

On the Dynamic Atmospheric Response to the Chandler Wobble Forcing

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(Manuscript received 30 August 1982, in final form 9 December 1982)

ABSTRACT

A simple form of atmospheric tidal theory is used to deduce the dynamic atmospheric response to the nearly periodic 14 month precession of the Earth's rotation axis about its mean position, known as Chandler wobble. The departure of the tide from equilibrium, and the associated horizontal velocity field are calculated for an atmosphere initially at rest and having a constant adiabatic lapse rate with height. Results are in close agreement with those obtained from simple quasi-static approximations to the shallow water equations. The equilibrium tide in surface pressure is small, of order 10^{-3} mb, for typical polar displacements. The departure of the tide from equilibrium is still smaller, of order 10^{-6} mb, with associated horizontal velocities of order 10^{-6} m s⁻¹. Inclusion of a reasonable solid body rotation component in the basic state does not qualitatively alter these results, although the non-equilibrium components are enhanced by nearly an order of magnitude, relative to those for a basic state at rest. The conclusion, qualified by numerous assumptions, is that the atmospheric response in surface pressure and horizontal velocity to the Chandler wobble forcing is of negligible amplitude.

1. Introduction

It appears that Spitaler (1928) first raised the hypothesis that a detectable and climatically significant atmospheric tide might be produced by the Chandler wobble (small variations, of approximately 14 month period, in the apparent orientation of the earth's rotation axis, relative to a geographically fixed co-ordinate system). Maksimov (1958), Maksimov *et al.* (1967, 1970), and more recently Bryson and Starr (1977) have conducted empirical studies in search of this tide. Maksimov claimed to have found the tide in sea-level pressure, and in a related horizontal movement of the center of the Icelandic Low. However, the data he used were so sparse and of such short length that his definite conclusions regarding the tide's planetary scale and structure were not well supported. Bryson and Starr examined sea-level pressure variations of approximately 14 month period at 180 grid points in the Northern Hemisphere for the period 1899–1970. They found the spatial character of this frequency component to be very complex, in contradiction with the generally accepted view that a tide forced by the Chandler wobble would be of the equilibrium form, a simple wavenumber one pattern in longitude, propagating from west to east, with an antisymmetric north–south structure proportional to $\sin\theta \cos\theta$, where θ is colatitude. They argued that their results, in conjunction with accumulating evidence

that the oceanic pole tide was of non-equilibrium amplitude and form (cf., Hosoyama *et al.*, 1975, 1976; Naito, 1977), supported the conclusion that the equilibrium theory of an atmospheric Chandler tide was incorrect, and that a dynamical theory was necessary to account for the observations.

While there can be no question that air parcels do in fact experience the disturbing forces resulting from the Chandler wobble, disagreement and some confusion exists regarding both the amplitude and spatial form of the response (cf., Wilson, 1978; Starr and Bryson, 1978). This is partly due to the complexity of the problem, which is extraordinarily rich in geophysical, oceanographic, meteorological and statistical considerations. But it is also due to the absence to date of any theoretical treatment of the problem which allows for horizontal dynamics.

The only clear theoretical statement of what form the tidal response should take has been based on an equilibrium approach to the problem (cf., Munk and MacDonald, 1960). This approach requires either that the frequency of the imposed forcing be identically zero (the static case), or that the atmosphere's bottom boundary surface moves vertically in perfect synchrony with every air parcel in the column above it. In either case, as is demonstrated later, horizontal motion is completely absent from the solution. Thus, it is not possible to deduce from the equilibrium theory what form a non-equilibrium tide with attendant horizontal motions might take.

In this paper, the results from applying a simple form of atmospheric tidal theory to the Chandler

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wobble forcing are reported. Because the calculation allows for horizontal dynamics, and because the assumption of a stationary bottom boundary requires them, the resulting tide is of non-equilibrium form. However, for the simple model atmosphere used (one at rest and having a constant adiabatic lapse rate with height in the basic state or alternatively, in uniform solid body rotation), the departure of the tide from equilibrium is approximately two orders of magnitude smaller than the equilibrium tide itself, so that the equilibrium theory, while wrong in principle, is nearly right in practice. Since the equilibrium tide itself is very small, these results cast serious doubt on claims that the atmospheric pressure and horizontal velocity responses to the Chandler wobble forcing are significant, or even detectable, in data as variable as surface climate observations.

2. A dynamical formulation of the problem

a. The basic equations

We start with the equations governing the atmospheric response to a small external forcing given in Chapman and Lindzen (1970). As noted therein, numerous assumptions and approximations are invoked to reduce the problem's complexity. These are:

- 1) the motion of the atmosphere relative to a geographically fixed coordinate system is described by the linearized form of the Navier-Stokes equations for a compressible gas;
- 2) the atmosphere remains in local thermodynamic equilibrium;
- 3) the atmosphere is a perfect gas;
- 4) the depth of the atmosphere is shallow compared to the earth's mean radius;
- 5) the atmosphere remains in hydrostatic equilibrium;
- 6) the earth's ellipticity is ignored;
- 7) surface topography is ignored;
- 8) dissipative processes are ignored;
- 9) the undisturbed or basic state is assumed to be motion free so that basic state temperature, pressure and density fields are dependent only on height.

Considerable discussion of the validity of these assumptions and approximations appears in Siebert (1961) and Chapman and Lindzen (1970). The least satisfactory for this problem is the last, since the horizontal velocities associated with a tide of 14 month period will presumably be small compared to realistic mean zonal velocities at most latitudes. Later, assumption (9) will be relaxed to allow the basic state to be in uniform solid body rotation relative to the solid earth.

It is further assumed that the external forcing can be described solely in terms of a tidal potential (external heat sources are assumed absent), and that this

potential is independent of height. The basic linearized equations are then

$$\frac{\partial u}{\partial t} - 2\Omega v \cos\theta = -(a \sin\theta)^{-1} \frac{\partial}{\partial \phi} \left(\frac{\delta p}{\rho_0} + U \right), \quad (1)$$

$$\frac{\partial v}{\partial t} + 2\Omega u \cos\theta = a^{-1} \frac{\partial}{\partial \theta} \left(\frac{\delta p}{\rho_0} + U \right), \quad (2)$$

$$\frac{\partial \delta \rho}{\partial z} = -g \delta \rho, \quad \frac{d\rho_0}{dz} = -g\rho_0, \quad (3)$$

$$\frac{\partial \delta \rho}{\partial t} + w \frac{d\rho_0}{dz} = -\rho_0 \left[(a \sin\theta)^{-1} \left(\frac{\partial}{\partial \theta} (-v \sin\theta) + \frac{\partial u}{\partial \phi} \right) + \frac{\partial w}{\partial z} \right], \quad (4)$$

$$1/(\gamma - 1) \left(\frac{\partial \delta T}{\partial t} + w \frac{dT_0}{dz} \right) = \frac{T_0}{\rho_0} \left(\frac{\partial \delta \rho}{\partial t} + w \frac{d\rho_0}{dz} \right), \quad (5)$$

$$\frac{\delta p}{\rho_0} = \frac{\delta T}{T_0} + \frac{\delta \rho}{\rho_0}, \quad p_0 = RT_0\rho_0, \quad (6)$$

where

- θ colatitude,
- ϕ east longitude,
- z height above the earth's surface,
- u, v, w eastward, northward, and upward velocities,
- $\delta p, \delta \rho, \delta T$ pressure, density, and temperature perturbation fields,
- p_0, ρ_0, T_0 basic state pressure, density, and temperature fields,
- Ω, a earth's rotation rate, mean radius,
- g acceleration of gravity,
- R gas constant for dry air,
- γ $C_p/C_v = 1.4, \kappa = (\gamma - 1)/\gamma,$
- U external tidal potential.

