# Bounds on the Growth of Perturbations to Non-Parallel Steady Flow on the Barotropic Beta Plane

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(Manuscript received 9 March 1982, in final form 29 November 1982)

ABSTRACT

Based on consideration of the perturbation enstrophy and energy equations, we have derived a general family of bounds on the growth rates of perturbations to non-parallel (vortex-like or wave-like) flow on the barotropic beta-plane, allowing for the effects of forcing, Ekman friction, and topography. The family of bounds generalizes Arnol'd's stability criterion. A number of specific applications of the family of bounds are explored. In particular, the formulas are used to demonstrate that the growth rate of the perturbations must vanish if the perturbation length-scale approaches zero or infinity. The distinction between transient and sustained growth of perturbation energy is discussed in light of our results. It is suggested that the bounds are most useful for estimating transient growth rates.

## 1. Introduction

There has recently been a great deal of interest in the stability properties of non-parallel flows, that is, flows in which the streamlines are neither parallel lines nor concentric circles. Wave instabilities represent a well-known example of this class of problems, and Frederiksen (1979) and Niehaus (1980) have suggested that such instabilities may govern the global distribution of cyclogenesis and eddy fluxes. Pierrehumbert and Widnall (1982) have had some success in relating instabilities of a non-parallel basic state to various stages in the development of the free shear layer. However, very little is known mathematically about this class of problems. Arnol'd (1965) has derived a stability criterion for inviscid non-parallel flows, and Blumen (1968) has extended Arnol'd's result to the case of continuously stratified quasigeostrophic flows. Nevertheless, one would like to know more about the character of the instabilities in the unstable case. In this note, we hope to make a contribution along these lines by deriving some simple inequalities governing the growth rate of a two-dimensional perturbation to a two-dimensional steady state of the barotropic vorticity equation on the beta plane.

## 2. Derivation of the general growth rate inequality

The equation for flow on the beta plane in the presence of topography, forced by a vorticity source and damped by Ekman friction is

$$\partial_t \nabla^2 \psi + J(\psi, q) = -\nu \nabla^2(\psi - \psi^*), \tag{2.1}$$

where the vorticity $q$ is given by

$$q = \nabla^2 \psi + \beta y + \frac{f_0 h}{H}, \tag{2.2}$$

with topographic height $h(x, y)$, layer depth $H$, and Coriolis parameter $f_0$. Here $\nu$ is the damping coefficient and $\psi^*$ is a given function representing the external forcing. The remaining symbols have their customary meaning. The steady states are determined by the nonlinear equation

$$J(\psi_0, q_0) = -\nu \nabla^2(\psi_0 - \psi^*) \tag{2.3}$$

for the steady-state streamfunction $\psi_0$. In the inviscid unforced case ($\nu = 0$) this equation has the familiar solution $q_0 = f(\psi_0)$. Perturbing about the steady state defined by (2.3), we substitute $\psi = \psi_0 + \psi'$ into (2.1) and neglect terms that are second order in the perturbation quantities. This results in the perturbation equation

$$\partial_t \nabla^2 \psi' + J(\psi', q_0) + J(\psi_0, \nabla^2 \psi') = -\nu \nabla^2 \psi'. \tag{2.4}$$

Finally, substituting $\psi' = \tilde{\psi} \exp(-\nu t)$, we arrive at the equation

$$\partial_t \nabla^2 \tilde{\psi} + J(\tilde{\psi}, q_0) + J(\psi_0, \nabla^2 \tilde{\psi}) = 0. \tag{2.5}$$

It is noteworthy that this equation is identical in form to the perturbation equation for the unforced inviscid case, except for the important difference that in the undamped case $q_0 = f(\psi_0)$, whereas in the general case $q_0$ and $\psi_0$ are not functionally related.

We now form a weighted enstrophy and an energy equation from (2.5). To form the enstrophy equation,

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we multiply (2.5) by $G(\psi_0)\nabla^2\tilde{\psi}$, where $G$ is an arbitrary function, and integrate over the flow domain. Using integration by parts, the contribution from the second Jacobian vanishes in a domain that is closed, periodic in both directions, periodic in one direction and bounded by rigid boundaries in the other, or in an infinite domain in which the mean and perturbation quantities vanish sufficiently rapidly at infinity.[2] We are left with the perturbation enstrophy equation

$$\frac{d}{dt}\frac{1}{2}\iint [G(\psi_0)(\nabla^2\tilde{\psi})^2] + \iint [G(\psi_0)\nabla^2\tilde{\psi}J(\tilde{\psi}, q_0)] = 0. \quad (2.6)$$

The perturbation energy equation is formed by multiplying (2.5) by $\tilde{\psi}$ and integrating, whereupon the contribution from the first Jacobian vanishes[2] under the above-stated conditions, leaving

$$\frac{d}{dt}\frac{1}{2}\iint [\nabla\tilde{\psi}\cdot\nabla\tilde{\psi}] - \iint [\tilde{\psi}J(\psi_0, \nabla^2\tilde{\psi})] = 0. \quad (2.7)$$

Adding the two conservation equations, we obtain

$$\frac{d}{dt}\frac{1}{2}\iint [G(\psi_0)(\nabla^2\tilde{\psi})^2 + \nabla\tilde{\psi}\cdot\nabla\tilde{\psi}] = \iint [\tilde{\psi}J(\psi_0, \nabla^2\tilde{\psi}) - G(\psi_0)\nabla^2\tilde{\psi}J(\tilde{\psi}, q_0)]. \quad (2.8)$$

But the first Jacobian may be written

$$J(\psi_0, \nabla^2\tilde{\psi}) = \nabla\cdot(\mathbf{v}_0\nabla^2\tilde{\psi}), \quad (2.9)$$

where $\mathbf{v}_0$ is the divergenceless velocity field of the steady state. Using this expression and integration by parts, (2.8) may be re-written as

$$\frac{d}{dt}\frac{1}{2}\iint [G(\psi_0)(\nabla^2\tilde{\psi})^2 + \nabla\tilde{\psi}\cdot\nabla\tilde{\psi}] = -\iint (\nabla^2\tilde{\psi})\nabla\tilde{\psi}\cdot[\mathbf{v}_0 - G(\psi_0)\mathbf{z}\times\nabla q_0], \quad (2.10)$$

where $\mathbf{z}$ is the unit vector in the vertical direction. Denoting the integral on the left-hand side by $Z$,

$$\frac{1}{Z}\frac{d}{dt}\frac{1}{2}Z = \frac{-\iint (\nabla^2\tilde{\psi})\nabla\tilde{\psi}\cdot[\mathbf{v}_0 - G(\psi_0)\mathbf{z}\times\nabla q_0]}{\iint (G(\psi_0)(\nabla^2\tilde{\psi})^2 + \nabla\tilde{\psi}\cdot\nabla\tilde{\psi})}. \quad (2.11)$$

