Gravity Wave Turbulence Interaction in the Presence of a Critical Level

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ABSTRACT

The analysis of an earlier work on the interaction between an internal gravity wave and the wave-induced turbulence is extended here to the case where the wave is generated by vertical wind shear. The initial system is described by a background wind and a temperature distribution such that the Richardson number is less than $\frac{1}{4}$. A gravity wave is generated by such a dynamically unstable system and grows exponentially in time. The wave modifies the Richardson number and lowers it, particularly in the neighborhood of the critical level. When the generalized Richardson number falls below an assumed critical level, turbulence is assumed to develop and is described by a "$1\frac{1}{2}$th order" scheme. A diffusion coefficient can then be calculated which has a mean and a fluctuating part. It is the latter which turns out to be responsible for the positive feedback between the wave and the wave-induced turbulence resulting in the wave growing at a faster rate than the one predicted by linear theory.

1. Introduction

In a recent paper, hereafter referred to as I, Fuà *et al.* (1982) analyze the interaction between a nonsingular neutral gravity wave and wave-induced turbulence in a statically and dynamically stable atmospheric background. Their system is characterized by an initial Richardson number Ri_0, larger than $\frac{1}{4}$ throughout the flow, but only slightly larger than $\frac{1}{4}$ over some height regions. At time $t = 0$, a monochromatic neutral gravity wave starts propagating in the system, modifies Ri and lowers it below $\frac{1}{4}$ over portions of the flow domain. As soon as this occurs, they assume that turbulence develops and can be parameterized by a so-called "$1\frac{1}{2}$th order" scheme in which the evolution of the eddy diffusion coefficient ($K \propto \nu^2 = \frac{1}{2}\langle u_i'u_i'\rangle$, u_i' being the ith component of turbulent velocity) is derived from the equation for the turbulence kinetic energy. This scheme allows them to calculate the mean and fluctuating part of K, $\bar{K}$ and $\tilde{K}$, respectively, and to show that, at times, the latter can overcome the damping effects of the former and force the wave to grow in time.

A number of assumptions are made in I such that: i) the occurrence of turbulence can be related to a local, wave-modified, Richardson number; ii) the wave-turbulence-mean field system is dominated, at least initially, by a wave-turbulence coupling described via an eddy diffusion coefficient depending on both time and spatial coordinates; iii) the fundamental harmonic of the fluctuating part of K coincides with the period and wavenumber of the initial gravity wave and dominates all the others; iv) the gravity wave is initially monochromatic and thus the various forms of weak and strong nonlinear interactions discussed by Phillips (1980) and Orlanski and Cerasoli (1981) are excluded; and v) the wave has no critical level where the horizontal phase velocity would match the background wind in direction and amplitude.

In this paper, we eliminate the last assumption. The initial background, still statically stable, is dynamically unstable, with exponentially growing waves having a critical level where Ri_0 is less than a ¼. This is an important case since vertical shear is a common mechanism for generation of gravity waves and wave-turbulence coupling has often been invoked to explain the occurrence of turbulence in the ocean and the atmosphere. Various references to this subject are given in I and in Einaudi *et al.* (1978/79). We show in this paper that the wave-turbulence feedback leads to a transfer of energy to the wave which, in turn, grows at a rate much larger than the linear prediction.

The equations of motion and the parameterization of turbulence are discussed in Section 2, while the analytical solution and its approximations are presented in Section 3. In both sections, the reader is referred to I for details. The numerical results are presented in Section 4.

2. The governing equations

A wave-turbulence-mean flow system can often be described effectively with the introduction of a triple decomposition, whereby each variable $b(t, x_i)$ is written as

$$b(t, x_i) = \bar{b}(x_i) + \tilde{b}(t, x_i) + b'(t, x_i), \qquad (2.1)$$

where x_i, $i = 1, 2, 3$, are the Cartesian coordinates, with the x_3 axis taken vertically upward; $\bar{b}$ is the mean value defined as

$$\bar{b} = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} b(t')dt'; \qquad (2.2)$$

$\tilde{b}$ is the wave component and b' the turbulent component. The presence in the system of a periodic disturbance with period τ allows the introduction of a phase averaging operator

$$\langle b \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} b(t + n\tau) = \bar{b} + \tilde{b}, \qquad (2.3)$$

which averages over a large ensemble of points having the same phase with respect to the gravity wave. The gravity wave, therefore, plays the role of a reference oscillator (Reynolds and Hussain, 1972). In the case of a growing wave, periodicity is not strictly true and care must be exercised in phase averaging and choosing small enough growth rates.

