

Buoyant Production and Consumption of Turbulence Kinetic Energy in Cloud-Topped Mixed Layers

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ABSTRACT

Entrainment closure theories for mixed-layer models entail assumptions about how the net rate of buoyant production of turbulence kinetic energy is partitioned into gross production and consumption. Three alternative partitioning theories are examined in this paper: Eulerian partitioning, process partitioning and Lagrangian partitioning. Lagrangian partitioning provides a *definition* of the gross production rate, but is difficult to implement directly. Eulerian and process partitioning are attempts to implement Lagrangian partitioning indirectly.

For the buoyancy fluxes due to a single family of plumes, Eulerian and Lagrangian partitioning are shown to be equivalent. Recent observations reported by Wilczak and Businger rule out such a model. However, it serves as a useful conceptual link between Eulerian and Lagrangian partitioning.

Process partitioning can be formulated in a variety of ways. Examples show that mixed-layer model results are very sensitive to the way in which radiative cooling is assumed to influence the production and consumption rates. A quantitative relationship between process partitioning and Lagrangian partitioning has yet to be established. The observations of Wilczak and Businger show that consumption and entrainment are not as closely linked as current versions of process partitioning suggest.

1. Introduction

Studies of the entraining planetary boundary layer (PBL) have generally emphasized the role of buoyancy fluxes in driving entrainment. The buoyancy flux is proportional to the rate of conversion of the potential energy of the mean flow into the kinetic energy of the turbulence. Of course, turbulence kinetic energy (TKE) can also be produced by the interaction of the Reynolds stress with the shear of the mean wind. For the PBL as a whole, the net rate of TKE production is closely balanced by dissipation (Fig. 1).

Entrainment theories have focused on this energy cycle. The entrainment rate can be determined if an assumption is made to relate the dissipation rate to the other parameters of the problem. Generally it is assumed that the dissipation rate is a constant fraction of the gross rate of TKE production. However, in order to proceed, it is necessary to define the gross production rate. Although everyone agrees that shear generation contributes to gross production, there is some disagreement concerning the important contribution due to buoyancy.

Buoyancy fluxes can either produce or consume TKE, as indicated by the pair of arrows in Fig. 1. In some cases the conversion may proceed in only one direction, e.g., in the nocturnal boundary layer. However, considering the PBL as a whole, it is not at all

unusual for conversion to proceed in both directions simultaneously. This occurs, for example, in both clear and cloudy convective mixed layers that are capped by inversions. If we can partition the net conversion into positive (i.e., TKE-generating) and negative (TKE-consuming) parts, we can include the positive part in the gross production rate, and closure will be achieved.

Three different approaches to partitioning have been proposed. Ball (1960) and Deardorff *et al.* (1969, 1974) considered the sign of the net buoyancy flux at each level (Fig. 2). If the net flux is positive, it is counted as TKE-producing; otherwise, it is counted as TKE-consuming. This approach, which can be called *Eulerian partitioning*, has been applied to the cloud-topped mixed layer by Kraus and Schaller (1978), Randall (1980b), Fravallo *et al.* (1981), Fiedler (1984), and Suarez *et al.* (1983). Similar closures, based on consideration of the net buoyancy flux, have been used by Lilly (1968), Schubert (1976), and Deardorff (1976).

The second approach can be called *process partitioning*. It was pioneered by Manins and Turner (1978), extended by Stage and Businger (1981a,b), advocated by Hanson (1982), and discussed by Wilczak and Businger (1983). It has also been employed in models of the mixed layer of the upper ocean, as reviewed by Niiler and Kraus (1977). The idea is that the buoyancy flux at any particular level can be considered to be forced by various processes acting at that and other

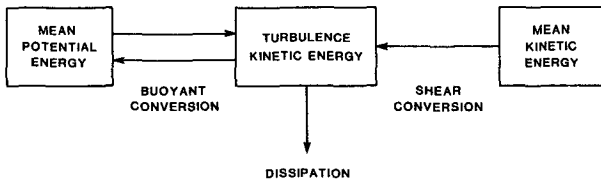


FIG. 1. The flow of energy from the mean flow through the turbulence, and to dissipation.

levels throughout the PBL. Each of these forcing processes is viewed as TKE-producing or consuming¹ (Fig. 3). It is assumed that each of the various processes acting in concert produces and consumes the same energy as if it acted independently. The total rates of TKE production and consumption are obtained by summing the effects of all the forcing processes.

The third approach, which was discussed briefly by Stage and Businger (1981b), is *Lagrangian partitioning*. Each air parcel is considered as either producing or consuming TKE, according to the sign of the product of its density and vertical velocity anomalies (Fig. 4). For the PBL as a whole, the TKE production and consumption rates are obtained by summing over all parcels. Wilczak and Businger (1983) have used Boulder Atmospheric Observatory (BAO) tower data to study the distribution of productive and consumptive parcels in the clear convective mixed layer.

In this paper, we compare the three partitioning theories, in an effort to understand the relationships among them, and the relevance of each to entraining boundary layers. We emphasize the cloud-topped mixed layer because it is here that the differences between Eulerian and process partitioning become most apparent.

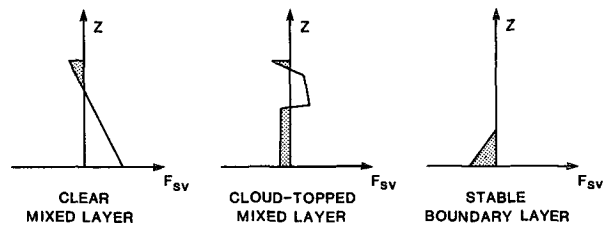
2. The basic equations

Before discussing the three partitioning theories, we present the basic equations to which they all relate.

The rate of TKE generation due to buoyancy is

$$B = \int_{p_{B+}}^{p_S} \kappa F_{sv}(p) / p dp, \tag{2.1}$$

¹ Although Manins and Turner (1978) and Hanson (1982) discussed process partitioning from the point of view of the potential energy budget, a completely equivalent analysis can be based on the TKE budget, since there is a one-to-one correspondence between buoyant production of TKE and buoyant destruction of mean potential energy; they are one and the same process. For example, Hanson's expression for the rate of release of potential energy by buoyancy in a convective mixed layer agrees exactly (as it must) with the rate of production of TKE by buoyancy in the same layer. (In fact, Hanson's result for potential energy release can be obtained more easily by considering TKE production.) In this paper we focus throughout on the TKE budget.