This expression can be generalized further by noting that, under some circumstances, a constant vector can be added to the bracketed term in the numerator of (2.11) without affecting the value of the integral. Specifically, if $\mathbf{C}$ is a constant vector, the identity

$$(\mathbf{C}\cdot\nabla\tilde{\psi})\nabla^2\tilde{\psi} = \nabla\cdot[(\nabla\tilde{\psi})(\mathbf{C}\cdot\nabla\tilde{\psi}) - \mathbf{C}(\tfrac{1}{2}\nabla\tilde{\psi}\cdot\nabla\tilde{\psi})] \quad (2.12)$$

implies that

$$\iint (\mathbf{C}\cdot\nabla\tilde{\psi})\nabla^2\tilde{\psi} = \iint_{\partial R} [(\mathbf{C}\cdot\nabla\tilde{\psi})\nabla\tilde{\psi}\cdot\mathbf{n} - (\tfrac{1}{2}\nabla\tilde{\psi}\cdot\nabla\tilde{\psi})\mathbf{C}\cdot\mathbf{n}], \quad (2.13)$$

where $\partial R$ represents the boundary curve of the flow domain and $\mathbf{n}$ is the unit outward normal to that curve. In the case of an unbounded domain, we consider the path integral to be the limit of the integral along a finite closed curve as the curve is removed to infinity. The right hand side of (2.13) vanishes under any of the following circumstances: 1) the domain is periodic in $x$ or unbounded in $x$ with sufficiently rapid decay of perturbation velocities, and $C_y = 0$, or 2) the domain is periodic in $y$ or unbounded in $y$ with sufficiently rapid decay of perturbation velocities, and $C_x = 0$, or 3) the domain is doubly periodic or unbounded in both directions with sufficiently rapid decay of perturbation velocities. Thus, the restrictions on $\mathbf{C}$ depend on the geometry under consideration. The above results may be summarized by saying that if a domain permits a mean flow in some direction, then the addition of a constant vector in that direction to $\mathbf{v}_0$ leaves (2.11) unchanged. With the above restrictions on $\mathbf{C}$, (2.11) may be rewritten

$$\frac{1}{Z}\frac{d}{dt}\frac{1}{2}Z = \frac{-\iint (\nabla^2\tilde{\psi})\nabla\tilde{\psi}\cdot[\mathbf{v}_0 - \mathbf{C} - G(\psi_0)\mathbf{z}\times\nabla q_0]}{\iint (G(\psi_0)(\nabla^2\tilde{\psi})^2 + \nabla\tilde{\psi}\cdot\nabla\tilde{\psi})}. \quad (2.14)$$

The significance of this statement is that the growth rate of $Z$ is insensitive to the spatial mean flow components of the basic flow $\mathbf{v}_0$, when such components are allowed by the geometry.

The lhs of (2.14) represents the growth rate of the quantity $\sqrt{Z}$. For different choices of $G$ we can prove various theorems. For example, in the inviscid case in which $q_0 = f(\psi_0)$, we have

$$\mathbf{z}\times\nabla q_0 = f'(\psi_0)\mathbf{z}\times\nabla\psi_0 = f'(\psi_0)\mathbf{v}_0. \quad (2.15)$$

If we then choose $G = 1/f'$, the right-hand side of (2.11) will vanish, and $Z$ will be time-invariant. If $f'$ is strictly positive and finite, $Z$ will be positive definite because $G$ will be positive and finite. If this constraint on $f$ is satisfied, the perturbation cannot grow indefinitely large. We have thus recovered the stability theorem of Arnol'd (1965). It is noteworthy that even

[2] The doubly periodic case requires special consideration, and some constraints apply to $G$. See Appendix A.

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a small amount of dissipation will disrupt the relation $q_0 = f(\psi_0)$ so that $Z$ will no longer be time-invariant. This suggests that a steady state of the equations of motion which is maintained against weak dissipation by a weak forcing may possess instabilities induced by the nonconservative effects, even if the state differs only slightly from a stable equilibrium state of the inviscid equations.

Useful bounds on the perturbation growth rate can be obtained only when $G$ is chosen so as to make $Z$ a positive definite quantity. In the simple case $G = 0$, $Z$ is proportional to perturbation kinetic energy, and (2.11) gives the growth rate of the square root of this quantity. Let us introduce the notation $\tilde{E} = \tfrac{1}{2}Z(\tilde{\psi})$, in which $Z$ is to be computed with $G = 0$. Through some elementary applications of the divergence theorem, (2.11) in this case can be written

$$\frac{d}{dt}\ln(\tilde{E}) = 2\frac{\iint \sum_{ij}\left(\frac{\partial\tilde{\psi}}{\partial x_i}\right)\left(\frac{\partial\tilde{\psi}}{\partial x_j}\right)S_{ij}}{\iint \nabla\tilde{\psi}\cdot\nabla\tilde{\psi}}, \quad (2.16)$$

where the deformation tensor $S_{ij}$ is given by

$$S_{ij} = \tfrac{1}{2}(\partial_{x_i}v_{0j} + \partial_{x_j}v_{0i}), \quad (2.17)$$

where $v_{0j}$ is the $j$th component of the steady-state velocity. The latin indices in these and other equations presented in this paper are to be understood as ranging from one to two, with the convention $x_1 = x$ and $x_2 = y$. Making use of the inequality

$$\iint \sum_{ij}\left(\frac{\partial\tilde{\psi}}{\partial x_i}\right)\left(\frac{\partial\tilde{\psi}}{\partial x_j}\right)S_{ij} \leqslant \sum_{ij} S_{ij}^M \iint \left|\frac{\partial\tilde{\psi}}{\partial x_i}\right|\left|\frac{\partial\tilde{\psi}}{\partial x_j}\right|, \quad (2.18)$$

where

$$S_{ij}^M = \underset{x,y}{\mathrm{MAX}}[S_{ij}(x, y)],$$

and the Bessel inequality

$$\iint |(\partial_x\tilde{\psi})||(\partial_y\tilde{\psi})| \leqslant \frac{1}{2}\left(\iint \nabla\tilde{\psi}\cdot\nabla\tilde{\psi}\right), \quad (2.19)$$

we obtain

$$\iint \sum_{ij}\left(\frac{\partial\tilde{\psi}}{\partial x_i}\right)\left(\frac{\partial\tilde{\psi}}{\partial x_j}\right)S_{ij} \leqslant 2S_M \iint \nabla\tilde{\psi}\cdot\nabla\tilde{\psi}, \quad (2.20)$$

where $S_M = \tfrac{1}{2}[S_{12}^M + \mathrm{MAX}(S_{11}, S_{22})]$. When this is substituted into (2.11) we obtain the following bound on the growth of perturbation kinetic energy:

$$\frac{d}{dt}\ln\tilde{E} \leqslant 4S_M. \quad (2.21)$$

We have shown elsewhere that in the non-rotating case, the same bound restricts the growth of three-dimensional perturbations to a two-dimensional steady state (Pierrehumbert, 1980). In the parallel-flow case for which $u_x = u(y)$ and $u_y = 0$, (2.21) reduces to

$$\frac{d}{dt}\ln(\tilde{E}) \leqslant \underset{y}{\mathrm{MAX}}\left|\frac{du}{dy}\right|. \quad (2.22)$$

which is identical to the barotropic limit of the bound given by Eq. (7.5.24) in Pedlosky (1979, p. 451). A similar result for the case of flow with circular streamlines can also be obtained. We note that in deriving (2.21) it was not necessary to assume that the flow is inviscid. In the dissipative case, however, $\tilde{E}$ is no longer proportional to kinetic energy. Instead, using the relationship between $\tilde{\psi}$ and $\psi'$, we find that the perturbation kinetic energy is given, up to a constant factor, by

$$E = \tilde{E}e^{-2\nu t}. \quad (2.23)$$

Substituting this into (2.21) we find that the bound on the growth rate of perturbation kinetic energy in the dissipative case is

$$\frac{d}{dt}\ln(E) \leqslant 4S_M - 2\nu. \quad (2.24)$$

This equation implies that if the dissipation is sufficiently large, specifically if $\nu > 2S_M$, then the energy of an arbitrary small perturbation will decay monotonically. Experience with the Orr–Somerfeld equation, however, indicates that energy arguments of this type are likely to greatly overestimate the critical dissipation required for stability (Lin, 1955).