Eqs. (1) and (2) are the horizontal components of the linearized equations of motion, (3) is the hydrostatic relation, (4) the continuity equation, (5) the adiabatic form of the first law of thermodynamics and (6) is the equation of state.

b. The Chandler wobble potential

The potential generated by the Chandler wobble has been discussed in detail by Munk and MacDonald (1960). It is given by

$$U = -(1 + k - \epsilon)\Omega^2 a^2 \sin\theta \cos\theta (m_1 \cos\phi + m_2 \sin\phi),$$

where m_1 and m_2 are the direction cosines of the instantaneous rotation axis with respect to the Greenwich meridian and 90°E. The parameters k and ϵ are the tidal effective Love numbers, (0.59 and 0.29, respectively), which allow for the deformation in the bottom boundary surface and the resulting change in

geopotential, caused by the response of the substrate to the wobble potential. Time series of m_1 and m_2 , after removal of their annual variation, are illustrated in Fig. 1. Power spectral analyses have revealed that these time series appear to consist primarily of very narrow band noise centered on a period of approximately 432 days (cf., Wilson and Vicente, 1980). The motion is a nearly circular precession of the rotation axis from west to east about its mean orientation. Maximum recorded displacements are of order 10^{-6} rad. To simplify the analysis, it will be assumed that the pole motion can be effectively represented as a purely circular precession with amplitude M of this order and period $2\pi/\sigma$ of 432 days. Thus, m_1 and m_2 can be written as:

$$m_1 = M \cos \sigma t, \\ m_2 = M \sin \sigma t,$$

and

$$U = CP_2^1(\cos \theta) \operatorname{Re}[e^{i(\phi - \sigma t)}], \quad (7)$$

with

$$C = \frac{1}{3}(1 + k - \epsilon)\Omega^2 a^2 M,$$

and where $P_2^1(\cos \theta) = -3 \sin \theta \cos \theta$, is the associated Legendre polynomial of order 1 and degree 2.

c. Conversion to the shallow water equations

Since the wobble potential is periodic in longitude and nearly periodic in time, steady-state solutions of Eqs. (1)–(6) are sought for which all the field variables are periodic in longitude and time

$$f = \operatorname{Re}[f(\theta, z)e^{i(s\phi - \sigma t)}], \quad (8)$$

where s is the zonal wave number, equal to unity for the wobble problem, $2\pi/\sigma$ the period of the tidal forcing, and $f(\theta, z)$ is a possibly complex function of θ and z which allows for phase shifts in the field variables relative to each other and the tidal potential.

Considerable manipulation of these equations as detailed in Siebert (1961) leads to the result that the velocity divergence

$$\chi = (a \sin \theta)^{-1} \left(\frac{-\partial}{\partial \theta} (v \sin \theta) + \frac{\partial u}{\partial \phi} \right) + \frac{\partial w}{\partial z}$$

must satisfy the following partial differential equation:

$$T_0 \frac{\partial^2 \chi}{\partial z^2} + \left(\frac{dT_0}{dz} - \frac{g}{R} \right) \frac{\partial \chi}{\partial z} - \left(\frac{g}{4a\Omega^2} \right) F_\lambda^s \left[\left(\frac{g\kappa}{R} + \frac{dT_0}{dz} \right) \chi \right] = 0, \quad (9)$$

where F_λ^s is Laplace's tidal operator:

$$F_\lambda^s = (\sin \theta)^{-1} \frac{\partial}{\partial \theta} \left(\frac{\sin \theta}{\lambda^2 - \cos^2 \theta} \frac{\partial}{\partial \theta} \right) + (\lambda^2 - \cos^2 \theta)^{-1} \left(\frac{s \lambda^2 + \cos^2 \theta}{\lambda \lambda^2 - \cos^2 \theta} - \frac{s^2}{\sin^2 \theta} \right), \quad (10)$$

where

$$\lambda = \frac{\sigma}{2\Omega}.$$

The last simplifying assumption involves the choice of a vertical structure for the basic state of the model atmosphere. A constant dry adiabatic lapse rate was selected:

$$T_0 = T_0(0) - \frac{g}{R} \kappa z, \quad (11)$$

which makes the operand of F_λ^s in (9) identically zero. From this it follows that the only bounded solution of (9) is one in which the divergence is independent of height

$$\chi = \chi(\theta, \phi, 0), \quad (12)$$

and further, that the contributions to χ from vertical and horizontal motions are separately independent of height:

$$\frac{\partial w}{\partial z} = -(\gamma - 1)\chi(\theta, \phi, 0), \quad (13)$$

$$\chi_H = (a \sin \theta)^{-1} \left[-\frac{\partial}{\partial \theta} (v \sin \theta) + \frac{\partial u}{\partial \phi} \right] = \gamma \chi(\theta, \phi, 0). \quad (14)$$

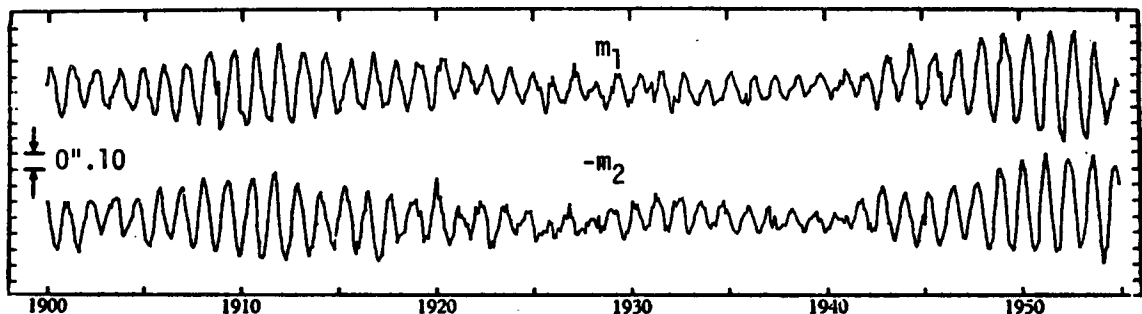


FIG. 1. Time series of the m_1 and m_2 components of Earth's polar motion after removal of the annual cycle (adapted from Munk and MacDonald, 1960).