EULERIAN PARTITIONING

FIG. 2. Diagram illustrating the vertical profile of the net buoyancy flux for the clear mixed layer, the cloud-topped mixed layer, and the stable PBL. In each case the stippled area represents TKE consumption.

while that due to shear is

$$S = \int_{p_{B+}}^{p_S} F_V \cdot \partial V / \partial p dp. \tag{2.2}$$

Here p is pressure; κ is Poisson's constant; F_{sv} and F_V are the turbulent fluxes of virtual dry static energy and horizontal momentum, respectively; V is the horizontal velocity; and subscripts $B+$ and S denote levels just above the PBL top and just above the earth's surface, respectively. Neglecting storage and loss into gravity waves, the conservation of TKE is expressed by

$$B + S - D = 0, \tag{2.3}$$

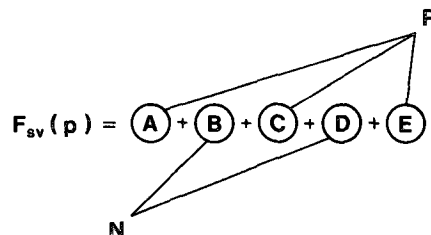
where D is the vertically integrated dissipation rate.

We can write

$$B + S = P - N, \tag{2.4}$$

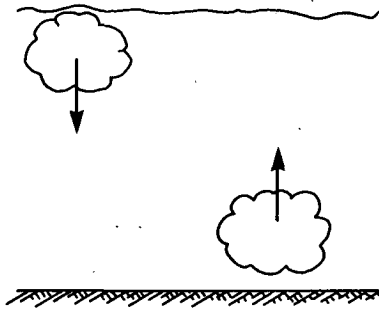
where P and N are the gross production and consumption rates, which must be determined on the basis of a partitioning theory. Correspondingly, we divide the net buoyancy flux into productive and consumptive parts:

$$F_{sv}(p) = [F_{sv}(p)]_{prod} + [F_{sv}(p)]_{cons}, \tag{2.5}$$



PROCESS PARTITIONING

FIG. 3. Diagram illustrating process partitioning, in which certain terms in the expression for the net buoyancy flux are assumed to contribute to gross production, and others are assumed to contribute to consumption.



LAGRANGIAN PARTITIONING

FIG. 4. Diagram illustrating Lagrangian partitioning. Each parcel is regarded as productive or consumptive.

and write

$$P - S = \int_{p_B}^{p_S} \kappa [F_{sv}(p)]_{\text{prod}} / p dp, \quad (2.6)$$

$$N = - \int_{p_B}^{p_S} \kappa [F_{sv}(p)]_{\text{cons}} / p dp. \quad (2.7)$$

The definitions of $[F_{sv}(p)]_{\text{prod}}$ and $[F_{sv}(p)]_{\text{cons}}$ depend on the partitioning theory. All three theories satisfy (2.5); in this sense, all three are consistent with the budgets of mean potential and turbulence kinetic energy.

The dissipation and gross production rates are assumed to be related by

$$D = (1 - A)P, \quad (2.8)$$

where A is an empirical constant or an empirical universal function, which must lie between zero and one; $A \ll 1$ is usually recommended. Using (2.4) and (2.8) in (2.3), we obtain

$$AP - N = 0. \quad (2.9)$$

If P and N can be expressed in terms of the entrainment mass flux E , we can solve (2.9) for E .

The assumption (2.8) is usually described as the closure assumption to determine the entrainment rate. However, the problem is not really closed until P is defined. In order to test (2.8) by direct comparison with observations, it would be necessary to measure P and D separately, so that their ratio could be determined, and its variability assessed. It is therefore necessary to explain how to measure P directly, at least in principle.

The concept of Lagrangian partitioning, suggested by Stage and Businger (1981b), provides a straightforward, unambiguous definition of P . According to this definition, the gross production rate is obtained by integrating the local rate of kinetic energy production over all those parcels for which it is positive. This

is consistent with the physical idea behind (2.8), since dissipation can begin, locally, as soon as kinetic energy has been locally generated. To implement Lagrangian partitioning as an entrainment closure assumption, it is necessary to devise a model in which the buoyancies and vertical velocities of individual parcels are explicitly represented. The only existing models that meet this requirement are the large-eddy models (e.g., Deardorff, 1980b), which require no entrainment assumptions. Simpler models that do require entrainment assumptions may not be able to use Lagrangian partitioning directly. *Eulerian partitioning and process partitioning can be viewed as attempts to implement Lagrangian partitioning indirectly.*

In this paper, we assume that (2.8) and (2.9) are valid for Lagrangian partitioning, with $A = A_L$, $P = P_L$, and $N = N_L$. In Sections 3 and 4, we compare Eulerian partitioning ($A = A_E$, $P = P_E$, $N = N_E$) with process partitioning ($A = A_P$, $P = P_P$, $N = N_P$). In Section 5, we investigate to what extent each is consistent with Lagrangian partitioning.

Apart from the choice of A , Eulerian and process partitioning give identical predictions for the clear buoyancy-driven mixed layer, and also for stably stratified shear layers. For the convective case, studies reviewed by Stull (1976) suggest $A_E \approx 0.04$, while Stage and Businger (1981a,b) recommend $A_P \approx 0.2$. Recent laboratory work by McEwan (1983) shows that $A_E = A_P \approx 0.2$ for the stable case. Thus, process partitioning can model both stable and unstable turbulent layers with a single choice of A_P , while Eulerian partitioning must either allow A_E to vary with stability, or else separate the shear and buoyancy contributions to P , assigning different values of A_E to each. Recent observations reported by Wilczak and Businger show that $A_L \approx 0.37$ for the clear convective mixed layer.

For both Eulerian and process partitioning, A is considerably less than one. For $A \ll 1$, (2.9) implies that *the entrainment rate is much more sensitive to N than to P* . To see this, let

$$P(E) = P_0 + P_1 E, \quad (2.10a)$$

$$N(E) = N_0 + N_1 E, \quad (2.10b)$$

where P_0 , P_1 , N_0 , and N_1 are independent of E . For some partitioning theories, P and N are exactly given by linear functions of E , as in (2.10a,b). For the remaining theories, (2.10a,b) can be used as approximations valid in the vicinity of some particular value of E . Suppose now that P_0 is perturbed about some given value; in order to satisfy (2.9), a perturbation in E will arise. From (2.8-9), we find that

$$(AP_1 - N_1)\delta E + A\delta P_0 = 0, \quad (2.11)$$

where δP_0 and δE are the perturbations. As these perturbations become small, we obtain

$$\partial E / \partial P_0 = -A / (AP_1 - N_1). \quad (2.12)$$

Similarly, we can show that

$$\partial E/\partial N_0 = 1/(AP_1 - N_1). \tag{2.13}$$

Comparison of (2.12–13) gives

$$\frac{\partial E/\partial N_0}{\partial E/\partial P_0} = \frac{-1}{A}. \tag{2.14}$$

This demonstrates that, for small A , E is more sensitive to N_0 than to P_0 . We can also show that

$$\frac{\partial E/\partial N_1}{\partial E/\partial P_1} = \frac{-1}{A}, \tag{2.15}$$

i.e., E is more sensitive to N_1 than to P_1 . The implications of this greater sensitivity to N are discussed in Section 4.