We now return to the general case and derive a family of bounds which incorporate the constraints imposed by the enstrophy equation that are implicitly contained in (2.14). First, we note that if $\mathbf{A}(x, y)$ is an arbitrary vector field and $F(x, y)$ is an arbitrary scalar function, and $D$ is an arbitrary region, we have

$$0 \leqslant \iint_D \left[(F)(\nabla^2\tilde{\psi})\mathbf{A} \pm \frac{\nabla\tilde{\psi}}{F}\right]^2 = \iint_D \left[(F^2\mathbf{A}\cdot\mathbf{A}(\nabla^2\tilde{\psi})^2 + \frac{\nabla\tilde{\psi}\cdot\nabla\tilde{\psi}}{F^2}\right] \pm 2\iint_D \nabla^2\tilde{\psi}\cdot(\mathbf{A}\cdot\nabla\tilde{\psi}), \quad (2.25)$$

so that

$$\left|\iint_D (\nabla^2\tilde{\psi})\nabla\tilde{\psi}\cdot\mathbf{A}\right| \leqslant \frac{1}{2}\iint_D \left[F^2\mathbf{A}^2(\nabla^2\tilde{\psi})^2 + \frac{\nabla\tilde{\psi}\cdot\nabla\tilde{\psi}}{F^2}\right]. \quad (2.26)$$

We now use (2.26) along with (2.14) to derive a series of growth rate bounds. If we choose

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$$\mathbf{A} = \mathbf{v}_0 - G(\psi_0)\mathbf{z} \times \nabla q_0 - \mathbf{C} \tag{2.27}$$

in (2.26) and apply (2.14), we arrive at the inequality

$$\sigma \equiv \left| \frac{1}{Z}\frac{d}{dt}\frac{1}{2}Z \right| \leq \frac{1}{2} \frac{\iint_D \left[ F^2\mathbf{A}^2(\nabla^2\tilde{\psi})^2 + \frac{\nabla\tilde{\psi}\cdot\nabla\tilde{\psi}}{F^2} \right]}{\iint [G(\psi_0)(\nabla^2\tilde{\psi})^2 + \nabla\tilde{\psi}\cdot\nabla\tilde{\psi}]}, \tag{2.28}$$

where $F(x, y)$, $G(\psi_0)$ and $\mathbf{C}$ are still arbitrary. The two-dimensional region $D$, however, is now required to be the region in which $\mathbf{A}$ is non-zero.

To proceed further, we must introduce the assumption that $G(\psi_0)$ is positive everywhere. Among other things this assures that the quantity $Z$ is positive definite in the perturbation quantities so that a bound to the growth rate of $Z$ yields a bound to the growth rate of the perturbation quantities themselves. If $G$ is positive, we have

$$\iint_D F^2\mathbf{A}^2(\nabla^2\tilde{\psi})^2 = \iint_D \left(\frac{F^2\mathbf{A}^2}{G}\right)(\nabla^2\tilde{\psi})^2 G \leq M_1 \iint_D (\nabla^2\tilde{\psi})^2 G \leq M_1 \iint (\nabla^2\tilde{\psi})^2 G, \tag{2.29}$$

$$\iint_D \frac{\nabla\tilde{\psi}\cdot\nabla\tilde{\psi}}{F^2} \leq M_2 \iint_D \nabla\tilde{\psi}\cdot\nabla\tilde{\psi} \leq M_2 \iint \nabla\tilde{\psi}\cdot\nabla\tilde{\psi}, \tag{2.30}$$

where $M_1$ and $M_2$ are the maxima of certain quantities over the domain $D$, *viz.*,

$$\left. \begin{aligned} M_1 &= \underset{x,y\in D}{\mathrm{MAX}} \left(\frac{F^2\mathbf{A}^2}{G}\right) \\ M_2 &= \underset{x,y\in D}{\mathrm{MAX}} \left(\frac{1}{F^2}\right) \end{aligned} \right\}. \tag{2.31}$$

Using (2.29) and (2.30) we may rewrite (2.28) as

$$\sigma \leq \frac{1}{2} \frac{M_1R_1 + M_2R_2}{R_1 + R_2}, \tag{2.32}$$

where

$$R_1 = \iint [(\nabla^2\tilde{\psi})^2 G], \tag{2.33a}$$

$$R_2 = \iint [\nabla\tilde{\psi}\cdot\nabla\tilde{\psi}]. \tag{2.33b}$$

It is evident that $R_1$ and $R_2$ are both non-negative. Letting $R = R_2/R_1$, the growth rate bound may be written in the more convenient form

$$\sigma \leq \frac{1}{2} \frac{M_1 + M_2R}{1 + R}. \tag{2.34}$$

The quantities $M_1$ and $M_2$ may be regarded as known, as they depend only on the externally specified functions $F$ and $G$ and on the basic-state flow field quantities involved in the definition of $\mathbf{A}$. The quantity $R$, on the other hand, is a function of the perturbation fields; fortunately it admits of a simple physical interpretation.

Were it not for the positive factor $G$ appearing in the integrand of (2.33a) $R$ would be proportional to the ratio of perturbation energy to perturbation enstrophy; this latter quantity is precisely $L_*^2$, where $L_*$ is the mean length scale of the perturbation field as defined by Fjørtoft (1953). Thus, $R^{1/2}$ may be regarded as being proportional to a weighted mean perturbation length scale, in which the enstrophy in certain regions of space is given greater weight than in others. As the weighting factor $G$ appears only in $R_1$ and not in $R_2$, the weighted mean scale cannot be regarded as the local length scale one would obtain by restricting attention to regions where $G$ is largest. In any event, if $G(\psi_0)$ is bounded below and above by $G_1$ and $G_2$, we have the inequalities

$$G_1 \iint (\nabla^2\tilde{\psi})^2 \leq R_1 \leq G_2 \iint (\nabla^2\tilde{\psi})^2. \tag{2.35}$$

Hence, as long as $G_2$ is finite and $G_1$ is non-zero, we have the result that $R$ becomes infinite whenever $L_*^2$ becomes infinite and $R$ approaches zero whenever $L_*^2$ approaches zero.

Our objective now is to choose $F$, $G$, and $\mathbf{C}$ so as to obtain the best possible bound. At this point, we will make use of only a small part of this arbitrariness by writing $F = b\bar{F}$, and varying the constant $b$ so as to obtain the optimal bound for fixed $R$. Substituting the expression for $F$ into (2.31) defining $M_1$ and $M_2$, and using the result in (2.34), we find

$$\sigma \leq \frac{1}{2} \frac{\bar{M}_1b^2 + \bar{M}_2R/b^2}{1 + R}, \tag{2.36}$$

where $\bar{M}_1$ and $\bar{M}_2$ are computed using $\bar{F}$. The right-hand side of (2.36) is minimized when $b^2 = (M_2R/M_1)^{1/2}$. Substituting this expression into (2.36) we obtain the bound

$$\sigma \leq (\bar{M}_1\bar{M}_2)^{1/2}R^{1/2}(1 + R)^{-1}. \tag{2.37}$$

Note that the rhs of (2.37) vanishes if $R$ becomes infinite or zero. Hence, we have proved that, provided functions $\bar{F}(x, y)$ and $G(\psi_0)$ can be chosen such that $M_1$ and $M_2$ are finite and $G$ is bounded above and away from zero below, *a perturbation stops growing if its mean length scale becomes infinitely large or vanishingly small.* In most cases, the conditions of the theorem can be met by taking $F$ and $G$ constant and in an unbounded domain choosing $\mathbf{C}$ to be the mean velocity at infinity.