Direct integration of (13) yields

$$w = -\kappa z \chi_H + w(0). \quad (15)$$

Eq. (11), together with the first law (5), the hydrostatic relation (3) and the equation of state (6) imply that

$$\delta p = RT_0 \delta \rho. \quad (16)$$

Substitution of this result and (15) in the continuity equation (4) finally yields

$$\frac{\partial}{\partial t} \left(\frac{\delta p}{\rho_0} \right) + RT_0 \chi_H = 0. \quad (17)$$

Note here the fact, first demonstrated by Bartels (1927, 1928), that equations (1), (2), and (17) are coupled equations in u , v , and $\delta p/\rho_0$ that are formally identical to the shallow water equations of Laplace. They describe the response of an incompressible fluid, of uniform undisturbed depth h and density ρ_0 , to a small imposed forcing. The identity is established by making the substitutions

$$RT_0 \rightarrow gh, \quad \delta p/\rho_0 \rightarrow g\zeta, \quad (18)$$

where ζ is the variation in the fluid's surface elevation.

d. The equilibrium tide

As mentioned earlier, one way to define the equilibrium tide is to set the frequency of the imposed forcing equal to zero. Then, if the *static* disturbing potential U is simply turned on at $t = 0$, a period of time will pass during which horizontal motion occurs, but this transient motion will inevitably decay to zero as the dissipative processes which have been ignored take effect, and a static horizontal pressure gradient results which exactly counter-balances the gradient of the tidal potential. This pressure distribution is the equilibrium tide given by

$$\overline{\delta p} = -\rho_0 U, \quad \sigma = 0. \quad (19)$$

It is instructive to examine the conditions under which such a tide can be a steady-state solution of the basic dissipation-free equations when the frequency of the imposed forcing is not identically zero. Substitution of (19) into (1) and (2) leads directly to the result that u and v must be everywhere identically zero. (Here u and v are required to be regular functions of colatitude to reach this conclusion.) Thus, if the equilibrium tide (19) is to be a solution, there can be no horizontal motion associated with it. Furthermore, Eq. (13) requires that w be independent of z , and hence the total velocity divergence must vanish. Differentiating (19) with respect to z and using the hydrostatic relation (3) leads to

$$\overline{\delta \rho} = \frac{U}{g} \frac{d\rho_0}{dz}, \quad (20)$$

and substitution of this result in the continuity equation finally yields

$$\overline{w} = -\frac{i\sigma}{gU}. \quad (21)$$

Thus, the equilibrium tide defined by (19) can be a steady-state solution of the basic equations with a non-zero forcing frequency, but only if the atmosphere's bottom boundary surface moves with the same vertical velocity, given by (21), as that of every air parcel in the column above it.

If $w(0)$ is permitted to have any value different from that given by (21), e.g., $w(0) = 0$ (a standard boundary condition for tidal problems), then the equilibrium tide specified by (19) cannot be a solution unless the frequency of the imposed forcing is identically zero, and only then, as noted earlier, if frictional dissipation is invoked to attenuate the horizontal currents induced by turning the potential on at $t = 0$.

Analytical solutions to the shallow water equations (1), (2) and (19) have been vigorously pursued from the time of Laplace to the present day. In the late nineteenth century they were used to model oceanic responses to the purely zonal ($s = 0$) long-period constituents of the solar and lunar tidal potentials (cf., Darwin, 1886; Poincare, 1895; and Hough, 1897). For many years, a lively debate persisted on whether or not these tides were of the equilibrium form. [Wunsch (1967) provides an excellent review of the subject.] The discussion focused on how large frictional dissipation had to be in order to significantly attenuate the horizontal motions, primarily steady-state zonal geostrophic currents, whose presence, of necessity, would result in non-equilibrium tidal responses.

More recently, Proudman (1960) examined this question by using the equivalent forms of the shallow water equations for systems with a finite number of degrees of freedom. He concluded that the limiting response, as the forcing frequency tends to zero, depends critically on the ratio of the forcing frequency to the linear friction coefficient. If this ratio remains finite as σ tends to zero, then the response to purely zonal forcing contains non-vanishing horizontal currents and is necessarily non-equilibrium. If the ratio tends to zero, the limiting response is the static equilibrium tide. Note that Proudman's results imply that for any finite frequency, even with friction present, the response will not be exactly of the equilibrium form, and that the nature of the departure from equilibrium remains unspecified.

3. The forced problem in terms of free mode solutions

In the twentieth century, the shallow water equations, or their equivalent forms for isothermal and more complicated basic states, have been used to

model atmospheric responses to short period constituents of the solar and lunar gravitational potentials, and also to identify the discrete frequencies at which dissipation free model atmospheres resonate (cf., Chapman and Lindzen, 1970). Longuet-Higgins (1968) performed an elegant and thorough study of the free modes of a global ocean governed by Laplace's tidal equations. The free modes are solutions to the unforced or homogeneous form of (1), (2) and (17). When (1) and (2) with $U = 0$ are solved for u and v in terms of $\delta p/\rho_0$, using (8) we obtain

$$v = i\sigma/[4a\Omega^2(\lambda^2 - \cos^2\theta)]\left(\frac{\partial}{\partial\theta} - \frac{s}{\lambda \cot\theta}\right)\frac{\delta p}{\rho_0}, \quad (22)$$

$$u = \sigma[4a\Omega^2(\lambda^2 - \cos^2\theta)]^{-1} \times \left(-\cos\theta\lambda^{-1}\frac{\partial}{\partial\theta} + \frac{s}{\sin\theta}\right)\delta p/\rho_0. \quad (23)$$

Substituting these expressions in (17) yields the free mode equation which the colatitudinal dependence of $\delta p/\rho_0$ must satisfy:

$$F_\lambda^s(\delta p/\rho_0) = -\epsilon\delta p/\rho_0, \quad (24)$$

with F_λ^s given by (10) and $\epsilon = 4a^2\Omega^2/RT_0$, sometimes referred to as Lamb's parameter. For fixed ϵ , regular solutions of (24) exist only for discrete values of the dimensionless frequency parameter $\lambda = \sigma/2\Omega$. Each of the associated free mode solutions, termed a Hough function, has its own characteristic colatitudinal dependence.