As discussed by Randall (1980a), the virtual dry static energy flux satisfies

$$F_{sv}(p) = F_h - (1 - \delta\epsilon)L F_r, \tag{2.16}$$

below cloud base, and

$$F_{sv}(p) = \beta F_h - \epsilon L F_r, \tag{2.17}$$

above cloud base. Here, F_h and F_r are the turbulent fluxes of moist static energy and total moisture (vapor

and liquid), respectively. The definitions of the positive, quasi-constant thermodynamic coefficients β , δ , and ϵ are given by Randall (1980a). For a given E , the height-variations of F_h and F_r are given by

$$F_h = (F_h)_S \xi - E \Delta h (1 - \xi) + R_S \xi + R_B (1 - \xi) - R, \tag{2.18}$$

$$F_r = (F_r)_S \xi - E \Delta r (1 - \xi), \tag{2.19}$$

where

$$\xi \equiv (p - p_B)/\delta p_M. \tag{2.20}$$

Here R is the radiative flux, subscript B denotes a level just below the PBL top, δp_M is the pressure-thickness of the PBL, and $\Delta(\)$ denotes a jump across the PBL top. In this paper, we assume that

$$R(p) = \begin{cases} 0, & p_R \leq p \leq p_S \\ R_B(p_R - p)/\delta p_R, & p_B \leq p \leq p_R \\ R_B, & p < p_B, \end{cases} \tag{2.21}$$

where, as in Randall (1980b), $\delta p_R = p_R - p_B$. Using (2.18–2.19) and (2.21) in (2.16) and (2.17), we can write

$$F_{sv}(p) = \begin{cases} [(F_h)_S - (1 - \delta\epsilon)L(F_r)_S]\xi + R_B(1 - \xi) - E[\Delta h - (1 - \delta\epsilon)L\Delta r](1 - \xi), & p_C \leq p \leq p_S \\ [\beta(F_h)_S - \epsilon L(F_r)_S]\xi + \beta R_B(1 - \xi) - E(\beta\Delta h - \epsilon L\Delta r)(1 - \xi), & p_R \leq p \leq p_C \\ [\beta(F_h)_S - \epsilon L(F_r)_S]\xi + \beta R_B[(1 - \xi) - (\xi_R - \xi)/\xi_R] - E(\beta\Delta h - \epsilon L\Delta r)(1 - \xi), & p_B \leq p \leq p_R \\ 0, & p < p_B. \end{cases} \tag{2.22}$$

where subscript C denotes the cloud-base level. From (2.22) it is evident that there are several distinct processes acting to promote or retard buoyant production of TKE in a cloud-topped mixed layer. These include surface fluxes of moist static energy and moisture, entrainment related fluxes of these same two properties, and cloud-top radiative cooling.

3. Eulerian partitioning

As proposed by Ball (1960), Lilly (1968), and Dear-dorff *et al.* (1969, 1974), Eulerian partitioning is the assumption that

$$[F_{sv}(p)]_{\text{prod}} = H[F_{sv}(p)], \tag{3.1}$$

$$-[F_{sv}(p)]_{\text{cons}} = H[-F_{sv}(p)], \tag{3.2}$$

where H is the Heaviside function, i.e., $H(x) = x$ for $x \geq 0$, and $H(x) = 0$ for $x < 0$. This assumption is motivated by the fact that turbulent motions in the mixed layer inevitably transfer heat both up from the surface and down from the inversion; both are entailed by the physical situation. This suggests that it is the combined effects of the productive and consumptive parcels, i.e., the net buoyancy flux, that characterizes

the layer energetics. Of course, it is this net buoyancy flux that appears in the turbulence kinetic energy equation. Further discussion is given in Section 5.

In order to investigate the dependence of P and N on surface fluxes, radiative cooling, and entrainment, as implied by Eulerian partitioning, we restrict ourselves to a relatively simple special case, in which F_{sv} does not change sign in the subcloud layer, and is positive throughout the cloud layer, except for a possible layer of negative F_{sv} induced near cloud top by distributed radiative cooling. These restrictions do not imply horizontal homogeneity or a steady state. The shear contribution to P is neglected here and in the remainder of this paper. Under these restrictions, we can show (see Randall, 1980b, p. 153) that

$$\begin{aligned} \Pi^{-1}P \approx & (1 - \xi_C)H[(F_{sv})_S + (F_{sv})_{C-}] + (\xi_C - \xi_R) \\ & \times [(F_{sv})_{C+} + (F_{sv})_R] + \xi_R(F_{sv})_{C+}^2 / \\ & [(F_{sv})_{C+} - (F_{sv})_B], \end{aligned} \tag{3.3}$$

$$\begin{aligned} \Pi^{-1}N \approx & (1 - \xi_C)H[-(F_{sv})_S - (F_{sv})_{C-}] \\ & + \xi_R(F_{sv})_B^2 / [(F_{sv})_{C+} - (F_{sv})_B]. \end{aligned} \tag{3.4}$$

Subscripts $C-$ and $C+$ denote levels just below and just above cloud base, respectively; and $\Pi \equiv \frac{1}{2}\kappa\delta p_M/p_S$. We have assumed that $\Pi \ll 1$. We now substitute from (2.22), neglecting the height-variation of β and ϵ , to obtain

$$\begin{aligned} \Pi^{-1}P \approx & (1 - \xi_C)H\{[(F_h)_S - (1 - \delta\epsilon)L(F_r)_S](1 + \xi_C) \\ & + R_B(1 - \xi_C) - E[\Delta h - (1 - \delta\epsilon)L\Delta r](1 - \xi_C)\} \\ & + (\xi_C - \xi_R)\{[\beta(F_h)_S - \epsilon L(F_r)_S](\xi_C + \xi_R) + \beta R_B \\ & \times [2 - (\xi_C + \xi_R)] - E(\beta\Delta h - \epsilon L\Delta r)[2 - (\xi_C + \xi_R)]\} \\ & + \xi_R\{[\beta(F_h)_S - \epsilon L(F_r)_S]\xi_R + \beta R_B(1 - \xi_R) \\ & - E(\beta\Delta h - \epsilon L\Delta r)(1 - \xi_R)\}^2/[\{\beta(F_h)_S - \epsilon L(F_r)_S\}\xi_R \\ & + \beta R_B(1 - \xi_R) + \xi_R E(\beta\Delta h - \epsilon L\Delta r)], \quad (3.5) \end{aligned}$$

$$\begin{aligned} \Pi^{-1}N \approx & (1 - \xi_C)H\{-[(F_h)_S - (1 - \delta\epsilon)L(F_r)_S](1 + \xi_C) \\ & - R_B(1 - \xi_C) + E[\Delta h - (1 - \delta\epsilon)L\Delta r](1 - \xi_C)\} \\ & + \xi_R E^2(\beta\Delta h - \epsilon L\Delta r)^2/[\{\beta(F_h)_S - \epsilon L(F_r)_S\}\xi_R \\ & + \beta R_B(1 - \xi_R) + \xi_R E(\beta\Delta h - \epsilon L\Delta r)]. \quad (3.6) \end{aligned}$$

These expressions for P and N are quite complicated, in spite of our simplifying assumptions. Nevertheless, we can draw the conclusions summarized in Table 1. Entrainment generally acts to increase N and decrease P . By tending to warm descending currents, it creates consumptive sinking parcels, and reduces the buoyancy of productive rising parcels. Regardless of the value of ξ_R , cloud-top radiative cooling increases P by creating cold sinking parcels, and decreases N by reducing the positive buoyancy of warm sinking parcels. Finally, ξ_R acts to increase N , as well as to decrease P . When the cooling extends deep into the cloud layer, rising positively buoyant parcels are cooled before reaching p_B , and sinking positively buoyant parcels descend through a finite depth before being cooled appreciably.