If some estimate of the length scale of the perturbation is available, (2.37) can be used to obtain a family of bounds on the growth rate of the perturbation. However, in many cases no such estimate is

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available, and we are only interested in the maximum growth rate over all possible perturbations. To obtain this we choose $R$ so as to maximize the rhs of (2.37), which yields the bound for the "worst case" perturbation. As this occurs for $R = 1$, we have the perturbation-independent bound

$$\sigma \leq \tfrac{1}{2}(\bar{M}_1 \bar{M}_2)^{1/2}. \tag{2.38}$$

The sharpness of this and the preceding bound depends, of course, on the choice of $F$, $G$, and $\mathbf{C}$; the most appropriate choice for these quantities depends on the situation under consideration. A few specific examples will be given in the subsequent section.

Before discussing the examples, however, it will be useful to develop a slight generalization of our stability bound. Until now, we have been assuming that the streamfunction $\tilde{\psi}$ was real. Because the governing equation is linear, though, there is no reason not to allow it to be complex. Indeed, for some applications, such as providing bounds on the growth rate of eigenmodes, it is essential for $\tilde{\psi}$ to be complex. In this case, squared quantities can no longer be assumed positive, and our derivation requires certain minor modifications. In the derivation leading to (2.11) at every juncture at which we formerly multiplied an equation by a quantity, we now multiply by the complex conjugate of that quantity and take the real part of the resulting equation. The generalized version of (2.11) becomes

$$\frac{1}{Z}\frac{d}{dt}\frac{1}{2}Z = -\frac{\iint \frac{1}{2}((\nabla^2\tilde{\psi})\nabla\tilde{\psi}^* + (\nabla^2\tilde{\psi}^*)\nabla\tilde{\psi})\cdot\mathbf{A}}{Z}, \tag{2.39}$$

where an asterisk denotes complex conjugation, and

$$Z = \iint (G(\psi_0)(\nabla^2\tilde{\psi})(\nabla^2\tilde{\psi}^*) + \nabla\tilde{\psi}\cdot\nabla\tilde{\psi}^*). \tag{2.40}$$

$\mathbf{A}$ remains as defined in (2.27). In (2.25) it is only necessary to replace the squared integrand on the lhs with the product of the appropriate quantity by its complex conjugate, whereupon the lemma given in (2.28) becomes

$$\left|\iint_D \frac{1}{2}((\nabla^2\tilde{\psi})\nabla\tilde{\psi}^* + (\nabla^2\tilde{\psi}^*)\nabla\tilde{\psi})\cdot\mathbf{A}\right| \leq \frac{1}{2}\iint_D \left[F^2\mathbf{A}^2(\nabla^2\tilde{\psi})(\nabla^2\tilde{\psi}^*) + \frac{\nabla\tilde{\psi}\cdot\nabla\tilde{\psi}^*}{F^2}\right]. \tag{2.41}$$

The rest of the derivation leading to (2.37) remains essentially unchanged. Eq. (2.37) itself remains valid as it stands, except that the new form of $Z$ given in (2.40) must be used in defining the growth rate $\sigma$ and that the mean length scale parameter $R$ is now given by

$$R = \frac{\iint (\nabla^2\tilde{\psi})(\nabla^2\tilde{\psi}^*)G}{\iint \nabla\tilde{\psi}\cdot\nabla\tilde{\psi}^*}. \tag{2.42}$$

Now consider an eigenmode of the stability problem, for which

$$\tilde{\psi} = \psi_1(x, y)e^{-i\omega t}. \tag{2.43}$$

From (2.42) it is evident that $R$ becomes independent of time. Moreover, using (2.40) we may find

$$\sigma = \left|\frac{1}{Z}\frac{d}{dt}\frac{1}{2}Z\right| = |\mathrm{Im}(\omega)|, \tag{2.44}$$

so that $\sigma$ is indeed equal in this special case to the growth rate of the eigenmode as conventionally defined.

In applying either form of the growth rate bound, it should be kept in mind that the growth rate of the physical streamfunction $\psi'$ [as measured by the growth of $Z(\psi')$] differs from the growth rate of $\tilde{\psi}$ [as measured by the growth of $Z(\tilde{\psi})$] owing to the transformation used to eliminate the friction term from (2.4). In particular, since $\psi' = \tilde{\psi}\exp(-\nu t)$, we have

$$\sigma' \equiv \frac{1}{2}\frac{1}{Z(\psi')}\frac{dZ(\psi')}{dt} = \frac{1}{2}\frac{1}{Z(\tilde{\psi})}\frac{dZ(\tilde{\psi})}{dt} - \nu \leq \sigma - \nu. \tag{2.45}$$

In contrast to the bound on $\sigma$, the inequality can constrain the physical growth rate $\sigma'$ to be negative, in which case all perturbations would decay to zero with time.

## 3. Some applications of the instability bound

A particularly useful choice of $F$ is given by $F = G^{1/2}/|\mathbf{A}|$, since then $M_1 = 1$ and $M_2 = \mathrm{MAX}(\mathbf{A}^2/G) \equiv M$, whereupon the bound takes on a particularly simple form. There is still considerable latitude in the choice of $G$. As our first example, let us consider the stability properties of a steady flow that is a small perturbation of stable solution to the inviscid, unforced equations. Specifically, suppose that the steady state streamfunction satisfying (2.3) is given by

$$\psi_0 = \psi_s + \epsilon\psi_1, \tag{3.1}$$

where $\psi_s$ satisfies

$$q_s \equiv \nabla^2\psi_s + \frac{f_0 h}{H} + \beta y = f(\psi_s), \tag{3.2}$$

and we require $f'(\psi_s) > 0$ everywhere, so as to guarantee stability of $\psi_s$. In (3.2) $\epsilon$ is an arbitrary small parameter. Note that we do not require the perturbed steady state $\psi_0$ to be a solution to the inviscid, unforced steady equations. Thus, $\psi_0$ need not be functionally related to $q_0$.

The most natural way to obtain such a solution would be to perturb the inviscid problem with a weak

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forcing and weak damping. However, given a sufficiently long time to act, the weak perturbation will in general cause an order unity change in the flow even if the initial condition consists of an inviscid steady state. For example, the addition of a weak damping without the addition of any forcing will cause an arbitrary initial condition to eventually come to a state of rest. It is therefore unreasonable to assume that a given inviscid steady state will be only slightly perturbed by the introduction of a weak forcing and dissipation. We conjecture, however, that the weakly forced and damped system will generally have a solution that is a small perturbation of some inviscid steady state. If this inviscid state is stable according to Arnol'd's theorem, then the premises of our calculation are satisfied. Note that this "high amplitude" state need not be the only solution to the weakly forced and damped system. There may also be "low amplitude" solutions which are dominated by forcing and dissipation. A simple class of forced/damped problems admitting solutions near to inviscid steady states is constructed in Appendix B. A more general class of such solutions is currently being studied in connection with a theory of blocking patterns as local multiple equilibria (P. Malguzzi, personal communication, 1982).

In the problem at hand, it is convenient to choose $G(\psi_0) = 1/f'(\psi_0)$. This choice obviously satisfies the requirement that $G$ be a function of the steady state streamfunction alone. Moreover, assuming $f'$ is a continuous function, we may conclude that $f'(\psi_0)$ is positive everywhere for sufficiently small $\epsilon$, because $f'(\psi_s)$ is positive everywhere. Thus, for sufficiently small $\epsilon$, $G(\psi_0)$ is positive and $Z$ is a positive-definite functional, as required.