While being of considerable interest in and of themselves, solutions of the homogeneous equations have proven useful in the representation of the response of the atmosphere to external gravitational and/or thermal forces. If σ is fixed at the frequency of the forcing, then (20) has regular solutions only for certain discrete values of the dimensionless parameter ϵ . It is known that for fixed σ the Hough function $H_n^{\sigma,s}(\theta)$ associated with different ϵ_n are orthogonal, and that they constitute a complete set of regular colatitude dependent functions. Hence, any regular function of colatitude may be represented with a linear combination of them. Now the inhomogeneous counterpart to (24) is

$$F_\lambda^s(\delta p/\rho_0 + U) = -\epsilon\delta p/\rho_0. \quad (25)$$

Thus, if $\delta p/\rho_0$ and the disturbing potential U are both expanded in terms of the appropriate free modes,

$$\delta p/\rho_0 = \sum_n \delta p_n/\rho_0 H_n^{\sigma,s}(\theta), \quad (26)$$

we find that Eq. (25) reduces term by term to

$$\delta p_n/\rho_0 = (1 - \epsilon/\epsilon_n)^{-1}(-U_n). \quad (28)$$

If we represent the equilibrium tide (for which $\delta p/\rho_0 = -U$) by $\bar{\delta p}/\rho_0$, then (28) may be written as

$$\frac{\delta p_n}{\rho_0} - \frac{\bar{\delta p}_n}{\rho_0} = \frac{\epsilon/\epsilon_n}{(1 - \epsilon/\epsilon_n)} \left(\frac{\bar{\delta p}_n}{\rho_0}\right). \quad (29)$$

This result clearly identifies the only conditions under which the tidal response can be of the equilibrium form: the right hand side must vanish. Thus each ϵ_n must be infinite. The Longuet-Higgins investigation established that this occurs only when $s > 0$ and the forcing frequency σ approaches zero. This equation also identifies some of the conditions under which the tidal response will depart significantly from an equilibrium response. For example, if for some n , ϵ_n is very close to ϵ , then the denominator in (29) can become vanishingly small, and resonant excitation of the associated free mode will take place. More importantly, Eq. (29), together with (1) and (2), enable in principle a calculation of the horizontal velocity fields which are associated with the tidal response, whatever its form.

Fig. 2, adapted from Longuet-Higgins (1968), illustrates the relationship between ϵ_n and the dimensionless frequency parameter λ for longitudinal wave-number one modes. We see that for eastward propagating modes of long period ($\lambda < 10^{-3}$), ϵ_n is extremely large in magnitude (the equivalent depth, $h_n = 4a^2\Omega^2/g\epsilon_n$ is extremely small). Under these conditions, Longuet-Higgins has shown that the modes may be classified into three types:

- 1) a zonally symmetric Kelvin wave trapped near the equator by Coriolis forces to the north and south;
- 2) symmetric and antisymmetric Type 1 modes for $\epsilon_n > 0$, concentrated within an angular distance of order $\epsilon_n^{-1/4}$ from the equator, in which the kinetic energy of horizontal motion exceeds the potential energy by a factor of 3;
- 3) symmetric and antisymmetric Type 6 modes, for $\epsilon_n < 0$, concentrated within an angular distance of order $(-\epsilon_n)^{-1/4}$ from the poles, in which the energy is primarily potential energy and the horizontal motion is in inertial circles.

In order to solve the forced problem by using the free modes, one must first be able to represent their north-south structure accurately in terms of known functions. Longuet-Higgins derived approximate asymptotic forms for the free modes in the limits $\epsilon_n \rightarrow \pm\infty$, but the accuracy of his results, especially for large north-south wavenumbers, is uncertain. A more straightforward numerical approach involves representation of the modes with associated Legendre polynomials (cf., Chapman and Lindzen, 1970). As discussed therein, the ϵ_n , and associated mode structures, are determined by solving a series of successively larger finite-matrix eigenvalue problems which are truncated forms of the exact infinite-matrix eigenvalue problem. Calculation is terminated when the results for a given matrix size are judged to differ insignificantly from those of the previous iteration.

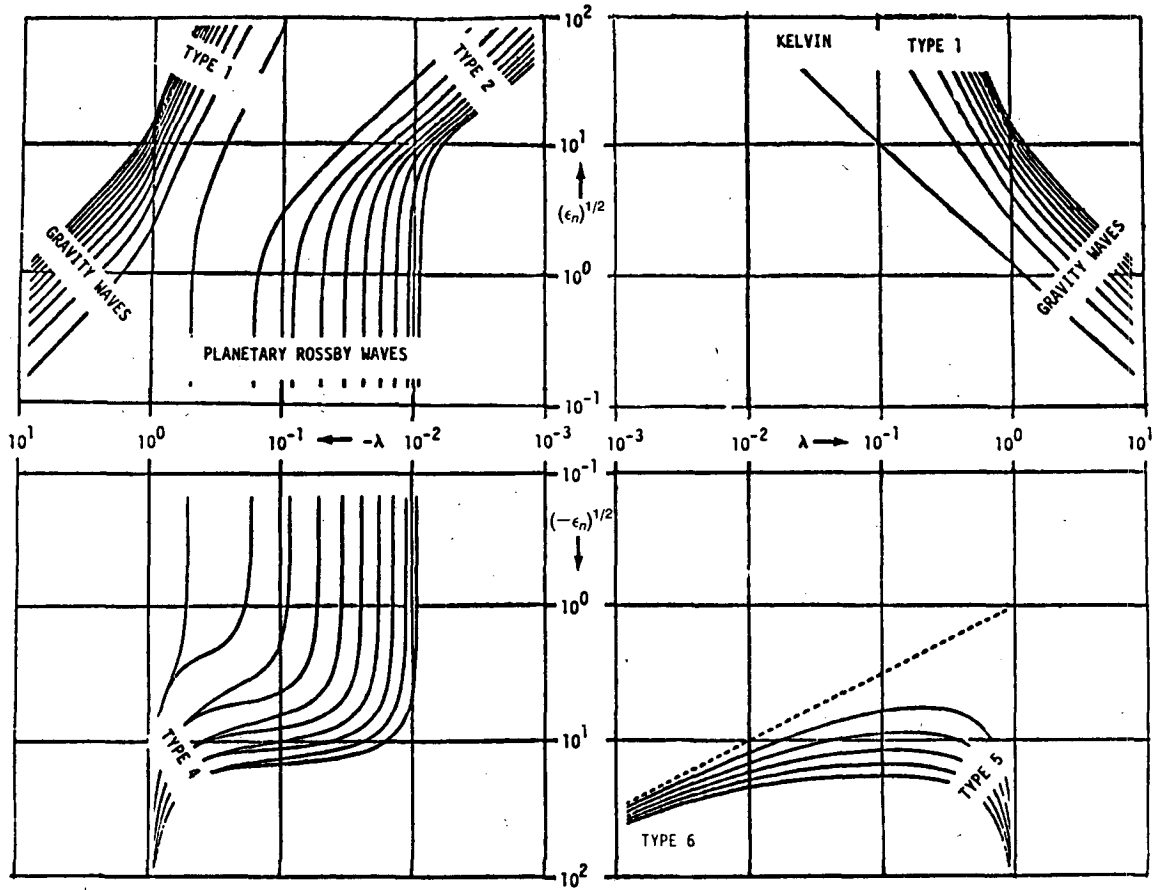


FIG. 2. Illustrating the relationship between ϵ_n and λ for different free mode types (adapted from Longuet-Higgins, 1968).