Further discussion of Eulerian partitioning is given in Appendix A, and in Section 5.

4. Process partitioning

a. Basic concepts

Process partitioning is the assumption that each of various processes acting in concert produces and consumes the same energy as if it acted independently.

Manins and Turner (1978, hereafter MT) argued that, for an entraining mixed layer, the physically relevant energy ratio is the work done by the turbulence to redistribute the initial density profile, divided by the TKE generated by heating from below. They illustrated this idea by considering a tank of water initially isothermal but salt-stratified, and heated from below. Since both temperature and salt are conservative properties of the system, both are well-mixed by the

TABLE 1. Sensitivity of P and N to E , R_B and ξ_R for each partitioning theory.

	Eulerian partitioning	Process partitioning	Lagrangian partitioning
$\Pi^{-1}(\partial P/\partial E)_{E_R=0}$	-	0	?
$\Pi^{-1}(\partial N/\partial E)_{E_R=0}$	+	+	?
$\Pi^{-1}(\partial P/\partial R_B)_{E_R=0}$	+	+	?
$\Pi^{-1}(\partial N/\partial R_B)_{E_R=0}$	-	0	?
$\Pi^{-1}\partial P/\partial \xi_R$	-	-	?
$\Pi^{-1}\partial N/\partial \xi_R$	+	0	?

convective turbulence, and so both flux profiles are linear with height. The temperature flux vanishes at the mixed-layer top, and the salt flux vanishes at the lower boundary. Entrainment does not lead to any TKE-consuming downward temperature flux, but it does lead to a TKE-consuming upward salt flux. The surface heat flux clearly acts to generate TKE.

For their heated, salty mixed layer, MT argued that the gross production rate depends only on the heat flux (is independent of the salt flux), and that the consumption rate depends only on the salt flux (is independent of the heat flux). This is an example of process partitioning. Clearly, if there were salt stratification but no heating, both P and N would vanish, because there would be no turbulence; and if there were heating but no salt stratification, the net buoyancy flux would be non-negative at all levels. Nevertheless, warm rising parcels will be saltier, on the average, than cool sinking parcels; the salt stratification can reduce P as well as increase N . Moreover, fresh, sinking parcels will be cooler, on the average, than salty rising parcels; the heat flux can reduce N as well as increase P .

Similar but more striking ambiguities arise in the application of process partitioning to the cloud-topped mixed layer, as proposed by Stage and Businger (1981a,b; hereafter SB). Whereas MT had only one process to generate TKE (surface heating) and only one to consume it (salt mixing), SB have as TKE sources surface heating, surface evaporation, radiative cooling aloft, and radiative warming at low levels; and as TKE sinks they have surface cooling, surface drying, radiative warming aloft, and radiative cooling at low levels. Betts (1983) has pointed out that these various processes interact strongly to determine the buoyancy of individual parcels. For example, cloud-top radiative cooling reduces the positive buoyancy of entrained parcels, and so tends to reduce the buoyant consumption due to entrainment (Brost *et al.*, 1982). On the other hand, the entrainment of warm air reduces the ability of radiative cooling to promote convection by creating negatively buoyant parcels near cloud top. The production and consumption due to entrainment and radiative cooling are thus closely related.

Nevertheless, SB boldly assume that the buoyant

consumption due to the entrainment of warm air is independent of the cloud-top radiative cooling rate,

and that the buoyant production due to cloud-top radiative cooling is independent of the rate of entrainment of warm air. They write the production rate as

$$[F_{sv}(p)]_{\text{prod}} = \begin{cases} H[(F_h)_S - (1 - \delta\epsilon)L(F_r)_S] \cdot \xi + R_B(1 - \xi), & p_C \leq p \leq p_S \\ H[\beta(F_h)_S - \epsilon L(F_r)_S] \cdot \xi + \beta R_B(1 - \xi) + EH(-\beta\Delta h + \epsilon L\Delta r)(1 - \xi), & p_R \leq p \leq p_C \\ H[\beta(F_h)_S - \epsilon L(F_r)_S] \xi + \beta R_B[(1 - \xi) - (\xi_R - \xi)/\xi_R] + EH(-\beta\Delta h + \epsilon L\Delta r)(1 - \xi), & p_B \leq p \leq p_R, \end{cases} \quad (4.1)$$

and the consumption rate as

$$-[F_{sv}(p)]_{\text{cons}} = \begin{cases} H[-(F_h)_S + (1 - \delta\epsilon)L(F_r)_S] \cdot \xi + E[\Delta h - (1 - \delta\epsilon)L\Delta r](1 - \xi), & p_C \leq p \leq p_S \\ H[-\beta(F_h)_S + \epsilon L(F_r)_S] \cdot \xi + EH(\beta\Delta h - \epsilon L\Delta r)(1 - \xi), & p_B \leq p \leq p_C. \end{cases} \quad (4.2)$$

We have simplified SB's equations slightly by assuming that $R_B \geq 0$. Using (2.18-19) in (4.1-2), and neglecting the height variation of β and ϵ , we find that, for $\Pi \ll 1$,

$$\Pi^{-1}P \approx H[(F_h)_S - (1 - \delta\epsilon)L(F_r)_S](1 - \xi_C^2) + R_B(1 - \xi_C)^2 + H[\beta(F_h)_S - \epsilon L(F_r)_S]\xi_C^2 + \beta R_B[\xi_C(2 - \xi_C) - \xi_R] + EH(-\beta\Delta h + \epsilon L\Delta r)\xi_C(2 - \xi_C), \quad (4.3)$$

$$\Pi^{-1}N = H[-(F_h)_S + (1 - \delta\epsilon)L(F_r)_S](1 - \xi_C)^2 + H[-\beta(F_h)_S + \epsilon L(F_r)_S]\xi_C^2 + E\{\Delta h - (1 - \delta\epsilon)L\Delta r\} \times (1 - \xi_C^2) + H(\beta\Delta h - \epsilon L\Delta r)\xi_C(2 - \xi_C). \quad (4.4)$$

It is useful to compare (4.3-4) with (3.5-6), which are the corresponding results for Eulerian partitioning. The dependencies of P and N on E , R_B , and ξ_R are qualitatively different. Process partitioning makes P and N simple linear functions of E (by assumption), whereas Eulerian partitioning leads to more complex functions. This certainly makes process partitioning easier to work with. For $(\beta\Delta h - \epsilon L\Delta r) > 0$, which is the usual case, process partitioning says that P is independent of E , and that N increases with E . For $(\beta\Delta h - \epsilon L\Delta r) < 0$, which implies cloud-top entrainment instability (Randall, 1980a; Deardorff, 1980a), process partitioning allows P to increase with E . In contrast, Eulerian partitioning allows both P and N to change with E , regardless of the sign of $(\beta\Delta h - \epsilon L\Delta r)$.