Following the general approach of turbulence theory (see Lumley and Panofsky, 1964), straightforward manipulation of the equations of conservation of momentum, mass and energy leads to a set of coupled equations for the mean, turbulent and wave components of the flow field. Of interest to us here are the equations for the periodic components:

$$\bar{\rho}\left[\frac{\partial \tilde{u}_i}{\partial t} + \bar{u}_j \frac{\partial}{\partial x_j} \tilde{u}_i + \tilde{u}_j \frac{\partial}{\partial x_j} \bar{u}_i\right] = -\frac{\partial \tilde{p}}{\partial x_i} - g\tilde{\rho}\delta_{i3} - \frac{\partial \tilde{R}_{ij}}{\partial x_j} - \frac{\partial \tilde{r}_{ij}}{\partial x_j}, \qquad (2.4)$$

$$\frac{\partial}{\partial x_i}(\bar{\rho}\tilde{u}_i) = 0, \qquad (2.5)$$

$$\bar{\rho}c_p\left[\frac{\partial \tilde{\theta}}{\partial t} + \bar{u}_i \frac{\partial \tilde{\theta}}{\partial x_i} + \tilde{u}_i \frac{\partial \bar{\theta}}{\partial x_i}\right] = -\frac{\partial}{\partial x_i}[\tilde{E}_i + \tilde{e}_i], \qquad (2.6)$$

with

$$\tilde{R}_{ij} = \bar{\rho}[\tilde{u}_i\tilde{u}_j - \overline{\tilde{u}_i\tilde{u}_j}], \quad \tilde{r}_{ij} = \bar{\rho}[\langle u'_i u'_j\rangle - \overline{u'_i u'_j}], \qquad (2.7)$$

$$\tilde{E}_i = c_p\bar{\rho}[\tilde{u}_i\tilde{\theta} - \overline{\tilde{u}_i\tilde{\theta}}], \quad \tilde{e}_i = c_p\bar{\rho}[\langle u'_i\theta'\rangle - \overline{u'_i\theta'}], \qquad (2.8)$$

and where u_i is the velocity component in the i-direction, θ the potential temperature, p the pressure, ρ the density, g the acceleration of gravity, c_p the specific heat at constant pressure, δ_{ij} the Kronecker delta and the Einstein summation convention has been adopted. The functions $-\tilde{r}_{ij}$ and $-\tilde{e}_i$ can be interpreted as the oscillation of the turbulent Reynolds stress and heat flux, respectively, due to the presence of the wave. Equivalently, $-\tilde{R}_{ij}$ and $-\tilde{E}_i$ represent the oscillating part of the nonlinear wave contribution to the Reynolds stress and heat flux. In these equations, the molecular viscous and conductive terms have been neglected; we are treating flows with large Reynolds and Péclet numbers.

The evolution of the $\tilde{r}_{ij}$'s and the $\tilde{e}_i$'s is given by a set of equations which contain many unknown terms. Thus, the solution of (2.4)–(2.6) will have to rely on an appropriate parameterization for the $\tilde{r}_{ij}$'s and the $\tilde{e}_i$'s. We make the working hypothesis that turbulence and wave interact through a process that can be represented formally as in the constant K theory, except that now the eddy viscosity coefficient K will have a fluctuating part due to the presence of the wave and the strain rate tensor S_{ij} will depend on the gravity wave as well as on the gradients of the background.