We attempted such an expansion for the antisymmetric modes which are required to represent the Chandler-wobble potential. A 64×64 matrix involving associated Legendre polynomials of order 1 and degree up to 128 produced reasonably accurate eigenvalues ϵ_n for the first 10 negative equivalent depth modes (of type 6 in Fig. 2), but none of the eigenvalues corresponding to positive equivalent depth modes of type 1 were recovered. Agreement between the colatitudinal structure of any of the modes calculated in this way and the asymptotic forms derived by Longuet-Higgins was rather poor. The reason for poor agreement is clear. As noted earlier, for $\lambda < 10^{-3}$, both Types 1 and 6 modes are well-localized near the equator and poles respectively. Longuet-Higgins' asymptotic forms indicate that only modes of extremely large index n have non-negligible amplitudes in midlatitudes. The associated Legendre polynomials do not share this property: many terms of very large degree are required to reflect it. Thus, the first step in a solution of the forced problem in terms of the free modes, namely, accurately representing their colatitudinal dependence, is limited by the very slow convergence of their expansion

in terms of associated Legendre polynomials. Similar convergence problems are to be expected for expansions in terms of sines and cosines of colatitude.

The second step in this solution process would have involved representation of the tidal potential and the response in terms of the free modes as discussed earlier. A complementary difficulty arises here. The Chandler potential is of north-south wavenumber one form. Its maximum and minimum values occur at 45° north and south. To represent such a slowly varying function of colatitude over the entire pole to pole range would require retention of both Types 1 and 6 modes of extremely high index.

Thus, the free mode approach, which has worked extremely well for forced problems in which λ is of order unity, fails due to two complementary convergence problems when it is applied to the Chandler-wobble forcing.

4. The forced problem directly in terms of associated Legendre polynomials

Following Longuet-Higgins (1968), Eqs. (1), (2) and (17) can be simplified somewhat by use of the substitutions

$$\left. \begin{aligned} u^* &= u \sin\theta \\ v^* &= iv \sin\theta \\ \zeta^* &= (2\Omega a)^{-1} \delta p / \rho_0 \\ \bar{\zeta}^* &= (2\Omega a)^{-1} \overline{\delta p} / \rho_0 = -U/2\Omega a \end{aligned} \right\} \quad (30)$$

$$DP_n^1 = (2n + 1)^{-1} [(n + 1)^2 P_{n-1}^1 - n^2 P_{n+1}^1], \quad (40)$$

$$\nabla^2 P_n^1 = -n(n + 1) P_n^1, \quad (41)$$

Using (8) with $s = 1$ and the fact that u, v and $\delta p / \rho_0$ are independent of height for our choice of basic state vertical structure, we obtain

$$\lambda u^* - \mu v^* - (\zeta^* - \bar{\zeta}^*) = 0, \quad (31)$$

$$\mu u^* - \lambda v^* + D(\zeta^* - \bar{\zeta}^*) = 0, \quad (32)$$

$$u^* - Dv^* - \epsilon\lambda(1 - \mu^2)(\zeta^* - \bar{\zeta}^*) = \epsilon\lambda(1 - \mu^2)\bar{\zeta}^*, \quad (33)$$

where $\mu = \cos\theta$, and $D = (1 - \mu^2)d/d\mu$. Elimination of u^* and v^* with the use of (31) and (32), and multiplication by $(\lambda^2 - \mu^2)$ converts (33) to

$$[(\lambda^2 - \mu^2)(\lambda\nabla^2 + 1) + 2\mu(\lambda D + \mu) + \epsilon\lambda(\lambda^2 - \mu^2)](\zeta^* - \bar{\zeta}^*) = -\epsilon\lambda(\lambda^2 - \mu^2)\bar{\zeta}^*.$$

This equation expresses the departure of the tide from equilibrium directly in terms of the forcing on the right-hand side. Once (34) is solved, both u^* and v^* may be expressed in terms of $(\zeta^* - \bar{\zeta}^*)$ as well:

$$(\lambda^2 - \mu^2)v^* = (\lambda D + \mu)(\zeta^* - \bar{\zeta}^*), \quad (35)$$

$$\mu u^* = \lambda v^* - D(\zeta^* - \bar{\zeta}^*). \quad (36)$$

Now the colatitudinal structure of $\bar{\zeta}^*$ is given by

$$\bar{\zeta}^* = KP_{\frac{1}{2}}^1(\mu), \quad \text{where} \quad K = -(1 + k - \ell)\Omega a/6M \quad (37)$$

from (7). So, we expand $(\zeta^* - \bar{\zeta}^*)$, u^* , v^* and the right hand side of (34) in terms of associated Legendre polynomials:

$$(\zeta^* - \bar{\zeta}^*) = \epsilon\lambda K \sum_{\substack{n=2 \\ \text{even}}}^{\infty} a_n(\epsilon, \lambda) P_n^1(\mu), \quad (38)$$

$$u^* = \epsilon\lambda K \sum_{\substack{n=2 \\ \text{even}}}^{\infty} u_n(\epsilon, \lambda) P_n^1(\mu),$$

$$v^* = \epsilon\lambda K \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} v_n(\epsilon, \lambda) P_n^1(\mu),$$

$$-\epsilon\lambda(\lambda^2 - \mu^2)\bar{\zeta}^* = \epsilon\lambda K \sum_{\substack{n=2 \\ \text{even}}}^{\infty} f_n(\epsilon, \lambda) P_n^1(\mu).$$

Making use of the recurrence relation

$$\mu P_n^1 = (2n + 1)^{-1} [(n + 1)P_{n-1}^1 + nP_{n+1}^1], \quad (39)$$

and the facts that

we obtain the following recurrence relations among the a_n, u_n, v_n and f_n :

$$\begin{aligned} &\epsilon\lambda(q(n - 2)q(n))a_{n-4} \\ &+ q(n)[(p(n - 2) + p(n - 1)) \\ &+ p(n) + p(n + 1)]\epsilon\lambda + (1 - 2\epsilon\lambda^3) \\ &+ (n - 2)(n - 3)\lambda]a_{n-2} + \{(p(n) + p(n + 1)) \\ &\times [(p(n) + p(n + 1))\epsilon\lambda + (1 - 2\epsilon\lambda^3) + n(n + 1)\lambda] \\ &+ (p(n - 1)p(n) + p(n + 1)p(n + 2))\epsilon\lambda \\ &+ 2\lambda(p(n) - p(n + 1)) - n(n + 1)\lambda^3 + \lambda^2 + \epsilon\lambda^5\}a_n \\ &+ r(n)[(p(n) + p(n + 1) + p(n + 2) + p(n + 3))\epsilon\lambda \\ &+ (1 - 2\epsilon\lambda^3) + (n + 3)(n + 4)\lambda]a_{n+2} \\ &+ \epsilon\lambda[r(n)r(n + 2)]a_{n+4} = f_n(\lambda); \quad (42) \end{aligned}$$

where

$$p(n) = \frac{(n - 1)(n + 1)}{(2n - 1)(2n + 1)}, \quad q(n) = \frac{(n - 2)(n - 1)}{(2n - 3)(2n - 1)},$$