Under process partitioning, cloud-top radiative cooling increases P , but does not influence N , while under Eulerian partitioning, R_B influences both P and N , regardless of the value of ξ_R . Finally, under process partitioning, ξ_R decreases P but does not influence N , while under Eulerian partitioning, both P and N depend on ξ_R . Table 1 summarizes these results, for the case $(\beta\Delta h - \epsilon L\Delta r) > 0$. Subsections *b* and *c* below give examples illustrating some alternative versions of process partitioning.

b. Offensive and defensive cooling

SB's assumption that cloud-top radiative cooling acts as a positive contribution to P is based on the plausible idea that the cooling helps to drive convection in the mixed layer. However, a second, equally plausible possibility is that the cooling reduces N by making it easier to entrain warm air from the inversion layer (Brost et

al., 1982), i.e., it "defends" the turbulence against buoyant consumption. For convenience, we refer to these alternative assumptions as "offensive" and "defensive" cooling, respectively. Offensive cooling promotes production; defensive cooling prevents consumption. In either case, stronger cooling will produce stronger entrainment. Both possibilities are consistent with the profile of the net buoyancy flux given by (2.22). However, the two alternatives lead to very different model results.

Figure 5 shows the equilibrium F_{sv} profiles obtained with the mixed-layer model and parameter settings described in Appendix B, for three different partitioning choices: Eulerian partitioning (with $A = 0.04$), process partitioning with offensive cooling (as implemented by SB, with $A = 0.2$), and process partitioning with defensive cooling (again, with $A = 0.2$). In all cases, $\delta p_R = 0$. For process partitioning, offensive cooling gives a relatively shallow, foggy mixed layer, while defensive cooling gives a deep mixed layer with elevated cloud. Eulerian partitioning gives a mixed layer depth comparable to that of process partitioning with defensive cooling, but with a smaller $(F_{sv})_S$.

c. Sensitivity to the depth of the radiatively cooled layer

Randall (1980b) and Fravalto *et al.* (1981) show that a mixed-layer model based on Eulerian partitioning is sensitive to δp_R . The reason is that the net buoyancy flux is smaller, in the upper part of the cloud layer, than it would be if the cooling did not extend down to that level; for Eulerian partitioning, this increases N and decreases P . On the other hand, SB show that

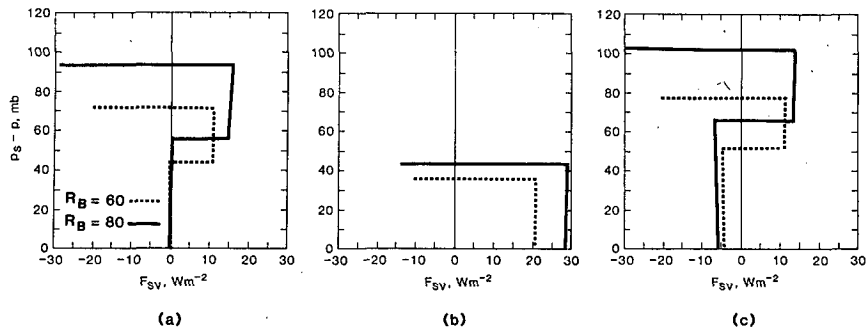


FIG. 5. Buoyancy flux profile for $R_B = 60$ and 80 W m^{-2} , for: (a) Eulerian partitioning; (b) process partitioning with offensive cooling; (c) process partitioning with defensive cooling.

their model is insensitive to δp_R . As seen in (4.3), they included the effects of δp_R as a negative contribution to P , but they did not allow δp_R to influence N . Fig. 6 shows the δp_R -sensitivity of three models: one based on Eulerian partitioning, the SB model, and a modified version of the SB model in which the δp_R -term is included as a positive contribution to N rather than as a negative contribution to P . For all three cases, $R_B = 70 \text{ W m}^{-2}$. The results show that the first and third models are sensitive to δp_R ; only the SB model is insensitive. Evidently models based on process partitioning may or may not be sensitive to δp_R , depending on whether the effects of $\delta p_R > 0$ are considered as reducing production or increasing consumption.

d. Discussion

As illustrated above, a process which contributes positively (say) to the net buoyancy flux can be regarded as a positive contribution to P , or as a negative contribution to N , or as contributing to both P and N . As further illustrated, model results are quite sensitive to such choices, simply because the entrainment rate is more sensitive to N than to P . The point is that there are serious ambiguities in the process partitioning theory. No such ambiguities occur with Eulerian parti-

tioning, which, as a rule, allows each process to influence both P and N .

Following MT and SB, we have appealed to Lagrangian partitioning in our attempts to decide whether a particular process produces or consumes TKE, since the rate of buoyant production or consumption of kinetic energy can be unambiguously determined (in principle) for individual parcels. However, these arguments have been strictly qualitative. To provide a firm basis for the evaluation of process partitioning, we need to quantitatively relate it to Lagrangian partitioning.

5. Lagrangian partitioning

a. Comparison with Eulerian and process partitioning

The Lagrangian partitioning theory defines P and N in terms of the buoyancy fluxes due to productive and consumptive parcels. A rising parcel is productive if it is positively buoyant, and consumptive otherwise. Similarly, a sinking parcel is productive if it is negatively buoyant, and consumptive otherwise. Consider N classes of parcels, such that all parcels within a class have identical vertical velocities and buoyancies. The classes are defined so that together they include all parcels. We can write

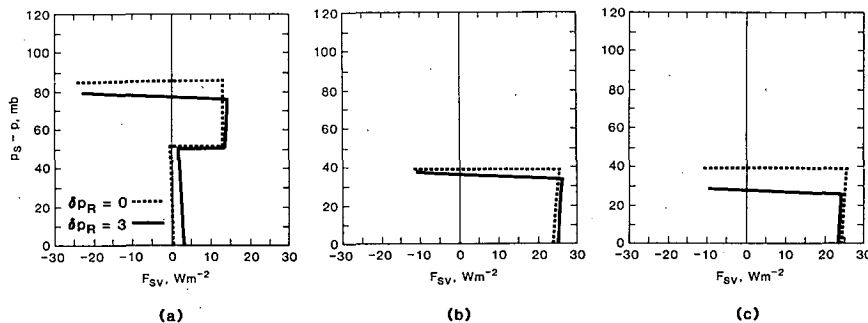


FIG. 6. Buoyancy flux profile for $\delta p_R = 0$ and 3 mb , for: (a) Eulerian partitioning; (b) process partitioning as implemented by Stage and Businger; (c) process partitioning modified so that δp_R increases N rather than decreases P . See text for details.

