

Multimode Radiative Transfer in Finite Optical Media. I: Fundamentals

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ABSTRACT

In this paper we develop a new method for solving the transfer of radiation within a laterally finite optical medium. A new radiative transfer equation, based on a multimode approach, is developed which includes the explicit effects of the sides of the medium. This equation, derived for a box shaped medium, is exactly analogous to the plane parallel radiative transfer equation with a source term. Accordingly, the new equation is solved using the familiar plane-parallel techniques based on invariant imbedding relationships in the form of doubling and adding. The additional terms in the newly derived radiative transfer equation can be interpreted as apparent source and sink terms which arise from the lateral finiteness of the medium. The geometric and physical aspects of these source-sink terms and their influence on the solutions are discussed. Results also show that the multimode solutions compare well with the Monte Carlo simulations.

1. Introduction

The solution of the transfer of radiation in media other than plane parallel is now recognized as being of fundamental importance not only to radiative transfer theory but significantly to the practical problem of atmospheric radiation budgets in and around individual cloud elements or cloud fields. Most studies of radiative transfer in "finite" clouds have been based on Monte Carlo simulations (e.g., McKee and Cox, 1974; Davis *et al.*, 1979). Less accurate analytic treatments have been forwarded by Gube *et al.* (1980), Liou and Ou (1979) and Davies (1978). A significant problem with these models (Davies, 1978 apart) is the inclusion of the direct (unscattered) solar beam.

There are several elegant and fast techniques for solving the radiative transfer equation in a plane parallel medium. A large number of these have been well summarized by Van de Hulst (1980). In general most of the solution techniques appeal either explicitly or implicitly to the important principles of invariance, the interaction principle and the invariant imbedding relationships to establish their basic framework. A clear example of such is the "doubling/adding" technique (e.g., Stephens, 1980).

In this paper we develop a new method for solving the transfer of radiation within a medium which is laterally finite. The object of the paper is to develop

an alternate form of the radiative transfer equation for a finite box shaped optical medium; a form which is strongly analogous to its more familiar plane-parallel counterpart. In this way, the solutions can be based on the elegant plane-parallel techniques with their rich physical imagery. There are two underlying assumptions associated with the method. First, the spatial distribution of radiance is represented by a cosine series. Second, we assume that the optical properties (volume attenuation and scattering phase function) are functions only of depth within the medium.²

In this paper, the multimode approach is introduced, and a new radiative transfer equation is developed initially for the spatial modes and then extended to include the azimuthal modes. The end result is contained within Section 5 which presents the multimode radiative transfer equation with the explicit inclusion of the effects of the sides of the box. Azimuthally averaged solutions are presented and comparisons with Monte Carlo solutions are provided. In part II, the direct beam solution is presented. The multimode procedure is then applied to simulated box like clouds

² This is not necessarily a restrictive assumption. Since the medium chosen is boxlike it is possible to join the faces of several boxes together to develop solutions for a horizontally or generally inhomogeneous medium. This would be done by applying composition rules to contiguous boxes, each homogeneous, but of different attenuation and scattering properties. The general theory of such composition rules is developed in Preisendorfer (1965). The present paper, incidentally, provides a potential solution procedure for problem VIII of the above reference.

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with more realistic input radiances around the faces of the box and for variable solar elevation. Solution techniques are more fully discussed.

2. Geometric setting

Consider a rectangular box of dimension L (length), W (width) and H (height) as shown in Fig. 1. The box represents an optical medium $X(a, b)$ with internal surfaces X_y , produced by slicing the box parallel to the $L \times W$ dimensions.

A general point P on surface X_y is located in $X(a, b)$ by the right handed cartesian coordinates (u, v, y) . Moreover, the unit vector ξ has the alternate representation

$$\xi = (\xi_u, \xi_v, \xi_y),$$

where

$$\left. \begin{aligned} \xi_u &= \eta \cos \phi, & 0 \leq \theta \leq \pi \\ \xi_v &= \eta \sin \phi, & 0 \leq \phi \leq 2\pi \\ \xi_y &= \mu = \cos \theta - 1 < \mu < 1 \end{aligned} \right\}, \quad (1)$$

with

$$\eta = \sin \theta = (1 - \mu^2)^{1/2}$$

where θ is the zenith angle and ϕ the azimuth angle.