$$r(n) = \frac{(n + 2)(n + 3)}{(2n + 3)(2n + 5)},$$

and $f_n = 0$ except for

$$f_2 = [-\lambda^4 + \frac{6}{7}\lambda^2 - \frac{5}{21}], \quad f_4 = [^{12}/_{35}\lambda^2 - ^{12}/_{77}],$$

$$f_6 = [-8/(11 \cdot 21)];$$

$$\begin{aligned} &-q(n)v_{n-2} + [\lambda^2 - (p(n) + p(n + 1))]v_n - r(n)v_{n+2} \\ &= \left(\frac{n - 1}{2n - 1}\right)[1 - (n - 1)\lambda]a_{n-1} \\ &+ \left(\frac{n + 2}{2n + 3}\right)[1 + (n + 2)\lambda]a_{n+1}; \quad (43) \end{aligned}$$

$$\begin{aligned} &\left(\frac{n - 1}{2n - 1}\right)u_{n-1} + \left(\frac{n + 2}{2n + 3}\right)u_{n+1} \\ &= \lambda v_n + \left(\frac{n - 1}{2n - 1}\right)^2 a_{n-1} - \left(\frac{n + 2}{2n + 3}\right)^2 a_{n+1}. \quad (44) \end{aligned}$$

These equations form an infinite coupled linear set. Approximate solutions are obtained by truncation at some finite n as discussed earlier for the free mode representation. However, in this case, the expansion converges very rapidly for λ of order 10^{-3} . In fact, results for a 3×3 representation were identical, within the limits of accuracy imposed by single precision truncation error, to those for a 50×50 representation.

5. Results and discussion

Table 1 presents the coefficient values calculated from the 50 × 50 representation of Eqs. (42)–(44) for both the Chandler frequency (λ = 1.15740 × 10⁻³) and the zero frequency limit. (The fourth column will be discussed later.) T₀(0) was taken to be 288 K, resulting in a value for ε of 10.438154 and a corresponding equivalent depth or scale height at the surface (z = 0) of 8.44 km for the autobarotropic atmosphere considered here. Coefficients of degree higher than 6 for the Chandler frequency case are not presented because they are of order 10⁻⁶(a_n), 10⁻⁵(u_n), and 10⁻⁶(v_n) or less, and they are contaminated with truncation error. For the zero frequency limit, all higher degree coefficients are identically zero.

Comparison of the two sets reveals that the non-vanishing zero frequency coefficients differ from the corresponding Chandler frequency coefficients by no more than ~2%. Thus, a reasonably accurate representation of both the departure of the Chandler tide from its equilibrium form and the associated horizontal velocity field may be obtained by using

$$\begin{aligned} a_n(\epsilon, \lambda) &\approx a_n(\epsilon, 0) \\ u_n(\epsilon, \lambda) &\approx u_n(\epsilon, 0) \\ v_n(\epsilon, \lambda) &\approx v_n(\epsilon, 0). \end{aligned}$$

(Note that in the zero frequency limit Eqs. (42)–(44) also become independent of ε.)

Although (ζ* - ζ̄*) tends to zero linearly as λ → 0, the ratio (ζ* - ζ̄*)/ελ remains finite. In fact, inspection of (34) for λ approaching zero reveals that

$$\lim_{\lambda \rightarrow 0} (\zeta^* - \bar{\zeta}^*)/\epsilon\lambda = -\mu^2 \bar{\zeta}^*, \quad (45)$$

and, using (35) and (36), we obtain

$$\begin{aligned} \lim_{\lambda \rightarrow 0} u^*/\epsilon\lambda &= -1/\mu D[(\zeta^* - \bar{\zeta}^*)/\epsilon\lambda] \\ &= (1 - \mu^2)(2 + \mu d/d\mu) \bar{\zeta}^*, \quad (46) \end{aligned}$$

and
$$\lim_{\lambda \rightarrow 0} v^*/\epsilon\lambda = -1/\mu(\zeta^* - \bar{\zeta}^*)/\epsilon\lambda = \mu \bar{\zeta}^*. \quad (47)$$

These results also follow simply and directly from the shallow water Eqs. (31)–(33), if what could be termed “quasi-static” approximations are made. These consist of ignoring the local time derivative terms in the horizontal momentum equations (31) and (32) and the local time derivative of the departure of the tide from equilibrium in the continuity Eq. (33). Except near the equator, the velocity field is then nearly in geostrophic balance with the departure of the tide from equilibrium, and the small horizontal divergence compensates for the equilibrium tide’s local time derivative.

The spatial patterns of the equilibrium tide, the departure of the tide from equilibrium, and the associated horizontal velocity field are displayed in Figs. 3–5 for the situation in which the earth’s rotation axis is displaced opposite the Greenwich meridian. As the axis precesses from west to east about its mean position, the patterns rotate from west to east at the same rate. Except for the northward velocity component, which is symmetric, the patterns are antisymmetric across the equator. The equilibrium tide has its maximum amplitude at 45°N on the meridian opposite the rotation axis. This maximum represents a surface pressure anomaly of 0.9254 × 10⁻³ mb for a polar displacement of 10⁻⁶ rad. The departure of the tide from equilibrium, which is always of opposite sign to the equilibrium tide, has its maximum amplitude at 60°N on the meridian opposite the rotation axis. For the same polar displacement, this maximum represents a surface pressure anomaly of 7.261 × 10⁻⁶ mb.

The velocity field has its maximum amplitude at the pole and also at 30°N on the meridian opposite the rotation axis. The cross-polar flow is at right angles to this meridian, leading it by 90° of longitude. Again, for a polar displacement of 10⁻⁶ rad, the maximum amplitudes represent velocities of 1.964 × 10⁻⁶ m s⁻¹. An air parcel initially at rest would be displaced only 23.27 m during a half-period of 216 days.

TABLE 1. Coefficients of P_n¹(cosθ) for (ζ* - ζ̄*)/ελK, u*/ελK, and v*/ελK, at the Chandler frequency and the zero frequency limit, for a basic state at rest, and at the Chandler frequency for a basic state in solid-body rotation with $\bar{u} = 10 \text{ m s}^{-1}$.