We now proceed to the evaluation of $M$. Expanding $G(\psi_0)$ about $\psi_s$, we obtain

$$G(\psi_0) = \frac{1}{f'(\psi_0)} = \frac{1}{f'(\psi_s)}\left[1 - \epsilon\frac{f''(\psi_s)}{f'(\psi_s)}\psi_1\right] + O(\epsilon^2). \quad (3.3)$$

We next evaluate $\mathbf{A}$ by substituting the above result and (3.1) into (2.27) with $\mathbf{C}$ chosen to be zero. The result is

$$\mathbf{A} = \left(\mathbf{v}_s - \frac{\mathbf{z}\times\nabla q_s}{f'(\psi_s)}\right) - \frac{\epsilon}{f'(\psi_s)}\left[\mathbf{z}\times\nabla q_1 - \left[\frac{f''(\psi_s)}{f'(\psi_s)}\mathbf{z}\times\nabla q_s\right]\psi_1 - f'(\psi_s)\mathbf{v}_1\right] + O(\epsilon^2) \quad (3.4a)$$

$$= -\frac{\epsilon}{f'(\psi_s)}[\mathbf{z}\times\nabla q_1 - (f''(\psi_s)\mathbf{v}_s)\psi_1 - f'(\psi_s)\mathbf{v}_1]. \quad (3.4b)$$

Eq. (3.2) has been used to eliminate the order unity term appearing in (3.4a). The expression for $M$ is then

$$M = \mathrm{MAX}[\mathbf{A}^2/G] = \epsilon^2\mathrm{MAX}[|\mathbf{z}\times\nabla q_1 - (f''(\psi_s)\mathbf{v}_s)\psi_1 - f'(\psi_s)\mathbf{v}_1|^2/f'(\psi_s)] + O(\epsilon^3) \equiv \epsilon^2 L + O(\epsilon^3), \quad (3.5)$$

where $L$ is an order unity quantity depending on $\psi_s$ and $\psi_1$. This value of $M$ may be used in any of the bounds derived in Section 2. In particular, substituting into the global bound given by (2.38) we arrive at

$$\sigma \leqslant \tfrac{1}{2}\epsilon L^{1/2} + O(\epsilon^2). \quad (3.6)$$

Thus, we have shown that *the instabilities of a steady flow that differs by $O(\epsilon)$ from a stable inviscid steady flow have growth rates that are at worst $O(\epsilon)$*. This is a non-trivial result, as *a priori*, the growth rates could have been as large as, say $O(\epsilon^{1/2})$. It is of particular interest to consider the application of this result to the case in which the deviation from the inviscid stable state is caused by a small damping and forcing, in which case the damping coefficient $\nu$ is an $O(\epsilon)$ quantity. Writing $\nu = \epsilon\nu_1$ and substituting (3.6) into (2.45) we find that the growth rate $\sigma'$ of the physical streamfunction is given by

$$\sigma' \leqslant \epsilon(\tfrac{1}{2}L^{1/2} - \nu_1). \quad (3.7)$$

Thus, it is possible for the direct damping effect to dominate the destabilizing viscous effect caused by disruption of the inviscid relation $q_0 = f(\psi_0)$, resulting in a spectrum of damped perturbations. Whether this in fact occurs depends on the relative magnitudes of $\nu_1$ and $\tfrac{1}{2}L^{1/2}$, the latter of which depends in turn on the details of the deviation of the steady state from the inviscid stable state.

The ability of the energy-enstrophy bound to predict that a state near to a stable state is at worst weakly unstable is a prime advantage of this bound over the energy bound given in (2.18). The energy bound allows order unity growth rates for energy $\tilde{E}$ even if the steady state satisfies Arnol'd's stability criterion. At first glance, one might think that the energy–enstrophy bound is simply a superior bound, by virtue of incorporating more information about the dynamics of the system. However, the reason for the difference between the bounds lies rather deeper than this, and in fact the behavior predicted by the energy bound is not incompatible with the behavior predicted by the energy–enstrophy bound. It should first be recalled that the energy–enstrophy bound limits the growth of the quantity $Z$, which is different from the quantity $\tilde{E}$ to which the energy bound applies. The two quantities are related by

$$\tilde{E}(t) = \frac{1}{2}Z(t)\frac{R(t)}{R(t)+1}, \quad (3.8)$$

where $R(t)$ is the measure of perturbation length scale introduced previously (see (2.33a), (2.33b)). Thus, the evolution of perturbation energy is given by

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$$\frac{E(t)}{E(t_0)} = \frac{Z(t)}{Z(t_0)} \frac{1 + 1/R(t_0)}{1 + 1/R(t)}. \tag{3.9}$$

From this formula, it is evident that even in the classically stable case, for which $Z$ is time independent, the perturbation energy can increase arbitrarily rapidly if the length scale $R$ is increasing rapidly. In the context of parallel flow, this sort of behavior was explored elegantly and in great detail by Farrell (1982). It is clear from (3.9) that such energy growth is a transient phenomenon, as it cannot be sustained once $1/R$ approaches zero. However, Farrell has found that physically plausible initial conditions can produce substantial energy growth within the initial stage alone. The growth of $Z(t)$ in the classically unstable case represents a *sustainable* mechanism of energy growth (with one caveat to be noted below) which modulates the transient energy changes associated with changes in the perturbation length scale. It is to the sustainable portion of the growth that the energy–enstrophy bound applies. For *eigenmodes,* though, $R$ is time independent and the growth rates of energy and $Z$ are identical; in this case the energy-enstrophy bound will generally provide the superior estimate of the energy growth rate.

Our final class of applications deals with the stability properties of inviscid steady states which violate Arnol'd's criterion by an order unity amount. Such flows have $f'(\psi_0) < 0$ somewhere and may also have $f' = 0$ along one or more curves. Because of the latter possibility, a multiple valued function would in general be necessary to describe the dependence of vorticity on streamfunction; however, for the sake of simplicity we will here confine attention to the case in which $f(\psi_0)$ is single-valued. Since $f'$ can be negative, we can no longer choose $G = 1/f'$ and still have $Z$ be a positive definite quantity. However, the choice retains a $G = 1/|f'|$ number of useful properties. Recall that with the choice of $F$ stated at the outset of the present section, $M_1 = 1$ and $M_2 = M = \mathrm{MAX}(\mathbf{A}^2/G)$ is the fundamental quantity appearing in all the growth rate bounds. With the stated choice of $G$ we have

$$\mathbf{A} = \begin{cases} 0, & \text{where} \quad f' \geqslant 0 \\ 2\mathbf{v}_0, & \text{where} \quad f' < 0. \end{cases} \tag{3.10}$$

Hence,

$$M = 4 \,\underset{D}{\mathrm{MAX}}(\mathbf{v}_0^2 |f'(\psi_0)|) \tag{3.11}$$

in which $D$ is the region of the $x$-$y$ plane where $f' < 0$. This value may be used in the formula for growth rate bound for fixed scale $R$ (2.37) or for the maximum growth rate over all scales $R$ (2.38). The latter case may be regarded as a generalization to non-parallel flow of the barotropic form of the bound given in Eq. 7.5.25 of Pedlosky (1979, p. 451).