3. Multimode representation of the radiance function

If f is a function satisfying Dirichlet's conditions (cf. Carslaw, 1930, p. 235) then over an interval $[0, L]$ where it has these properties, the function can be represented by

$$f(x) = \sum_{l=0}^{\infty} \hat{f}(l) \cos \frac{l\pi x}{L}, \quad 0 \leq x \leq L, \quad (2)$$

where

$$\hat{f}(l) = \frac{1}{L(1 + \delta_l)} \int_0^L f(x) \cos \frac{l\pi x}{L} dx, \quad l = 0, 1, 2, \dots, \quad (3)$$

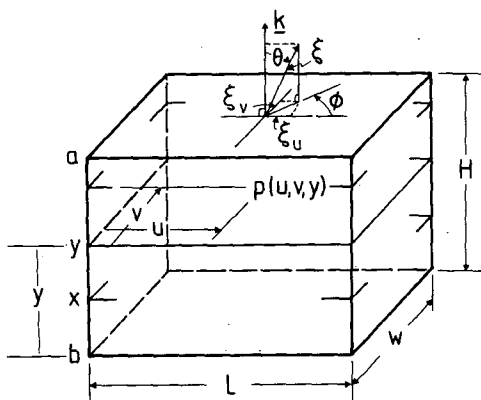


FIG. 1. Schematic view of the optical medium $X(a, b)$ with the imbedded parameter surface X_y .

and δ_l is Kronecker's delta. The series (2) converges to f at all continuity points including the ends of the interval, provided $\lim_{x \rightarrow 0} f(x)$, $\lim_{x \rightarrow L} f(x)$ exist (as x approaches the end points from within $[0, L]$). This is the great practical value of the cosine series employed here: it allows us to produce accurate representations of the radiance field right out to the sides of the box depicted in Fig. 1.

This analysis can be applied to the general (unpolarized) radiance function N ($W \cdot m^{-2} \text{ ster}^{-1}$ for fixed wavelength) defined within the medium $X(a, b)$ possessing the full five-argument structure $N(u, v, y, \mu, \phi)$. We assume that the radiance function, as a function of u and v , is described by the multimode representation

$$N(u, v, y, \xi) = \sum_{w=0}^{\infty} \sum_{l=0}^{\infty} \hat{N}(l, w, y, \xi) \cos \frac{l\pi u}{L} \cos \frac{w\pi v}{W}, \quad (4)$$

where the $\hat{N}(l, w, y, \xi)$ for the fixed integers l, w , are multimode amplitudes and are functions of y and ξ . We obtain (4) by applying (2) and (3) twice to $N(u, v, y, \xi)$. Thus following (3), the amplitudes $\hat{N}(l, w, y, \xi)$ are obtained from $N(u, v, y, \xi)$ via

$$\hat{N}(l, w, y, \xi) = \frac{4}{L_l W_w} \int_0^W \int_0^L N(u, v, y, \xi) \times \cos \frac{l\pi u}{L} \cos \frac{w\pi v}{W} dudv \quad (5)$$

with $L_l = (1 + \delta_l)L$, $W_w = (1 + \delta_w)W$. The main objective of the following analysis is to derive the equations of transfer which govern the modal amplitudes $\hat{N}(l, w, y, \xi)$ and their azimuthal associates $\hat{N}_j(l, w, y, \mu, a)$, $j = 1, 2$.