Coefficients	At rest ε = 10.438154		Solid-body rotation ε' = 10.892554 $\bar{u} = 10 \text{ m s}^{-1}$ λ' = -9.40999 × 10 ⁻³
	λ = 1.15740 × 10 ⁻³	λ = 0	
a ₂	-0.419920	-0.428571 (-3/7)	-0.516375
a ₄	-0.167280	-0.171429 (-6/35)	-0.217966
a ₆	0.000392	0 (0)	-0.006971
u ₂	-1.258614	-1.285714 (-9/7)	-1.559130
u ₄	0.675293	0.685714 (24/35)	0.767749
u ₆	-0.002349	0 (0)	0.0368761
v ₁	0.596265	0.600000 (3/5)	0.637824
v ₃	0.389592	0.400000 (2/5)	0.506247
v ₅	-0.000860	0 (0)	0.15131

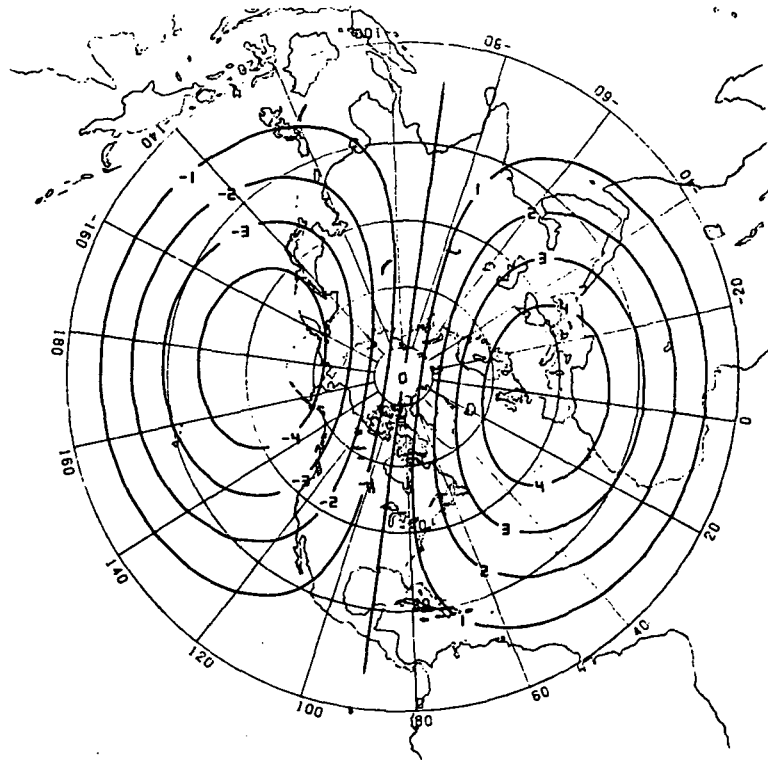


FIG. 3. The equilibrium tide pattern resulting from a polar displacement opposite the Greenwich meridian. The largest amplitude (at 45°N , 0°E) corresponds to a surface pressure anomaly of 0.9254×10^{-3} mb for a polar displacement of 10^{-6} rad.

We therefore conclude, subject to the validity of our assumptions, that the atmospheric Chandler tide and its associated horizontal motion field are of negligible amplitude and, in fact, lie well below the limits of detectability, given current instrumental resolution and the short record length (80 y). Assuming no instrumental or observational error at all, one would need a record of $\sim 10^6$ consecutive monthly observations to identify the maximum equilibrium tide, with 90% confidence, in an otherwise random time series having a standard deviation of 1 mb. Alternatively, assuming the same standard deviation, the tidal response to a displacement ~ 50 times larger than the maximum observed value would be identifiable, with 90% confidence, in the modern instrumental period of the last 80 years.

Finally, it is important to recognize that the model atmosphere used for these calculations has been constructed with numerous simplifying assumptions, and consequently does not correspond very accurately to the real atmosphere. The most telling discrepancies arise in the basic state itself. The model atmosphere used herein is at rest, while the real one is not. The assumption of a basic state at rest or nearly so is much more appropriate for forcing frequencies of the same order as the diurnal and, as detailed in Chapman and

Lindzen (1970), it has been used effectively to calculate atmospheric responses to semidiurnal gravitational and thermal forcings. Inclusion of a realistic mean zonal flow with shear in the basic state could produce significant changes in our results. Unfortunately, it also destroys the separability (in colatitude and height) of the basic equations. A fully numerical solution as described, e.g., in Kasahara (1980), is then the only recourse.

It is possible, however, without losing complete sight of an analytical solution, to add to the basic state a purely zonal solid-body rotation component, in geostrophic balance with the meridional gradient of the scale height. Advective terms must be retained in the linearized horizontal momentum and continuity equations. The zonal advection Doppler shifts the forcing frequency from σ to $\sigma' = \sigma - \bar{u}/a$, where $u_0(\theta) = \bar{u} \sin\theta$ is the zonal solid-body rotation velocity. A meridional advection term in the continuity equation must be retained as well.

Additionally, the curvature terms which appear in the horizontal momentum equations shift the atmosphere's basic state angular velocity from Ω to $\Omega' = \Omega + \bar{u}/a$. Finally, the dependence of h on colatitude in the basic state converts Lamb's parameter ϵ , through its dependence on h , to a colatitude de-

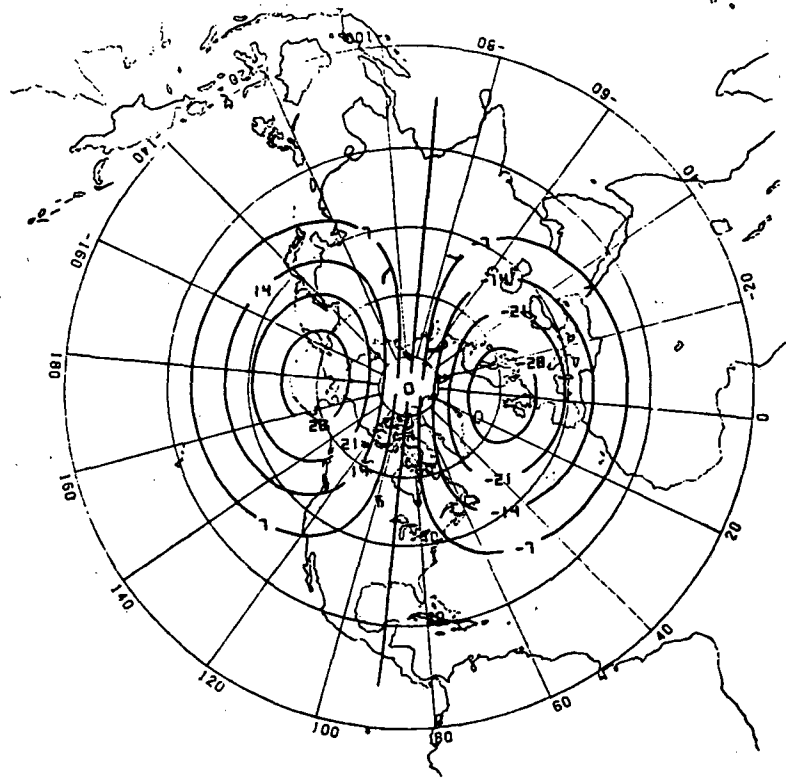


FIG. 4. The departure of the Chandler tide from equilibrium for the same conditions as in Fig. 3. The largest amplitude (at 60°N, 0°E) corresponds to a surface pressure anomaly of 7.264×10^{-6} mb for a polar displacement of 10^{-6} rad.

pendent function $\epsilon' = 4a^2\Omega^2/gh$. As a result of all these modifications equations (31)–(33) must be replaced by

$$\lambda'u^* - \mu v^* - (\zeta^* - \bar{\zeta}^*) = 0, \tag{48}$$

$$\mu u^* - \lambda'v^* + D(\zeta^* - \bar{\zeta}^*) = 0, \tag{49}$$

$$u^* + h^{-1}D(hv^*) - \epsilon'\lambda'(1 - \mu^2)(\zeta^* - \bar{\zeta}^*) = \epsilon'\lambda'(1 - \mu^2)\bar{\zeta}^*, \tag{50}$$

where the height variables ζ^* and $\bar{\zeta}^*$ are now scaled relative to ζ and $\bar{\zeta}$ by $g/2\Omega'a$.