Formally, we will write (see also Monin and Yaglom, 1971)

$$\langle u'_i u'_j\rangle = \frac{1}{3}\langle u'_m u'_m\rangle\delta_{ij} - (\bar{K}\bar{S}_{ij} + \tilde{K}\bar{S}_{ij} + \bar{K}\tilde{S}_{ij} + \tilde{K}\tilde{S}_{ij}), \qquad (2.9)$$

$$\overline{u'_i u'_j} = \frac{1}{3}\overline{u'_m u'_m}\delta_{ij} - (\bar{K}\bar{S}_{ij} + \overline{\tilde{K}\tilde{S}_{ij}}), \qquad (2.10)$$

$$\langle \theta' u'_i\rangle = -\alpha_0(\bar{K} + \tilde{K})\frac{\partial}{\partial x_i}(\bar{\theta} + \tilde{\theta}), \qquad (2.11)$$

$$\overline{\theta' u'_i} = -\alpha_0\left[\bar{K}\frac{\partial \bar{\theta}}{\partial x_i} + \overline{\tilde{K}\frac{\partial \tilde{\theta}}{\partial x_i}}\right], \qquad (2.12)$$

$$K = \bar{K} + \tilde{K}, \qquad (2.13)$$

$$S_{ij} = \bar{S}_{ij} + \tilde{S}_{ij}; \quad \bar{S}_{ij} = \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}; \quad \tilde{S}_{ij} = \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}, \qquad (2.14)$$

so that

$$\tilde{r}_{ij} = \bar{\rho}\left\{\frac{\delta_{ij}}{3}[\langle u'_m u'_m\rangle - \overline{u'_m u'_m}] - [\bar{K}\tilde{S}_{ij} + \tilde{K}\bar{S}_{ij} + (\tilde{K}\tilde{S}_{ij} - \overline{\tilde{K}\tilde{S}_{ij}})]\right\}, \qquad (2.15)$$

$$\tilde{e}_i = -c_p\bar{\rho}\alpha_0\left[\bar{K}\frac{\partial \tilde{\theta}}{\partial x_i} + \tilde{K}\frac{\partial \bar{\theta}}{\partial x_i} + \left(\tilde{K}\frac{\partial \tilde{\theta}}{\partial x_i} - \overline{\tilde{K}\frac{\partial \tilde{\theta}}{\partial x_i}}\right)\right]. \qquad (2.16)$$

The quantity α_0 is the ratio of the eddy diffusivities for temperature and momentum and will be assumed constant (Monin and Yaglom, 1971).

The final step consists in determining an equation for K. Such an equation is obtained from the turbulent kinetic energy equation once the constitutive relations have been chosen for K and for the rate of turbulent energy dissipation ϵ_d, in terms of the turbulent kinetic energy itself and the characteristic length scale l of turbulence. In a homogeneous turbulent field, the equation for the turbulent kinetic energy [see also Monin and Yaglom (1971) and Zeman (1981)] can be written as

$$\frac{\partial}{\partial t}\nu^2 + (\bar{u}_j + \tilde{u}_j)\frac{\partial}{\partial x_j}\nu^2 = -\frac{2}{3}\nu^2\frac{\partial}{\partial x_j}(\bar{u}_j + \tilde{u}_j) + KS_{ij}\frac{\partial}{\partial x_j}(\bar{u}_i + \tilde{u}_i) \times\left[1 + \frac{g}{\bar{\theta} + \tilde{\theta}}\frac{\langle u'_3\theta'\rangle}{KS_{mn}\dfrac{\partial}{\partial x_n}(\bar{u}_m + \tilde{u}_m)}\right] - \epsilon_d, \quad (2.17)$$

where

$$\nu^2 = \frac{1}{2}\langle u'_i u'_i\rangle, \quad (2.18)$$

and ϵ_d is the rate of turbulence energy dissipation. The neglect of pressure and turbulent transport implied in the use of the homogeneous equations requires that at least the turbulence be of much smaller scale than the wave motion.

In analogy with molecular transport, K is assumed to depend on a characteristic velocity, taken as ν, and on a length scale l

$$K = \alpha_K \nu l, \quad (2.19)$$

while the dissipation rate is given by dimensional considerations (Batchelor, 1953)

$$\epsilon_d = \alpha_\epsilon \nu^3 l^{-1}. \quad (2.20)$$

Both quantities α_K and α_ϵ are taken to be empirical constants. We choose l in the following two ways: let Ri_0 be less than ¼ over a height range Λ_0; we assume that no turbulence exists unless the gravity wave is present and the wave-dependent Richardson number reaches a value Ri_c