We obtain the latter as follows. The amplitude $\hat{N}(l, w, y, \xi)$ for fixed l, w, y and μ is a function of the azimuth angle ϕ . As such it can be given as a Fourier series:

$$\hat{N}(l, w, y, \mu, \phi) = \sum_{a=0}^{\infty} \hat{N}_1(l, w, y, \mu, a) \times \cos a\phi + \hat{N}_2(l, w, y, \mu, a) \sin a\phi$$

and together with (4), the radiance function becomes

$$N(u, v, y, \mu, \phi) = \sum_{a=0}^{\infty} \sum_{l=0}^{\infty} \sum_{w=0}^{\infty} [\hat{N}_1(l, w, y, \mu, a) \times \cos a\phi + \hat{N}_2(l, w, y, \mu, a) \sin a\phi] \times \cos \frac{l\pi u}{L} \cos \frac{w\pi v}{W}, \quad (6)$$

where each amplitude $\hat{N}_j(l, w, y, \mu, a)$ for fixed l, w and a numbers is a function only of depth (y) and

cosine factor (μ). Hence we need only deal with functions of two variables rather than five.

The resultant equations will be formally equivalent to the matrix generalization of the equation of transfer in a plane parallel medium. Hence the virtue of the derivation is to eliminate the horizontal location variables u and v and produce a resultant solution procedure formally equivalent to the imbedding solution procedures (i.e., including "doubling and adding") for a plane parallel medium. The price for this conceptual simplification is a larger set of simultaneous integral-differential equations governing the modal amplitudes.

4. Spatial mode form of the equation of transfer

The equation of transfer in a cartesian coordinate frame of the kind depicted in Fig. 1 is

$$\begin{aligned} \frac{dN}{dr}(u, v, y, \xi) &= \xi_u \frac{\partial N}{\partial u}(u, v, y, \xi) + \xi_v \frac{\partial N}{\partial v}(u, v, y, \xi) \\ &+ \xi_y \frac{\partial N}{\partial y}(u, v, y, \xi) \\ &= -\alpha(y)N(u, v, y, \xi) + s(y) \\ &\times \int_{\Xi} N(u, v, y, \xi')p(y, \xi', \xi)d\Omega(\xi') \\ &+ N_c(u, v, y, \xi'), \end{aligned} \quad (7)$$

where r is the distance along the unit vector ξ , Ξ the unit sphere, Ω a solid angle measure, N_c some emission function, α and s are volume attenuation and volume

scattering coefficients and p the scattering phase function. Notice that α , s and p depend only on y , which is an assumption adopted for the present³ multimode reduction of (7).

a. The general multimode procedure

The multimode procedure begins with the construction of a family of orthogonal functions over the parameter surfaces X_y . These are

$$\left(\frac{4}{L_l W_w} \cos \frac{l\pi u}{L} \cos \frac{w\pi v}{W}, l, w = 0, 1, 2 \dots \right). \quad (8)$$

We use these to analyze a function of (u, v) on X_y as in (5) and then use the family

$$\left(\cos \frac{l\pi u}{L} \cos \frac{w\pi v}{W}, l, w = 0, 1, \dots \right) \quad (9)$$

to synthesize a function of (u, v) on X_y as in (4). Together, these two families have the orthonormality property

$$\begin{aligned} \frac{4}{L_l W_l} \int_0^W \int_0^L \left(\cos \frac{l\pi u}{L} \cos \frac{w\pi v}{W} \right) \\ \times \left(\cos \frac{l'\pi u}{L} \cos \frac{w'\pi v}{W} \right) dudv = \delta_{l-l'} \delta_{w-w'}, \end{aligned} \quad (10)$$

for all integers $l, l', w, w' = 0, 1, 2 \dots$.

The multimode procedure requires multiplication of each side of (7) by a general function of (u, v) given in (8), and then integrating the resultant product terms over X_y . When this is carried out, term by term, the resultant multimode equation of transfer becomes