Two features of $M$ are particularly noteworthy. The first is that $M$ (and hence the growth rate bound) depends only on the properties of the steady flow in the region in which Arnol'd's criterion is violated. This is in marked contrast to the energy bound. The second feature is that $M$ remains finite even though $G = 1/|f'|$ can become infinite along certain curves in the plane. Although the singularities of $G$ do not affect the growth rate bounds *per se,* they do restrict the class of initial conditions for which $Z$ can be defined and they do affect the interpretation of the quantity $R$. In the most common case, in which $f'(\psi_0)$ vanishes linearly as some curve $\Gamma$ is approached, and the tangential velocity along that curve is finite, $1/f'$ will be non-integrable; that is

$$\iint_{D'} 1/|f'| = \infty, \tag{3.12}$$

where $D'$ is any small region including the curve $\Gamma$. Hence, in order for $Z$ to be defined, we are restricted to initial conditions which have vanishing vorticity along the curve $\Gamma$ where $f'$ vanishes. Moreover, $R_1$ (see (2.33a)) becomes infinite as perturbation vorticity moves toward the curve of vanishing $f'$. In this manner $R = R_2/R_1$ can approach zero even if the length scale as defined by Fjørtoft remains finite, and $R$ is no longer equivalent to the Fjørtoft length scale. The use of $R$ in (2.37) remains valid, however. We must simply keep in mind the additional effect that a movement of perturbation vorticity into a region of small $f'$ has the same effect as a decrease of the Fjørtoft length scale, and conversely.

The above results can be used to obtain some insights regarding the evolution of perturbation energy in a system in which $f' = 0$ along some curve. Consider an initial condition with vanishing perturbation vorticity where $f' = 0$, so that $R$ is non-zero initially. If some of the vorticity moves toward the curve of vanishing $f'$ so that order unity vorticities develop nearer and nearer to the curve, then $R$ decreases toward zero. As this happens, (2.37) implies that the exponential growth rate of $Z$ falls to zero, thus limiting the extent to which $Z$ can increase. Moreover, the second factor in (3.9) tends to reduce the perturbation vorticity as $R$ approaches zero. In fact, it is clear from (3.9) that if $R$ reaches zero in a *finite* time, the perturbation energy must decay to zero. However, a vanishing energy implies that the velocity vanishes identically which in turn implies that vorticity vanishes identically. As this state of affairs is inconsistent with the assumption that order unity vorticity exists in the neighborhood of $f' = 0$, we conclude that order unity vorticity cannot reach the curve of vanishing $f'$ in a finite time. If $R$ vanishes only asymptotically in time, the situation is rather more complicated, as it may be possible for $Z$ to increase indefinitely with time even though the exponential growth rate tends to zero. When $R$ is small, (2.37) reduces to

$$\frac{1}{Z}\frac{d}{dt}\frac{1}{2} Z \leqslant M^{1/2}(R(t))^{1/2}. \tag{3.13}$$

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Integrating this in time, we find

$$\frac{Z(t)}{Z(t_0)} \leqslant \exp\left[2M^{1/2}\int_{t_0}^{t}(R(t))^{1/2}dt\right]. \quad (3.14)$$

Hence, if $R(t)$ decays to zero faster than $1/t^2$, (3.13) implies that $Z(t)$ remains finite as $t$ becomes infinite. Using (3.9) we again find that perturbation energy decays to zero which leads us to the same contradiction as before. If $R(t)$ decays more slowly than $1/t^2$, then the growth of $Z$ can dominate the decay of $R$ in (3.9), allowing the perturbation energy to increase indefinitely with time. If $R(t)$ decays exactly like $1/t^2$, then the growth of $Z$ can offset the decay in $R$ only if $M > 1$. Together, these results limit how quickly perturbation vorticity can approach the curve of vanishing $f'$. They also exhibit the principle that *perturbation energy growth is strongly inhibited or even reversed as perturbation vorticity moves toward the curve where $f'$ vanishes.*

The reverse process is potentially more important physically. If perturbation vorticity initially has order unity values in a region where $f'$ is small, then $R$ is small initially and consequently the growth rate of $Z$ is small initially. If the vorticity subsequently moves away from the region of small $f'$, then $R$ increases, and hence $\sigma$ increases, so that $Z(t)$ can grow faster than exponentially during this time. Referring to (3.9), we see that the energy growth caused by this increase in $Z$ is assisted by the direct effects of increasing $R$. Thus, *movement of perturbation vorticity from a region of small $f'$ to a region of large $f'$ may be associated with very rapid increases in perturbation energy.* In applying the last two results, it may be convenient to note that the curve along which $f'$ vanishes is identical to the curve along which the gradient of steady-state vorticity in the direction normal to the steady-state streamlines vanishes.

It is instructive to apply the bound implied by (3.11) to the Rossby wave instability problem. We consider the stability of a free Rossby wave embedded in a mean flow chosen so as to make the wave stationary. The steady flow, described by

$$\psi_0 = -Uy + a\cos(kx + ly), \quad (3.15)$$

satisfies the equation

$$q_0 = \nabla^2\psi_0 + \beta y = \frac{-\beta}{U}\psi_0 \quad (3.16)$$

provided $\beta/U = k^2 + l^2$. Thus, $f(\psi_0) = -(\beta/U)\psi_0$, and $f' = -(\beta/U)$ is negative everywhere. In calculating the velocity $\mathbf{v}_0$ appearing in (3.11), we are free to first make use of the constant vector $\mathbf{C}$ appearing in the definition of $\mathbf{A}$ [see (2.27)] to subtract the mean zonal flow $U$ from the steady state. Applying (3.11), we find

$$M = 4\lambda^4 a^2, \quad (3.17)$$

where $\lambda = (k^2 + l^2)^{1/2}$ is the total basic-state wavenumber. Substituting into (2.38), we find $\sigma \leqslant -a\lambda^2$. It is convenient to non-dimensionalize $\sigma$ using the time scale $\lambda/\beta$, so as to put our bound in the same units as those used by Mied (1978) in his numerical study of Rossby wave instability. In these units, the bound becomes

$$\hat{\sigma} \equiv \sigma\frac{\lambda}{\beta} \leqslant a\frac{\lambda^3}{\beta} = \hat{M}, \quad (3.18)$$

where $\hat{M}$ is the steepness parameter as defined by Mied ("M" in Mied's notation). In these units, the bound states that the maximum growth rate equals the steepness parameter. From either Fig. 3 or Fig. 4 in Mied (1978), we find that the eigenmodes have maximum growth rates of approximately $0.1\hat{M}$. Thus, the energy–enstrophy bound appears to overestimate the actual maximum growth rates of eigenmodes by a factor of 10.

There is good reason to expect the energy–enstrophy bound to appreciably overestimate the growth rates of eigenmodes. Although the energy–enstrophy bound is in general closer to the sustainable growth rate than the energy bound, it must nevertheless be a bound for the *initial* growth rate of $Z$ as well as the *asymptotic* growth rate as $t$ approaches infinity. Even if the initial perturbation is configured so as to allow $Z$ to grow rapidly at $t = 0$, there is no guarantee that the dynamics would allow the perturbation to remain so configured as it evolves. As Farrell (1982) pointed out, though, the initial growth phase can be physically more important than the asymptotic growth phase. Farrell found that the initial growth rates of energy could approach the maximum predicted by the energy bound. We conjecture that, similarly, the initial growth rate of $Z$ can approach the energy–enstrophy bound. There are thus two types of transient initial behavior to be considered: 1) transient energy growth associated with increasing $R$ and fixed $Z$ and 2) transient energy growth associated with an initially large growth rate of $Z$.