$$\mathrm{Ri}_c = (\mathrm{Ri}_0)_{\min} - \delta, \quad (2.21)$$

where $(\mathrm{Ri}_0)_{\min}$ = minimum of Ri_0 and δ is positive and much less than 1. In the region of vertical extent Λ where the wave-modified Richardson number is less than Ri_c, we choose $l = \Lambda$. Because of relaxation effects, the turbulence will not disappear as soon as $\mathrm{Ri} > \mathrm{Ri}_c$ (Woods, 1969). In such regions, l is taken to be equal to

$$\Lambda' = \nu/n, \quad (2.22)$$

with n the Brunt-Väisälä frequency. If there were no mixing and the turbulence were isotropic, Λ' would be the vertical distance a parcel had to cover in a uniform mean density gradient to convert its vertical turbulent kinetic energy into potential energy. It is clear that l will depend on the wave structure and therefore, on time and spatial coordinates.

Using (2.19) and (2.20), Eq. (2.17) can be written as

$$\frac{\partial\kappa}{\partial t} + (\bar{u}_j + \tilde{u}_j)\frac{\partial}{\partial x_j}\kappa = -\frac{\kappa}{3}\frac{\partial}{\partial x_j}(\bar{u}_j + \tilde{u}_j) + \frac{l}{2}\alpha_K{}^2 S_{ij}\frac{\partial}{\partial x_j}(\bar{u}_i + \tilde{u}_i) \times [1 - \alpha_0 \mathrm{Ri}] - \epsilon_d\frac{\kappa^2}{2l\alpha_K}, \quad (2.23)$$

where

$$\kappa = K/l, \quad n^2 = \frac{g}{\bar{\theta} + \tilde{\theta}}\frac{\partial}{\partial x_3}(\bar{\theta} + \tilde{\theta}), \quad (2.24)$$

$$\mathrm{Ri} = n^2\left[\frac{1}{2}S_{ij}S_{ij}\right]^{-1}. \quad (2.25)$$

Eq. (2.23) describes the temporal and spatial evolution of κ and hence of K since l is determined once $\tilde{u}_i$ is known.

Two issues which remain to be resolved are the role of the strain rate tensor S_{ij} in (2.23) and the relative significance of the various terms present in the generalized form of the Richardson number (2.25).

3. An analytical solution

Using a Green's function approach, we can demonstrate that, to a first approximation and for a sufficiently short time, the evolution of the wave-turbulence system can be described by an expression of the form

$$q(x_1, x_3, t) = q(x_3)[1 + F(t)]\exp(\omega_0 t - ik_0 x_1) + \text{c.c.}, \quad (3.1)$$

where c.c. stands for the complex conjugate, $q(x_3)$, ω_0 and k_0 are the amplitude, frequency and wavenumber of the vertical displacements associated with the initial gravity wave and $F(t)$, in general complex, is zero at $t = 0$ and depends on the feedback between the wave and the wave-induced turbulence.

a. *Solution for the initial gravity wave*

The initial gravity wave is assumed to be generated locally by vertical shear. The initial state is similar to the one analyzed by Lalas and Einaudi (1976), characterized by a constant n^2 and a hyperbolic tangent profile for the mean wind $\bar{u}_1$:

$$\bar{u}_1 = u_0\tanh(y), \quad y = x_3/h. \quad (3.2)$$

The calculations are carried out for h = 120 m, u_0

$= 5$ m s^{-1}, the ground is located at -1200 m, the value of n^2 corresponds to a background temperature of 276 K and the minimum Richardson number is 0.2. The initial Richardson number remains below $\frac{1}{4}$ over an interval of 86 m. The value of δ in (2.21) is chosen to be equal to 0.002.