The requirement of geostrophic balance between the zonal solid-body rotation velocity and the meridional height gradient in the basic state may be expressed as

$$\partial h/\partial\theta = h_0\epsilon(1 + \bar{u}/a/2\Omega)(\bar{u}/a/2\Omega) \sin\theta \cos\theta, \tag{51}$$

where h_0 represents the uniform fluid depth in the corresponding at rest basic state. Direct integration of (51) yields

$$h = \frac{1}{2}h_0\epsilon\left(\frac{1 + \bar{u}/a}{2\Omega}\right)\left(\frac{\bar{u}/a}{2\Omega}\right) \sin^2\theta + \bar{h}. \tag{52}$$

Requiring the fluid mass to be the same as for the at rest basic state leads to

$$\bar{h} = h_0\left[1 - \frac{1}{3}\epsilon\left(\frac{1 + \bar{u}/a}{2\Omega}\right)\left(\frac{\bar{u}/a}{2\Omega}\right)\right], \tag{53}$$

and thus

$$h = h_0\left[1 + \epsilon\left[\frac{1 + \bar{u}/a}{2\Omega}\right]\left(\frac{\bar{u}/a}{2\Omega}\right) \times (-\frac{1}{3} + \frac{1}{2} \sin^2\theta)\right]. \tag{54}$$

The largest departure of h from h_0 occurs at the poles where, for $\bar{u} = 10 \text{ m s}^{-1}$ and $\epsilon = 10.438154$ (see Table 1),

$$(h - h_0)/h_0 = -0.0379.$$

This difference is small enough to justify neglecting the dependence of h on colatitude. For larger u , however, this factor should be explicitly taken into account.

Treating h as a constant, Eqs. (48)–(50) reduce to the original set (30)–(33) for an at-rest basic state, except that λ and ϵ are replaced by λ' and ϵ' . Results of the calculations for $\bar{u} = 10 \text{ m s}^{-1}$ are presented in the last column of Table 1. Here λ' is nearly an order of magnitude larger than λ and is also negative, indicating that the potential appears to propagate west-

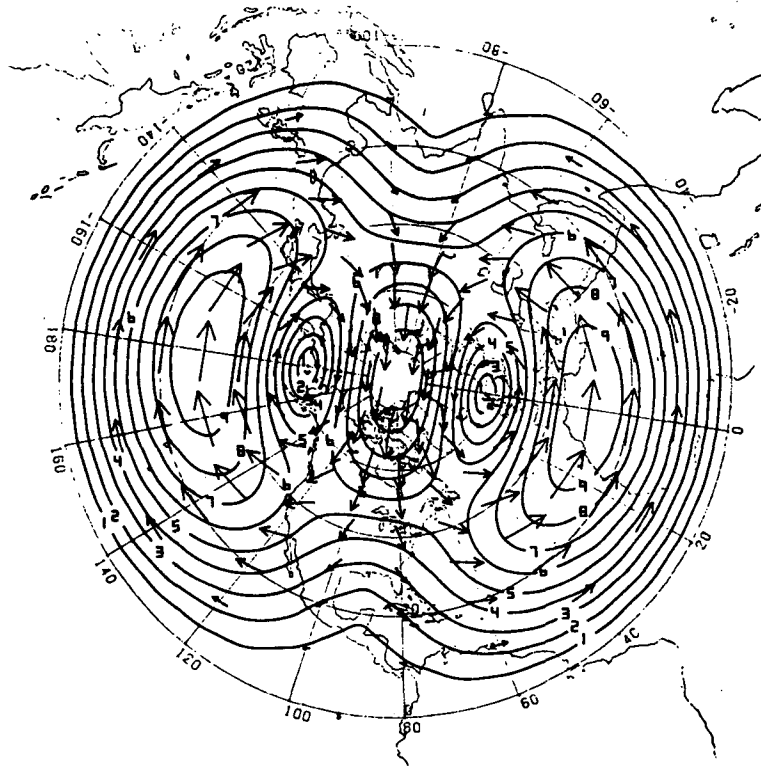


FIG. 5. Isotachs and wind vectors of the velocity field associated with the atmospheric Chandler tide for the same conditions as in Fig. 4. The largest amplitude (at 90°N) corresponds to a horizontal velocity of $1.964 \times 10^{-6} \text{ m s}^{-1}$.

ward with a period of ~ 52 days relative to the solid body zonal flow. Amplitudes of the dominant terms in the expansions are increased relative to those for the at-rest basic state by as much as 30%. Since the unscaled height departure and horizontal velocity fields are proportional to $\epsilon'\lambda'$, the inclusion of a reasonable solid-body rotation component in the basic state thus increases the dynamic components of the Chandler tide by approximately one order of magnitude. However, this rather large increase still falls far short of producing a tide which departs significantly from equilibrium. The equatorial velocity in the basic state would have to be increased to nearly 60 m s^{-1} before a large resonant response could occur. [At this very high solid body rotation rate, σ' is very nearly equal to the frequency of the $l_R = 2, s = 1$ free Rossby mode identified by Kasahara (1980). Examination of his Fig. 4 reveals the strong similarity between the spatial structure of this free mode and that of the non-equilibrium components of the response described herein. Thus a resonant response in this case could be expected.]

It must also be noted that inclusion of more realistic vertical structure in the basic state leads to the existence of internal modes, which might be excited by the forcing indirectly through interaction with surface topography. Furthermore, boundary conditions

have also been simplified in our analysis. Surface topography has been ignored, and its inclusion could lead to larger local effects. In addition, the oceans respond to the Chandler wobble forcing, and as noted earlier, observations of sea level variations suggest that the oceanic Chandler tide is not of equilibrium amplitude or form. In some coastal areas such as near the Baltic and North Seas, the observed tide is several times larger than its equilibrium value. Sawada (1965) has examined the effects of oceanic tides, regarded as an external forcing, on the atmosphere, and has raised the possibility of resonant amplification.

To address these and other complicating factors in detail is beyond the scope of this paper. They are noted here only to appropriately qualify the conclusion. It would be extremely difficult to incorporate these factors into a purely analytical approach to the wobble problem. If serious concern persists that some or all of these factors could lead to a dramatically enhanced atmospheric Chandler tide, the issue might be resolved by numerical simulation with a general circulation model.

Acknowledgments. The author thanks John Young, Jon Ahlquist, Eric Nash, William O'Connor and Klaus Weickmann for helpful discussions and useful comments on this work, and Melanie Woodworth for

typing of the manuscript. This research was supported by NSF-Climate Dynamics Grant ATM77-15213.

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