$$(F_{sv})_{\text{prod}} = \rho \sum_{i=1}^M \sigma_i H[(w_i - \bar{w})(s_{vi} - \bar{s}_v)], \quad (5.1a)$$

$$-(F_{sv})_{\text{cons}} = \rho \sum_{i=1}^M \sigma_i H[-(w_i - \bar{w})(s_{vi} - \bar{s}_v)], \quad (5.1b)$$

where subscript i denotes a particular parcel, M is the total number of classes of parcels, ρ is density, σ is the fractional area occupied by a parcel, w is vertical velocity, and s_v is the virtual dry static energy. An overbar denotes an area average, i.e.,

$$(\bar{\quad}) = \sum_{i=1}^M \sigma_i (\quad)_i. \quad (5.2)$$

To explore the relationships between Lagrangian partitioning and Eulerian partitioning, we must compare (5.1a–b) with (3.1–2). The net buoyancy flux is given by

$$F_{sv} = \rho \sum_{i=1}^M \sigma_i [(w_i - \bar{w})(s_{vi} - \bar{s}_v)]. \quad (5.3)$$

As shown in Appendix C, we can rewrite (5.3) as

$$F_{sv} = \rho \sum_{i=1}^{M-1} \sum_{j=1}^M \sigma_i \sigma_j (w_i - w_j)(s_{vi} - s_{vj}). \quad (5.4)$$

Consider now the simple case $M = 2$, i.e. two classes of parcels, the first class consisting of rising parcels, and the second of sinking parcels. The turbulence is modeled here as a single family of “top-hat” plumes; all updrafts are the same, and all downdrafts are the same. For simplicity, let subscripts u and d denote ascending and descending parcels, respectively, and let σ (without a subscript) denote the fractional area covered by rising motion. From (5.4), the net buoyancy flux is given by

$$F_{sv} = \rho \sigma (1 - \sigma)(w_u - w_d)(s_{vu} - s_{vd}). \quad (5.5)$$

According to Lagrangian partitioning, the production flux is

$$(F_{sv})_{\text{prod}} = \rho \{ \sigma H[(w_u - \bar{w})(s_{vu} - \bar{s}_v)] + (1 - \sigma) H[(w_d - \bar{w})(s_{vd} - \bar{s}_v)] \}. \quad (5.6)$$

Using the identities

$$\begin{aligned} (\quad)_u - (\bar{\quad}) &= (1 - \sigma)[(\quad)_u - (\quad)_d], \\ (\quad)_d - (\bar{\quad}) &= -\sigma[(\quad)_u - (\quad)_d], \end{aligned} \quad (5.7)$$

and the fact that

$$0 \leq \sigma \leq 1, \quad (5.8)$$

we can rewrite (5.6) as

$$(F_{sv})_{\text{prod}} = \rho \sigma (1 - \sigma) \{ H[(w_u - w_d)(s_{vu} - \bar{s}_v)] + H[-(w_u - w_d)(s_{vd} - \bar{s}_v)] \}, \quad (5.9)$$

Since

$$w_u - w_d \geq 0, \quad (5.10)$$

we have

$$(F_{sv})_{\text{prod}} = \rho \sigma (1 - \sigma)(w_u - w_d) \{ H[(s_{vu} - s_v)] + H[-(s_{vd} - s_v)] \}. \quad (5.11)$$

Using (5.7) again, we can rewrite (5.10) as

$$(F_{sv})_{\text{prod}} = \rho \sigma (1 - \sigma)(w_u - w_d) \{ H[(1 - \sigma)(s_{vu} - s_{vd})] + H[\sigma(s_{vu} - s_{vd})] \}. \quad (5.12)$$

Invoking (5.8) and (5.10) again, and using (5.5), we finally obtain

$$(F_{sv})_{\text{prod}} = H[F_{sv}]. \quad (5.13)$$

Similarly, we can show that

$$-(F_{sv})_{\text{cons}} = H[-F_{sv}]. \quad (5.14)$$

These results show that, for $M = 2$, Lagrangian partitioning is equivalent to Eulerian partitioning. The reason is that, for $M = 2$, either all parcels at a given level are productive, or all are consumptive; productive and consumptive parcels cannot coexist at the same level. Of course, for $M > 2$ this conclusion does not hold.

We can also conclude from (5.13–14) that for $M = 2$, consumption occurs only where the net buoyancy flux is negative, and production occurs only where the net buoyancy flux is positive. This was recognized by Stage and Businger (1981b, p. 2233) and Hanson (1982, p. 471). Contrary to Manins and Turner (1978, p. 40), we were able to derive (5.13–14) without assuming that entrained fluid penetrates only so far so the net buoyancy flux is negative, or that fluid heated at the boundary penetrates only through the remaining fraction of the layer.

To explore the relationships between Lagrangian partitioning and process partitioning, we must compare (5.1a, b) with (4.1–2). Specifically, we must express the right-hand sides of (5.1a, b) in terms of the various cloud-topped mixed layer processes. In effect, we have already done this for the case $M = 2$. As discussed above, for $M = 2$ Lagrangian partitioning reduces to Eulerian partitioning, and we have compared Eulerian partitioning with process partitioning in Sections 3 and 4. The case $M > 2$ is pretty complicated; no attempt is made to analyze it here.

b. Discussion of observations

Wilczak and Businger (1983; hereafter, WB) examined BAO tower observations of the dry convectively driven mixed layer. They used conditional sampling techniques to identify, at each level, ascending positively buoyant parcels, ascending negatively buoyant parcels, descending positively buoyant parcels, and descending negatively buoyant parcels. Their results show that productive and consumptive parcels do indeed coexist at all levels; the $M = 2$ model does not apply. In the lower mixed layer, where production

dominates, consumption is primarily due to cool parcels carried upward within organized updrafts. Near the inversion, where consumption dominates, production is due to the collapse of cold penetrating domes, and/or the occasional, dynamically forced ascent of warm, newly entrained air. The consumption rate is nonzero right down to the surface, and the production rate is nonzero all the way up to the mixed-layer top.

These observations show that Eulerian partitioning is not strictly equivalent to Lagrangian partitioning, since such equivalence holds only if production and consumptive parcels do not coexist at any level. The observations also conflict with process partitioning, since the descent of warm, recently entrained air by no means dominates consumption; rising, cool air is of comparable importance. Contrary to the models of MT and SB, the observations show that consumption occurs even near the surface, and production occurs even near the mixed-layer top.

WB discuss the possibility that near-surface consumption and near-inversion production occur as the result of "temperature fluctuations being forcibly mixed by the TKE released through convection." They argue that such background consumption and production are independent of height, and irrelevant to the TKE balance of the mixed layer. They suggest that the observed remaining contributions to production and consumption are in fact those discussed in the process partitioning theory of SB, although this seems unlikely since the background consumption rate identified by WB is not large enough to account for all of the observed consumption due to rising cool parcels in the lower mixed layer.