The characteristics of the initial wave can be found from a stability analysis of (2.4)–(2.6) in which nonlinear and turbulent terms are neglected. The eigenfunctions and eigenvalues are derived from the solution of the Taylor-Goldstein equation

$$L(\hat{q}_k) = 0, \qquad (3.3)$$

where

$$\left.\begin{aligned} L &= f\frac{d^2}{dx_3^2} + \frac{df}{dx_3}\frac{d}{dx_3} + k^2 r(n_0^2 - \Omega^2) \\ f &= r\Omega^2, \quad r = \bar{\rho}\epsilon_0^2, \quad \Omega = k\bar{u}_i + i\omega \\ n_0^2 &= \frac{g}{\bar{\theta}}\frac{d\bar{\theta}}{dx_3}, \quad \epsilon_0 = \exp\left[g\int\frac{dx_3}{c^2}\right] \\ \gamma &= c_p/c_v, \quad R = c_p - c_v, \quad c^2 = \gamma\bar{p}/\bar{\rho} \end{aligned}\right\}, \qquad (3.4)$$

c_v is the specific heat at constant volume and $\hat{q}_k(x_3, k, \omega)$ is the Laplace and Fourier transform of $q(x_1, x_3, t)$, defined as

$$q_k(k, x_3, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} q(x_1, x_3, t) \times \exp(ikx_1)dx_1, \qquad (3.5)$$

$$\hat{q}(x_1, x_3, \omega) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} q(x_1, x_3, t) \times \exp(-\omega t)dt. \qquad (3.6)$$

The function $q(x_1, x_3, t)$ is essentially the vertical displacement of the disturbance. Here, $\hat{q}_k$ is related to the vertical velocity w by the equation

$$\hat{q}_k = -\hat{w}_k/(i\Omega\epsilon_0), \quad w = u_3. \qquad (3.7)$$

The eigenvalues ω and k of (3.3) will be such that the associated eigenfunctions satisfy the appropriate boundary conditions at the ground and at infinity. In Fig. 1, the amplitude and phase of the eigenfunction $\hat{q}_k$ are plotted as a function of height for k_0 and $\omega_0 = \omega_{0i} + i\omega_{0r}$ given by

$$k_0 h = 0.65, \quad \omega_{0i}/(k_0 u_0) = 0.061, \quad \omega_{0r}/(k_0 u_0) = 0.0036. \qquad (3.8)$$

The disturbance has a critical level very close to the inflection point of the velocity profile and grows exponentially in time. The amplitude of the disturbance plotted in Fig. 1 is chosen so that at $t = 0$, the minimum of the generalized Richardson number dips just below Ri_c [see Eq. (2.21)].

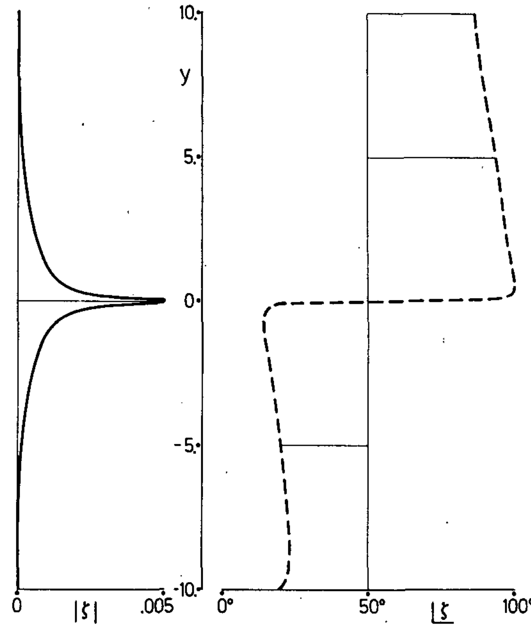

FIG. 1. Amplitude and phase of the normalized vertical displacement $\zeta = q/h$ as a function of the normalized vertical coordinate $y = x_3/h$. The normalizing length h is equal to 120 m.

b. The solution for $t > 0$

For $t > 0$, we are dealing with an initial value problem. Neglecting the nonlinear terms $\tilde{R}_{ij}$ and $\tilde{E}_i$ and neglecting terms proportional to $|\omega/kc|^2$, the Fourier and Laplace transforms of (2.4)–(2.6) produce the equation

$$L(\hat{q}_k) = \hat{F}_k + I_0, \qquad (3.9)$$

where I_0 is a function of the initial values of the field variables and $\hat{F}_k$ is given by

$$\hat{F}_k = k^2\epsilon_0\hat{Z}_k - \frac{\epsilon_0 g k^2}{i\Omega c^2}\hat{E}_k + \frac{d}{dx_3}\left[\frac{i\epsilon_0\Omega}{c^2}\left(\hat{E}_k - \frac{c^2k^2}{\Omega}\hat{X}_k\right)\right], \qquad (3.10)$$

where

$$X = -\partial\tilde{r}_{1j}/\partial x_j, \quad Z = -\partial\tilde{r}_{3j}/\partial x_j,$$
$$E = -\frac{\gamma R}{c_p}\partial\tilde{e}_j/\partial x_j. \qquad (3.11)$$