## 4. Conclusion

We have shown how the perturbation energy and enstrophy equations can be combined to yield a general family of constraints on the instability of nonparallel (vortex-like or wave-like) steady states. These bounds are applicable to forced, damped steady states as well as unforced inviscid ones. Through judicious choice of the free parameters of the family of bounds, a number of useful results on the evolution of the perturbations can be obtained. We have shown that: 1) if the Fjørtoft length scale of the perturbation approaches zero or infinity, the growth rate of the perturbation vanishes; 2) steady states that are close to stable inviscid steady states are at worst weakly unstable; and 3) perturbation energy can grow especially rapidly if perturbation vorticity moves from a region of small $f'$ to a region of larger $f'$, where $f$ is the function relating steady-state potential vorticity to the

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steady-state streamfunction. In the course of deriving these results, we have obtained a number of specific growth rate bounds which will generally be more stringent than bounds based on energetics alone.

Comparison of our results with the growth rates of normal modes in the Rossby-wave instability problem suggests that the bounds tend to appreciably overestimate the growth rate of eigenmodes. We believe our formulae are probably more useful for bounding transient growth than the asymptotic growth that is reached after a long time. In connection with this point, two kinds of transient energy growth are revealed; the first associated with increase of the Fjørtoft length scale and the second associated with initially large growth of energy and enstrophy at fixed length scale. The importance of the former mechanism has been discussed recently by Farrell (1982) in the parallel flow case.

The appearance of the perturbation length scale in the growth rate bound (2.37) calls to mind the growth rate bound provided by the semi-circle theorems in the case of parallel flow (see e.g., Pedlosky, 1979, Eq. 7.5.19 for the quasi-geostrophic case). In the semi-circle theorem, an upper bound is placed on the magnitude of $\sigma/k$, where $k$ is the zonal (or streamwise) wavenumber of the perturbation; thus, for large perturbation length scales $1/k$, the growth rate bound vanishes like $k$. This behavior is analogous to that of (2.37), since the growth rate vanishes like $1/R^{1/2}$ for large length scales and $1/R^{1/2}$ is proportional to an inverse length scale. However, an important difference between our result and the semi-circle result is that $R^{1/2}$ refers to a mean length scale taken over all spatial directions in the $x$-$y$ plane, whereas $1/k$ refers only to the perturbation length scale in the zonal direction. The separation of the zonal length scale in the semi-circle bound is no doubt a consequence of the translation invariance of the parallel flow problem in the zonal direction, and it seems unlikely that an analogous result can be obtained for the non-parallel problem. By way of compensation, though, (2.37) predicts vanishing growth rate also for short perturbation length scales, whereas the semi-circle bound does not.

The work presented here can be seen as an attempt to bring together two threads in current thought on stability theory, the first being that consideration of non-zonal states is necessary for the explanation of such phenomena as regional cyclogenesis, and the second that the transient behavior of perturbations can be physically more important than the asymptotic behavior evidenced by the eigenmodes. Application of the results to the problems of greatest interest probably requires extension to the baroclinic case, however. While this extension is in principle entirely straightforward, the interpretation of the results in the baroclinic case is more complicated, by virtue of the greater number of cases to be considered; this problem will be the subject of a future article.

*Acknowledgments.* During the early stages of this work, we benefitted from an illuminating conversation with Dr. William Young. We have also had the advantage of numerous discussions with Dr. Brian Farrell and are grateful to him for providing us with a preprint of his work on initial value problems.

The investigation reported above was partially supported by the National Science Foundation, under Grant 79-16877-CME in the earlier stages and ATM81-15394 in the later stages.

## APPENDIX A

### Special Considerations for the Doubly Periodic Domain

A problem appears to arise in the consideration of the doubly periodic case because the $\beta y$ term in (1.2) renders $q_0$ non-periodic in $y$. However, this difficulty is easily surmounted. First, we must consider what we mean by a "doubly periodic" solution. Obviously, it is necessary to require that $\psi^*$ and $h$ be periodic in both $x$ and $y$. However, $\psi_0$ and $q_0$ will not in general be periodic in $y$. Instead, we seek solutions of the form

$$\psi_0 = -Uy + \psi_{0p}, \tag{A1}$$

$$q_0 = \beta y + \nabla^2\psi_{0p}, \tag{A2}$$

where $\psi_{0p}$ is required to periodic in $y$. With this restriction, the steady-state equation (2.3) and the perturbation equation (2.4) have no coefficients which are not periodic in $y$, since the non-periodic terms in (A1) and (A2) are always differentiated by $y$, whereupon they become constants. Thus, it is consistent to seek solutions in which $\psi_{0p}$ and $\psi'$ are periodic in $y$ with the same period as $\psi^*$ and $h$. Of course, regardless of the boundary conditions chosen, it is never certain that a nonlinear equation such as (2.3) has a solution. We note, however, that the free Rossby wave considered in Section 3 provides a non-trivial example of a doubly periodic solution of the form assumed above.

It is instructive to consider the nature of doubly periodic steady states of the inviscid, unforced equation. A family of such states is defined by

$$\nabla^2\psi_{0p} + \beta y = f(\psi_{0p} - Uy), \tag{A3}$$

where $f(r)$ is a specified real function of a real number $r$. By making use of the periodicity of $\psi_{0p}$ and $h$ (with period $L_y$) we easily find that (A3) implies

$$f(\psi_{0p} - Uy \mp UL_y) = f(\psi_{0p} - Uy) \pm \beta L_y. \tag{A4}$$

Hence, the function $f$ is constrained by the relation

$$f(r \mp UL_y) = f(r) \pm \beta L_y. \tag{A5}$$

We can easily satisfy this constraint by defining $f$ in the range $0 \leqslant r < UL_y$ and extending the definition to the rest of the real axis using (A5). For continuity of $f$ and $df/dr$ (i.e., $f'$) we need to require that $f$ and

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$df/dr$ approach $f(0) - \beta L_y$ and $f'(0)$, respectively, as $r$ approaches $UL_y$ from below. This makes $f'$, and indeed any function of it, periodic with period $UL_y$. In the simple case in which $f(r)$ is linear, we may write $f(r) = br$. The periodicity constraint then requires that $UbL_y = -\beta L_y$, so that $b = -\beta/U$. This results in precisely the form of $f$ appropriate to the free Rossby wave case treated in Section 3. We see that periodicity in $y$ has determined the constant appearing on the rhs of (3.16).

We must now consider the vanishing of integrated Jacobians assumed in the derivation of the perturbation energy and enstrophy equations. Consider first the energy equation. Upon multiplying the first Jacobian in (2.5) by $\tilde{\psi}$, integrating over one period in $x$ and $y$ and making use of the divergence theorem, we find

$$\int_0^{L_x}\int_0^{L_y} dxdy\tilde{\psi}J(\tilde{\psi}, q_0) = \left[\int_0^{L_y} dy\tilde{\psi}\tilde{v}_x q_0\right]\Bigg|_{x=0}^{L_x} + \left[\int_0^{L_x} dx\tilde{\psi}\tilde{v}_y q_0\right]\Bigg|_{y=0}^{L_y}, \quad \text{(A6)}$$

where $\tilde{v}_x$ and $\tilde{v}_y$ are the perturbation zonal and meridional velocities and $L_x$ is the period in $x$. Recall that the perturbation streamfunction and velocities may be assumed periodic in $x$ and $y$. The first term on the right hand side of (A6) vanishes trivially by $x$-periodicity. The second term may be rewritten as

$$\left[\int_0^{L_x} dx\tilde{\psi}\tilde{v}_y q_0\right]\Bigg|_{y=0}^{L_y} = (\beta L_y)\int_0^{L_x}\tilde{\psi}\tilde{v}_y \quad \text{(A7)}$$

by making use of (A2) and the $y$-periodicity of $\psi_{0p}$. Since $\tilde{\psi}\tilde{v}_y = \partial_x(\frac{1}{2}\tilde{\psi}^2)$, it is evident that this term also vanishes by virtue of $x$-periodicity. Hence, the contribution of the first Jacobian to the energy equation vanishes as claimed.