Nevertheless, WB have advanced the idea that some part of the production and consumption rates, as defined by Lagrangian partitioning, may be due to "irrelevant" background processes. To examine the implications of this idea, let

$$P_L = P_P + \tilde{P}, \quad (5.15)$$

$$N_L = N_P + \tilde{N}, \quad (5.16)$$

where \tilde{P} and \tilde{N} are the background production and consumption rates, respectively. Following WB, we assume that

$$\tilde{P} - \tilde{N} = 0, \quad (5.17)$$

i.e., there is no net production due to background processes. Suppose now that

$$D = (1 - A_L)P_L, \quad (5.18)$$

$$D = (1 - A_P)P_P \quad (5.19)$$

both are valid. It follows that

$$\tilde{P} = P_L(A_L - A_P)/(1 - A_P). \quad (5.20)$$

The interpretation of (5.20) is that background production is simply proportional to total gross produc-

tion. If there is no gross production, there will be no background production. This seems quite plausible.

WB's attempt to identify background processes in their data was guided by the concept of process partitioning. Alternatively, it is possible to discuss background production and consumption from the point of view of Eulerian partitioning, as follows. The turbulent motions in the mixed layer can be modeled as the superposition of a "principal circulation" and small-scale eddies. The principal circulation is due to a single family of "top-hat" plumes, so that (5.5-14) apply to it. It can be defined in such a way that the net buoyancy flux due to the principal circulation is the same as the total net buoyancy flux. Then the gross production due to the principal circulation is given by (5.13), i.e., by Eulerian partitioning. The principal circulation contributes the relevant parts of the gross production and consumption. The small scale eddies contribute the irrelevant background production and consumption. By analogy with (5.20), we have

$$\tilde{P} = P_L(A_L - A_E)/(1 - A_E), \quad (5.21)$$

i.e., the background production due to the small-scale eddies is proportional to the total gross production. Thus, by acknowledging that the turbulence of the mixed layer consists of both large and small eddies, and using the concept of background production, we have been able to reconcile Eulerian partitioning with Lagrangian partitioning.

6. Summary and conclusions

Entrainment theories for mixed-layer models involve assumptions about how the net rate of turbulence kinetic energy production can be partitioned into gross production and consumption. Lagrangian partitioning provides a *definition* of gross production, but is difficult to implement directly in a model. Eulerian and process partitioning can be interpreted as attempts to implement Lagrangian partitioning indirectly.

For a simple plume model of the turbulence, Eulerian and Lagrangian partitioning are equivalent. The model can be reconciled with the observations of Wilczak and Businger (1983) by recognizing that small-scale eddies coexist with the plumes. It serves as an important conceptual link between Eulerian partitioning and Lagrangian partitioning.

Process partitioning can be formulated in many different ways. Examples given in this paper show that model results are sensitive to the choice of formulation. A quantitative link between process partitioning and Lagrangian partitioning has yet to be established.

The role of shear in determining the gross production and consumption rates has been touched on only briefly in this paper. Shear becomes dominant in the stably stratified PBL. It is possible that the efficiency with which TKE is generated depends on the overall stability of the PBL (i.e., on a bulk Richardson num-

ber). When the PBL is stably stratified, some of the kinetic energy generated through shear can be stored in gravity wave modes, which are only weakly dissipated. This suggests that A decreases as the stability increases. The dependence of A on stability is an interesting topic for further research.

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APPENDIX A

Approximate Solution for the Structure of the Steady, Horizontally Homogeneous Cloud-Topped Mixed Layer

From (B2-7), we find that the steady, horizontally homogeneous mixed-layer is described by

$$V(h_{SS}^* - h_M) = -E(h_{00} - \Gamma_h \delta p_M - h_M) + R_B, \quad (A1)$$

$$V(q_{SS}^* - r_M) = -E(q_{00} - \Gamma_q \delta p_M - r_M), \quad (A2)$$

and

$$\delta p_M (\nabla \cdot \mathbf{V}_M) = gE. \quad (A3)$$

For a cloud-topped mixed layer in equilibrium with moderate large-scale divergence, and for $\delta p_R = 0$, Eulerian partitioning tends to give a weak negative subcloud-layer buoyancy flux:

$$(F_{sv})_S = (F_h)_S - (1 - \delta\epsilon)L(F_r)_S < 0, \quad (A4)$$

$$(F_{sv})_S \ll (F_h)_S. \quad (A5)$$

The reason is that negative subcloud-layer buoyancy fluxes are needed to contribute to N , but only small negative values are allowed since $N \ll P$. The observations of Brost *et al.* (1982, Table 1) show that (A5) is relevant to the real world. For 7 cases with positive evaporation, the absolute value of $(F_{sv})_S/(F_h)_S$ ranged from 0.06 to 0.33, with a mean of 0.18. Eulerian partitioning provides an explanation for this.

We can take advantage of (A5) to obtain an approximate analytic solution for the equilibrium mixed-layer structure, along lines similar to those suggested by Lilly (1968) in his discussion of "minimum entrainment." It follows from (A5) that

$$h_{SS}^* - h_M \approx (1 - \delta\epsilon)L(q_{SS}^* - r_M). \quad (A6)$$

From (A1), (A2), and (A6), we can obtain

$$E[(s_{v00} - s_{vSS}^* - \Gamma_{sv} \delta p_M)] = R_B, \quad (A7)$$

where

$$s_{v00} - s_{vSS}^* \equiv (h_{00} - h_{SS}^*) - (1 - \delta\epsilon)L(q_{00} - q_{SS}^*), \quad (A8)$$

$$\Gamma_{sv} \equiv \Gamma_h - (1 - \delta\epsilon)L\Gamma_q. \quad (A9)$$

By using (A3) in (A7), we can obtain a quadratic equation for δp_M ; the solution is

$$\delta p_M = \{(s_{v00} - s_{vSS}^*) - [(s_{v00} - s_{vSS}^*)^2 - 4\Gamma_{sv} g R_B (\nabla \cdot \mathbf{V}_M)^{-1/2}]\} (2\Gamma_{sv})^{-1}. \quad (A10)$$

The sign of the discriminant has been chosen so that δp_M increases as R_B increases. Once δp_M has been obtained from (A9), we can solve (A3) for E , and then obtain h_M and r_M from (A1) and (A2), respectively.

Inspection of (A9) shows several interesting features. Only the ratio of R_B to $(\nabla \cdot \mathbf{V}_M)$ matters; these two quantities do not enter in any other combination. If the radiative cooling is sufficiently weak, the equilibrium PBL depth becomes independent of the large-scale divergence. The denominator of (A9) is essentially the static stability of the free atmosphere; as the static stability decreases, the equilibrium PBL depth increases. Deep mixed layers are favored by a warm sea surface and a cold, weakly stratified free atmosphere.

Somewhat similar results were obtained by Schubert (1976), using a different closure assumption.

In performing this analysis, the only "entrainment assumption" used was (A5). It wasn't even necessary to assume that the PBL is cloud-topped.

The results obtained above would fail, for example, in air-mass transformation situations, not only because (A5) is violated, but also because advective effects, which we have omitted, become important. They would also fail whenever cumulus clouds transport mass out of the PBL, since (A3) does not include the relevant mass sink term (Arakawa and Schubert, 1974).