The solution of (3.9) can be found using a Green's function technique as discussed in I. We limit ourselves here to making the following points:

i) The Green's function is determined by the solutions of the homogeneous part of (3.9), i.e., of (3.3), obeying the appropriate boundary conditions. It is calculated numerically while carrying out the stability analysis of (3.3).

ii) In calculating the inverse Laplace transform, we assume that the main contribution comes from the poles of the system, i.e., that the dynamics of the system is dominated by the modes of the mean atmosphere.

iii) The eddy diffusion coefficient is obtained by solving (2.23) numerically. Since the induced turbulence is expected to have a periodic structure in the x_1-direction, K can be represented as a Fourier series involving multiples of the wavenumber k_0. It is assumed that all wavenumbers higher than k_0 can be neglected. It follows that X, Z and E will be approximated by

$$X = \bar{X} + X^{(1)}, \quad Z = \bar{Z} + Z^{(1)},$$
$$E = \bar{E} + E^{(1)}, \tag{3.12}$$

where the suffix 1 refers to only the first component of the fluctuating part of X, Z and E.

Using the above approximations, the solution can be written as in (3.1) with $F(t)$ given by

$$F(t) = -\int_0^t d\tau \frac{\exp[-\omega_0 \tau]}{f(0)\left[\dfrac{\partial \hat{q}_k}{\partial x_3} \cdot \dfrac{\partial \hat{q}_k}{\partial \omega}\right]_{k_0,\omega_0}} \int_0^\infty d\zeta \epsilon_0$$
$$\left\{k_0^2 \hat{q}_{k_0}(\zeta)\left[Z^{(1)}(\zeta, \tau) + i\frac{E^{(1)}(\zeta, \tau)}{\Omega_0}\frac{1}{\epsilon_0}\frac{d\epsilon_0}{d\zeta}\right]\right.$$
$$\left. + ik_0 \frac{\partial \hat{q}_{k_0}}{\partial \zeta}\left[X^{(1)}(\zeta, \tau) - k_0 \frac{E^{(1)}(\zeta, \tau)}{\Omega_0}\frac{\Omega_0^2}{c^2 k_0^2}\right]\right\}, \tag{3.13}$$

where

$$\Omega_0 = k_0 \bar{u}_1 + i\omega_0. \tag{3.14}$$

4. Results and discussion

The amplitude and phase of $F(t)$ are calculated numerically from (3.13) in terms of the eigenvalues k_0 and ω_0 and the eigenfunction $\hat{q}_k$ specified in the previous section. The eddy diffusion coefficient is derived by solving (2.23) numerically with $\alpha_K = \alpha_\epsilon = 1$. The constant α_0 was chosen equal to $1/\mathrm{Ri}_c$ with Ri_c defined by (2.21) and $\delta = 0.002$. The results described in this section proved to be quite insensitive to the actual choice of these constants.

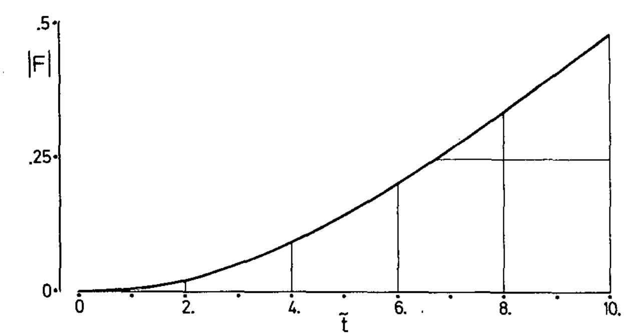

FIG. 2. The modulus of $F(\tilde{t})$ as a function of normalized time $\tilde{t} = tu_0/h$ and for wavenumber k_0 and frequency ω_0 specified by Eq. (3.8). The phase of $F(t)$ goes from $-3.7°$ for $\tilde{t} = 0$ to $-4.8°$ for $\tilde{t} = 10$.