In the enstrophy equation, it is the contribution of the second Jacobian in (2.5) that vanishes. Multiplying this Jacobian by $\nabla^2\tilde{\psi}G(\psi_0)$ and integrating by parts, we find

$$\int_0^{L_x}\int_0^{L_y} dxdy\nabla^2\tilde{\psi}G(\psi_0)J(\psi_0, \nabla^2\tilde{\psi}) = \left[\int_0^{L_y} dy\tfrac{1}{2}v_{0x}G(\psi_0)(\nabla^2\tilde{\psi})^2\right]\Bigg|_{x=0}^{L_x} + \left[\int_0^{L_x} dx\tfrac{1}{2}v_{0y}G(\psi_0)(\nabla^2\tilde{\psi})^2\right]\Bigg|_{y=0}^{L_y}. \quad \text{(A8)}$$

The first term on the right hand side vanishes trivially by $x$ periodicity. Turning attention to the second term, we note that $v_{0y} = \partial_x\psi_0 = \partial_x\psi_{0p}$ is periodic in $y$. By assumption, $\tilde{\psi}$, and hence $\nabla^2\tilde{\psi}$, is periodic in $y$. To make the integrand in the second term periodic in $y$ with period $L_y$, then, we need only require that

$$G(\psi_{0p}) = G(\psi_{0p} - U\cdot L_y). \quad \text{(A9)}$$

This condition is satisfied whenever $G(r)$ is periodic in $r$ with period $UL_y$. In the inviscid case, for example, the choice $G = 1/f'$ considered in Section 3 satisfies this constraint because $f'$ is periodic with period $UL_y$ in the doubly periodic domain.

Finally, we note that the derivation of (2.10) and (2.16) requires no special treatment in the doubly periodic domain, as the vanishing of the boundary terms arising in the $y$-integration follows in a completely straightforward manner from the $y$-periodicity of $v_{0y}$.

It may seem that by requiring the perturbation to be periodic with the *same* period as the basic state we have unduly restricted the stability problem, for instance, by precluding sub-harmonic instabilities. However, a simple artifice enables us to treat an arbitrary rational sub-harmonic perturbation. If $L_y$ is the period of the basic state relative vorticity, then, for the purposes of the above discussion, the system can also be regarded as periodic with period $mL_y$, where $m$ is any integer. Then, if $n$ is any integer, a perturbation with period $(m/n)L_y$ is also periodic with period $mL_y$, since $n$ perturbation waves fit in the same length $mL_y$ which encompasses $m$ basic state waves. In this manner, an arbitrary rational sub-harmonic perturbation of the basic state can be treated without violating the requirement that the geometry be periodic in $y$.

## APPENDIX B

### Equilibrium States of the Barotropic Vorticity Equation with Weak Forcing and Dissipation

In this appendix we construct a class of forcing functions $\psi^*$ for which (2.3) has solutions which approach inviscid unforced equilibria as the forcing and dissipation are made to vanish. We do not attempt to define here the most general set of circumstances under which this is the case. As an aside, we point out the existence of multiple equilibria in the weakly forced and dissipated system which are local in character and do not depend on resonance for their existence.

The approach we take is to specify a field $\psi_0$ and then to use (2.3) to find the forcing $\psi^*$ that is needed to make $\psi_0$ a steady state. This is much easier than the converse approach of specifying the forcing and attempting to use (2.3) to solve for $\psi_0$. The former approach, in fact, is widely used in the formulation of non-parallel steady states for use in stability studies. In Frederiksen (1979) for example, a streamfunction similar to the observed time-averaged streamfunction is chosen as the basic state and the net forcing needed to maintain this state is postulated to exist.

Let us now find the forcing needed to maintain a field $\psi_0$. We write the forcing function as

$$\psi^* = \psi_0 + B. \quad \text{(B1)}$$

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Substituting this expression into (2.3) we obtain

$$J(\psi_0, q_0) = \nu \nabla^2 B, \tag{B2}$$

which is just an alternate form of (2.3). Next, let $\psi_s$ and its associated potential vorticity $q_s$ be a steady state of the inviscid, unforced system, so that

$$J(\psi_s, q_s) = 0. \tag{B3}$$

We now restrict attention to fields of the form

$$\psi_0 = \psi_s(x, y) + \epsilon \psi_1(x, y), \tag{B4}$$

where $\psi_1$ is an arbitrarily specified streamfunction, and $\epsilon$ an adjustable constant. Substituting (B4) into (B2) and making use of (B3), we find that the necessary forcing is determined by

$$\nabla^2 B = \nu_1^{-1}[J(\psi_s, \nabla^2\psi_1) + J(\psi_1, q_s) + \epsilon \nu_1^{-1} J(\psi_1, \nabla^2 \psi_1)], \tag{B5}$$

where we have written $\nu = \epsilon \nu_1$. With this forcing, the form of $\psi_0$ given in (B4) will be an exact solution to (2.3) regardless of the magnitude of $\epsilon$. If we now allow $\epsilon$ to become small, the solution $\psi_0$ approaches the inviscid steady state, according to (B4). At the same time, it is clear from (B5) that $B$ remains of order unity in this limit so that the net forcing appearing on the rhs of (B2) is of order $\epsilon$. Thus, we have constructed a solution to a weakly forced and damped system which converges to an inviscid solution as the forcing and damping vanish.

It is interesting to note that (2.3) with forcing specified by (B1) and (B5) will in general have not only the high amplitude nonlinear solution $\psi_0$, but also, by virtue of the weakness of the forcing, a low amplitude linear solution. For simplicity, we consider a domain unbounded in $x$; we also assume that the topography $h(x, y)$ vanishes and that the zonal flow approaches a constant value $U_\infty$ far upstream of the forcing region. Expanding about the zonal basic state, we write

$$\psi = -U_\infty y + \epsilon \psi_L, \tag{B6}$$

$$q = \beta y + \epsilon \nabla^2 \psi_L. \tag{B7}$$

Upon substitution into (2.3) and dropping the $O(\epsilon^2)$ nonlinear terms, we find that the perturbation is governed by

$$U_\infty \partial_x \nabla^2 \psi_L + \beta \partial_x \psi_L + \epsilon \nu_1 \nabla^2 \psi_L = \nu_1 \nabla^2 \psi^*. \tag{B8}$$

Note that we have retained the damping term at this order even though it is formally of higher order than the other terms retained. This is simply a device to remove the well-known ambiguity of the inviscid wave equation, for which an arbitrary stationary homogeneous solution can be added to the forced solution. Eq. (B8) can be directly solved via the Fourier transform, and if no resonance exists, the solution $\psi_L$ will be of order unity. Hence the total streamfunction (B6) represents a small departure from the zonal state. Thus, for the same forcing $\psi^*$, (2.3) possesses both a high-amplitude state $\psi_0$ and a low-amplitude nearly zonal state given by (B6). It is significant that the existence of multiple equilibria in this case depends only on the existence of a *nonlinear* steady state of the inviscid, unforced system, and not on the possibility of resonant excitation of the linear system. This idea and its possible application to blocking states of the atmosphere are being explored in work currently in progress.

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