeneity in the equation of the temperature disturbance Θ' enable us to omit the first term of the left-hand side. The slope (third) term on the same side has the order $U_0\psi\gamma \sim 10^{-4}$. For convenience we omit the advection (second) term on the left-hand side of the temperature equation. We can do this if we assume that the potential temperature horizontal gradient is much smaller than $\gamma\psi$; i.e., only a weak baroclinicity is considered. On the right-hand side of this equation we have the difference of two compensating terms (turbulent and radiative) each of which have the order of $f\Delta \sim 10^{-3}$.

The final form of the model equations is given by set (1).

APPENDIX C

The Solution of the Governing Equations

The first two equations of (8) can be written in the form

$$\frac{d^2T}{dZ^2} - i\frac{T}{K} = -iasN, \qquad \text{(C1)}$$

where $i = \sqrt{-1}$ and

$$\left.\begin{aligned} T &= X + iS \\ S &= s(aY - s\eta H) \\ N &= \eta_x + ia\eta_y \end{aligned}\right\}. \qquad \text{(C2)}$$

The solution of (8c) has the form (Sorbjan, 1982)

$$H = s\frac{(a^2-1)}{a\eta}Y + \left[s\frac{(1-a^2)}{a\eta}Y_0 + \frac{1}{\alpha_H^0}\right]\left(1 - \frac{Z}{Z_T}\right), \qquad \text{(C3)}$$

where we assumed that the radiative heat flux R'_n is a linear function of height $R'_n = R'_n(0)(1 - Z/Z_T)$; from boundary conditions it follows that

$$R'_n(0) = \frac{1}{\alpha_n^0} - s\frac{(a^2-1)}{a\eta}\sin\delta.$$

We assume that fundamental solutions of the homogeneous equation (C1) are equal to T_1 and T_2. The functions T_1 and T_2 depend only on Z and μ_s since for the homogeneous form of (C1) the right hand side equals 0. The general solution of the homogeneous equation takes the form

$$T_a(Z, \mu_s) = c_1T_1 + c_2T_2. \qquad \text{(C4)}$$

The general solution of the nonhomogeneous equation can be found by the method of variable coefficients and has the form

$$T(Z, Z_0, \mu_s, \delta, \eta, a, N) = asNi \times\left[T_1\int_{Z_0}^{Z}\frac{T_2dZ}{T_1T'_2 - T_2T'_1} - T_2\int_{Z_0}^{Z}\frac{T_1dZ}{T_1T'_2 - T_2T'_1}\right] + a_1T_1 + a_2T_2, \qquad \text{(C5)}$$

where primes denote height derivatives and coefficients a_1 and a_2 depend on boundary condition formulations.

We can rewrite (C5) in the form

$$T(Z, \mu_s, \delta, \eta, a, N) = asN\varphi(Z, \mu_s) + T(Z, \mu_s, \delta, \eta, a, 0), \qquad \text{(C6)}$$

where by we have used $\varphi = \varphi_x + i\varphi_y$ to denote the term in the brackets in (C5). From (C6) and (C2) we get

$$\delta T \equiv T(Z, \mu_s, \delta, \eta, a, N) - T(Z, \mu_s, \delta, \eta, a, 0) = \delta X + i\delta S = asN\varphi(Z, \mu_s). \qquad \text{(C7)}$$

Variables δX, δS and δY, δH, δP, δQ, δX_0, δY_0 are defined analogous to δT as the differences between baroclinic and barotropic values. Moreover, with the help of (C2), (C3), (C7), we obtain

$$\left.\begin{aligned} \delta X &= as[\eta_x\varphi_x(Z, \mu_s) - a\eta_y\varphi_y(Z, \mu_s)] \\ \delta S &= as[\eta_x\varphi_y(Z, \mu_s) + a\eta_y\varphi_x(Z, \mu_s)] \\ \delta Y &= as\delta S + \delta Y_0(1 - a^2)\left(1 - \frac{Z}{Z_T}\right) \\ \delta H &= s\frac{(a^2-1)}{\eta}\left[s\delta S - a\delta Y_0\left(1 - \frac{Z}{Z_T}\right)\right] \end{aligned}\right\}, \qquad \text{(C8)}$$

and we assumed that $\delta\eta_0 = \psi\delta\mu_s/(\kappa^2\alpha_H^0) \approx 0$ since $\psi \sim 10^{-3}$. From (C3), (C8), (6), (7) it follows that

$$\left.\begin{aligned} \delta\left(\frac{\Theta_0}{T_*}\right) &= -\int_{Z_0}^{Z_T}\frac{\delta H}{K}dZ \\ &= s\frac{1-a^2}{\eta}\int_{Z_0}^{Z_T}\left[s\delta S - a\delta Y_0 \times\left(1 - \frac{Z}{Z_T}\right)\right]\frac{dZ}{K} \\ \delta P = \delta\left(\frac{dY}{dZ}\right) &= a^2(\eta_x\varphi'_y(Z, \mu_s) + a\eta_y\varphi'_x(Z, \mu_s)) - \frac{1-a^2}{Z_T}\delta Y_0 \\ \delta Q = \delta\left(\frac{dX}{dZ}\right) &= as[\eta_x\varphi'_x(Z, \mu_s) - a\eta_y\varphi'_y(Z, \mu_s)] \end{aligned}\right\}. \qquad \text{(C9)}$$

From (C3), (C8) we have also for $Z = Z_0$

$$\delta H_0 = 0, \tag{C10}$$

$$\delta Y_0 = sa^{-1}\delta S_0 = \eta_x \varphi_y(Z_0, \mu_s) + a\eta_x\varphi_x(Z_0, \mu_s). \tag{C11}$$

REFERENCES

Arya, S. P. S., 1981. Parametrizing the height of the stable atmospheric boundary layer. *J. Appl. Meteor.*, **20,** 1192–1202.

——, and J. C. Wyngaard, 1975. Effect of baroclinicity on wind profiles and the geostrophic drag law for the convective planetary boundary layer. *J. Atmos. Sci.*, **32,** 767–778.

Deardorff, J. W., 1972. Numerical investigation of neutral and unstable planetary boundary layers. *J. Atmos. Sci.*, **29,** 91–115.

Gutman, L. N., and J. W. Melgarejo, 1981. On the laws of momentum and heat transfer over a slightly inclined terrain. *J. Atmos. Sci.*, **38,** 1719–1729.

Kazanskii, A. B., and A. S. Monin, 1961. On the dynamical intersection between the atmosphere and earth's surface. *Izv. Acad. Sci. USSR Ser. Geophys.*, **5,** 514–515.

Sorbjan, Z., 1982. Rossby number similarity in the atmospheric boundary layer over a slightly inclined terrain. *J. Atmos. Sci.*, **40,** 718–728.

Wippermann, F., and D. Yordanov, 1972. A note on the Rossby similarity for flows of barotropic planetary boundary layer. *Beitr. Phys. Atmos.*, **45,** 66–71.

Yamada, T., 1976. On the similarity functions *A, B,* and *C* of the planetary boundary layer. *J. Atmos. Sci.*, **33,** 781–793.

Yordanov, D. L., 1973. Parameterization of baroclinicity effects in the planetary boundary layer. *C.R. Acad. Sci.*, **16,** 883–886.

Zilitinkevich, S. S., 1970. *The Dynamics of the Atmospheric Boundary Layer.* Gidrometeoizdat. Leningrad, 285 pp.

——, and J. W. Deardorff, 1974. Similarity theory of the planetary boundary layer of time-dependent height. *J. Atmos. Sci.*, **31,** 1449–1452.