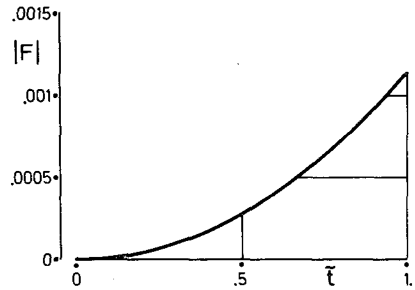

FIG. 3. As in Fig. 2, but with $K_1 = 0$. The phase of $F(t)$ is now 180°.

The amplitude of F is given in Fig. 2 as a function of the normalized variable $\tilde{t} = tu_0/h$. The rate of growth is substantial with the ratio of the logarithmic derivative of F to ω_{0i} progressively increasing, reaching the value of 1 at $\tilde{t} \cong 3.8$ and staying larger than 1 afterward. The phase of F varies from -3.7 to $-4.8°$ as $\tilde{t}$ goes from 0 to 10. Thus, the phase maintains essentially an optimal value for the feedback to be most effective. The amplitude of the initial disturbance is chosen small enough so that Ri remains positive throughout the flow domain and for all $\tilde{t}$'s investigated. While this may call for values of K smaller than we might obtain if Ri were to become negative, it also avoids the issue of the validity of the present treatment should the system involve convective overturning (Sykes and Lewellen, 1982; Davis and Peltier, 1979). In any event, it should be emphasized that because of the growth of the disturbance and the intrinsically nonlinear nature of the problem, the results presented here are valid for only small values of $\tilde{t}$. Had we not allowed for the fluctuating part of K to be included in the system, we would have obtained a negative feedback as evidenced in Fig. 3, where the modulus of $F(t)$ is plotted as a function of $\tilde{t}$ for the same parameters as for Fig. 2, but with $\tilde{K}$ set equal to zero. The phase of $F(t)$ is now 180°. Thus, the positive feedback due to $\tilde{K}$ is strong enough to overcome the negative one due to $\bar{K}$.

As in I, the shape of the regions where K is different from zero is quite elongated, as shown in Fig. 4, and may become continuous in $\tilde{x}$ as $\tilde{t}$ increases. This may lead to the thin layers of turbulence often present in the ocean and the atmosphere (Woods, 1968; Woods and Wiley, 1972; Brown and Hall, 1978).

The strength of the wave-induced turbulence varies substantially within a wavelength, as expected since the rates of strain also vary. A measure of this variation is provided in Figs. 5 and 6. The values of $\bar{K}$ and of the first harmonic component of $\tilde{K}$, i.e., K_1, are plotted

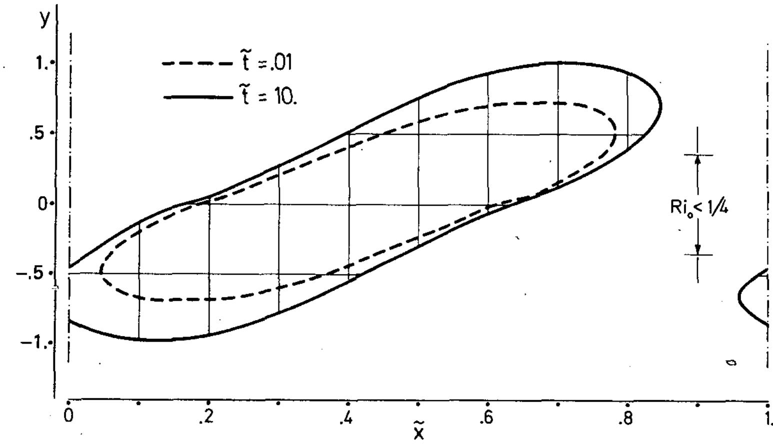

FIG. 4. Regions within which K is different from zero at $\tilde{t} = 0.01$ (dashed line) and $\tilde{t} = 10$ (solid line). The vertical coordinate is $y = x_3/h$; the horizontal coordinate is $\tilde{x} = (x_1 - t\omega_{0r}/k_0)/\lambda_0$, i.e., the horizontal distance (normalized with respect to $\lambda_0 = 2\pi/k_0$) measured in a frame of reference moving with the horizontal phase velocity of the wave ω_{0r}/k_0.

in Fig. 5, as a function of normalized height y. The values of $\bar{K}$ and $|K_1|$ peak at $y = 0$ where the background Richardson number has a minimum and the gravity wave has its critical level. The dependence of K on the y and $\tilde{x}$ coordinates, at a given time, is described in Fig. 6. The values of K appear to be antisymmetric about the $y = 0$ line. This is consistent with the nature of the background profiles and of the amplitude and phase of the disturbance, described in Fig. 1.