APPENDIX B

Model Description

In this paper, we use a mixed-layer model whose basic framework is almost identical to that described by Randall (1980b). The prognostic equations of the model express the conservation laws for the mixed layer mass, moist static energy, and total mixing ratio. These are given by

$$\partial/\partial t(\delta p_M) = -(\nabla \cdot \mathbf{V}_M)\delta p_M + gE, \quad (B1)$$

$$\partial h_M/\partial t = [V(h_{SS}^* - h_M) + E\Delta h - R_B]g/\delta p_M, \quad (B2)$$

$$\partial r_M/\partial t = [V(q_{SS}^* - r_M) + E\Delta r]g/\delta p_M. \quad (B3)$$

Here h_M and r_M are the mixed-layer moist static energy and total mixing ratio, respectively; $V \equiv \rho_s c_T |\mathbf{V}_M|$ is

a prescribed "ventilation mass flux"; and $(\nabla \cdot \mathbf{V}_M)$ is the prescribed large-scale divergence. In (B2) and (B3), we have omitted advection, for simplicity. The effective sea surface saturation moist static energy h_{SS}^* and the sea surface saturation mixing ratio q_{SS}^* are evaluated using an assumed sea surface temperature and surface pressure. The free-atmospheric profiles of h and q are assumed to be such that

$$h_{B+} = h_{00} - \Gamma_h \delta p_M, \tag{B4}$$

$$q_{B+} = q_{00} - \Gamma_q \delta p_M, \tag{B5}$$

where h_{00} , q_{00} , Γ_h and Γ_q are prescribed (Fig. 7). The jumps are obtained from

$$\Delta h = h_{B+} - h_M, \tag{B6}$$

$$\Delta r = q_{B+} - r_M. \tag{B7}$$

The cloud base level is determined through the method described by Randall (1980b).

For all runs discussed in this paper, we use $(\nabla \cdot \mathbf{V}_M) = 4 \times 10^{-6} \text{ s}^{-1}$, $h_{00} = 314.4 \text{ kJ kg}^{-1}$, $\Gamma_h = -17.3 \text{ J kg}^{-1} \text{ mb}^{-1}$, $q_{00} = 4.38 \text{ g kg}^{-1}$, and $\Gamma_q = 5.69 \times 10^{-3} \text{ g kg}^{-1} \text{ mb}^{-1}$. The sea surface temperature is given as 13°C , and the sea level pressure is 1020 mb . The ventilation mass flux is $1.303 \times 10^{-2} \text{ kg m}^{-2} \text{ s}^{-1}$.

APPENDIX C

Derivation of (5.4)

Given

$$\sum_{i=1}^M \sigma_i = 1, \tag{C1}$$

$$\sum_{i=1}^M \sigma_i A_i = \bar{A}, \tag{C2}$$

we shall show that

$$\overline{A'B'} = \sum_{i=1}^M \sigma_i (A_i - \bar{A})(B_i - \bar{B}) \tag{C3}$$

$$= \sum_{i=1}^{M-1} \sum_{j=i+1}^M \sigma_i \sigma_j (A_i - A_j)(B_i - B_j). \tag{C4}$$

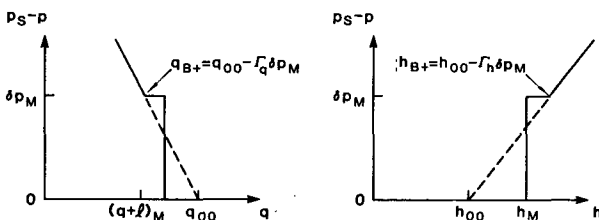


FIG. 7. Method to determine h_{B+} and q_{B+} .

From (C1) and (C3), we have

$$\begin{aligned} \overline{A'B'} &= \sum_{i=1}^M \sigma_i (A_i B_i - \bar{A} B_i - A_i \bar{B} + \bar{A} \bar{B}) \\ &= \sum_{i=1}^M \sigma_i A_i B_i - \bar{A} \sum_{i=1}^M \sigma_i B_i \\ &\quad - \bar{B} \sum_{i=1}^M \sigma_i A_i + \bar{A} \bar{B} \sum_{i=1}^M \sigma_i. \end{aligned} \tag{C5}$$

Using (C1) and (C2) in (C5)

$$\begin{aligned} \overline{A'B'} &= \sum_{i=1}^M \sigma_i A_i B_i - \bar{A} \bar{B} - \bar{B} \bar{A} + \bar{A} \bar{B} \\ &= \sum_{i=1}^M \sigma_i A_i B_i - \bar{A} \bar{B} \\ &= \sum_{i=1}^M \sigma_i A_i B_i - \left(\sum_{i=1}^M \sigma_i A_i \right) \left(\sum_{j=1}^M \sigma_j B_j \right). \end{aligned} \tag{C6}$$

The second equality above means that $\overline{A'B'} = \overline{AB} - \bar{A}\bar{B}$. But we have

$$\left(\sum_{i=1}^M \sigma_i A_i \right) \left(\sum_{j=1}^M \sigma_j B_j \right) = \sum_{i=1}^M \sum_{j=1}^M \sigma_i \sigma_j A_i B_j. \tag{C7}$$

Also, using (C1), we have

$$\sum_{i=1}^M \sigma_i A_i B_i = \sum_{i=1}^M \sum_{j=i}^M \sigma_i \sigma_j A_i B_i. \tag{C8}$$

Substituting (C7) and (C8) into (C6), we obtain

$$\overline{A'B'} = \sum_{i=1}^M \sum_{j=1}^M \sigma_i \sigma_j A_i B_i - \sum_{i=1}^M \sum_{j=1}^M \sigma_i \sigma_j A_i B_j. \tag{C9}$$

Now start with (C4) and work backwards. By symmetry,

$$\begin{aligned} &\sum_{i=1}^{M-1} \sum_{j=i+1}^M \sigma_i \sigma_j (A_i - A_j)(B_i - B_j) \\ &= \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^M \sigma_i \sigma_j (A_i - A_j)(B_i - B_j) \\ &= \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^M \sigma_i \sigma_j (A_i B_i - A_i B_j - A_j B_i + A_j B_j) \\ &= \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^M \sigma_i \sigma_j A_i B_i - \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^M \sigma_i \sigma_j A_i B_j \\ &\quad - \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^M \sigma_i \sigma_j A_j B_i + \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^M \sigma_i \sigma_j A_j B_j. \end{aligned} \tag{C10}$$

Now just rename subscripts, and combine terms:

$$\sum_{i=1}^{M-1} \sum_{j=i+1}^M \sigma_i \sigma_j (A_i - A_j)(B_i - B_j) \\ = \sum_{i=1}^M \sum_{j=1}^M \sigma_i \sigma_j A_i B_j - \sum_{i=1}^M \sum_{j=1}^M \sigma_i \sigma_j A_i B_i. \quad (\text{C11})$$

By comparing (C9) and (C11), we obtain (C4), which is the desired result.

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