The numerical values obtained here are, like those in I, well within those of a stably stratified boundary layer (Brost and Wyngaard, 1978; Finnigan and Einaudi, 1981).

Careful analysis of the terms which appear in the expression for the wave-dependent Richardson number, defined by (2.25), reveals that the major contribution comes from the term $\partial\tilde{u}_1/\partial x_3 + \partial\tilde{u}_1/\partial x_3$, so that the generalized Richardson number can be approximated by the expression

$$\mathrm{Ri} = n^2\left[\frac{1}{2} S_{13}S_{13}\right]^{-1} .$$

It is always difficult to assess in general the dependence of the results on the specific model adopted. Within this model, however, the results do not appear to depend on the particular choice of any numerical constant. Furthermore, the positive feedback is present throughout the range 0.55–0.85 of the normalized horizontal wavenumber k_0h.

The results presented here, like those in I, appear to confirm that a gravity wave and the accompanying wave-induced turbulence can establish an efficient

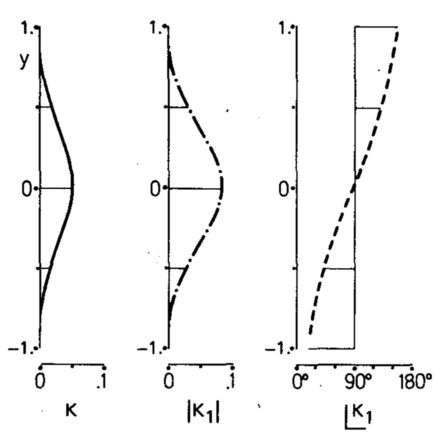

FIG. 5. Values of K and K_1 ($\mathrm{m^2 s^{-1}}$), the mean and the first harmonic of the diffusion coefficient K, as a function of $y = x_3/h$ for $\tilde{t} = 10$. The phase of K_1 is relative to the vertical displacements of the gravity wave.

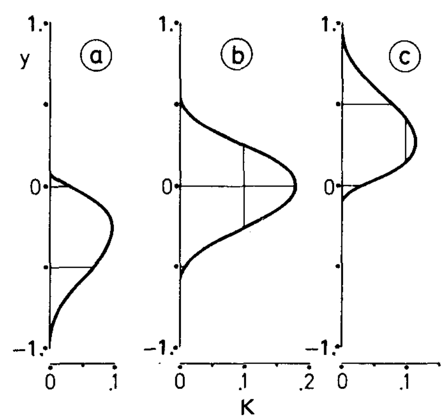

FIG. 6. Profiles of the actual diffusion coefficient ($\mathrm{m^2\ s^{-1}}$) as a function of $y = x_3/h$ for $\tilde{t} = 10$ and for three different horizontal positions: (a) $\tilde{x} = 0.2\lambda_0$, (b) $0.4\lambda_0$ and (c) $0.6\lambda_0$, with $\lambda_0 = 2\pi/k_0$. The maximum value of k occurs at $\tilde{x} = 0.4\lambda_0$.

positive feedback mechanism. Energy is transferred to the wave from the background via the turbulence. The wave grows further, induces more turbulence, and so on. The theory cannot describe the ultimate nonlinear evolution of the system, but it indicates that the initial stages of the interaction strengthen both the wave and the wave-generated turbulence. It may be that as time evolves, the system may reach some sort of equilibrium in which the roles of the mean and fluctuating part of the diffusion coefficient cancel each other. This may be one reason for the observed coexistence of waves and turbulence for several hours.

In other situations, the growth of the disturbance might lead to convective overturning with the wave-modified Richardson number falling below zero and eventual destruction of the periodic disturbance itself.

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