$$\begin{aligned} \xi_y \frac{\partial \hat{N}}{\partial y}(l, w, y, \xi) &= -\alpha(y)\hat{N}(l, w, y, \xi) + s(y) \int_{\Xi} \hat{N}(l, w, y, \xi')p(y, \xi', \xi)d\Omega(\xi') + \frac{4\xi_u}{L_l W_w} \\ &\times \int_0^W [N(0, v, y, \xi) - (-1)^l N(L, v, y, \xi)] \cos \frac{w\pi v}{W} dv + \frac{4\xi_v}{L_l W_w} \int_0^L [N(u, 0, y, \xi) - (-1)^w N(u, W, y, \xi)] \\ &\times \cos \frac{l\pi u}{L} du + \frac{4\xi_u}{L_l W_w} \int_0^W \left[\frac{l\pi}{L} \int_0^L N(u, v, y, \xi) \sin \frac{l\pi u}{L} du \right] \cos \frac{w\pi v}{W} dv + \frac{4\xi_v}{L_l W_w} \\ &\times \int_0^L \left[\frac{w\pi}{W} \int_0^W N(u, v, y, \xi) \sin \frac{w\pi v}{W} dv \right] \cos \frac{l\pi u}{L} du + \hat{N}_c(l, w, y, \xi) \end{aligned} \quad (11)$$

for $l, w = 0, 1, 2 \dots$, $a \leq y \leq b$, and ξ in Ξ . This is the required multimode form of the equation of transfer on a box (as shown in Fig. 1). The additional terms (third to sixth terms) in (11) result from the double integral of $\partial N/\partial u$ and $\partial N/\partial v$ over the surface X_y at the level y . The third and fourth terms describe the radiances over the sides of the box while the remaining two double integral terms are boundary divergence terms which characterize the net output of radiance over the boundaries.

The general form of the reduced equation of transfer is that of the equation of transfer for a plane parallel

medium. Just as $-\alpha N$ is a sink term (describing the disappearance of radiance), and the first integral term and N_c are sources of radiance, so too can the middle four integral terms in (11) involving N at the edge points of X_y , be interpreted as source and sink terms

³ It is possible, in principle, to carry out the multimode spatial decomposition of (17) when α , s and p depend on u, v . This would be achieved using an appropriate tensor formulation. However, the results would be numerically unwieldy and can be overcome in the manner discussed in the previous footnote.

of special types which we will distinguish by the terms *laterally incident*, *laterally emergent* and by the *boundary divergence*. Each of these terms are now analysed in more detail.

b. The lateral radiance fields

Each of the four boundary radiance distribution terms in (11) can be of the incident or emergent type depending on the direction of ξ . Fig. 2 depicts a schematic plan view of a general parameter surface X_y for the four different types of orientation of ξ .

Thus for ξ in the first quadrant, i.e., $\xi_u > 0, \xi_v > 0, N(0, v, y, \xi)$ is an incident radiance on the west side of the box while for ξ in the second quadrant (i.e., $\xi_u < 0, \xi_v > 0$), $N(0, v, y, \xi)$ is an emergent radiance from the west side of the box. Observe that the product $\xi_u N(0, v, y, \xi)$ is positive in the first quadrant (indicating a source of flux) and negative in the second quadrant (indicating a sink of flux). The term $-\xi_u (-1)^l N(L, v, y, \xi)$ also can be interpreted, but in a less obvious way using (4), as a source or sink term for ξ toward or away from the body of the medium. Hence the lateral radiance field can be split into two distinct parts: the incident field and the emergent field at any point of the sides of the box and for any orientation of ξ .

The incident lateral radiance field can be specified arbitrarily and, as such, constitutes a natural boundary condition much in the way incident radiance fields are specified at the upper or lower boundaries of a plane parallel medium. Thus, in addition to the usual incident radiance distributions on X_a and X_b in $X(a, b)$ we also must specify the incident radiance distributions on the sides of the box.

On the other hand, the emergent lateral radiance field cannot be specified arbitrarily but forms an

integral part of the entire radiance distribution in $X(a, b)$. Using (4), the lateral radiance field can be represented in modal form. For example, on the east and west faces

$$\left. \begin{aligned} N(0, v, y, \xi) &= \sum_{w=0}^{\infty} \left[\sum_{l=0}^{\infty} \hat{N}(l, w, y, \xi) \right] \cos \frac{w\pi v}{W} \\ N(L, v, y, \xi) &= \sum_{w=0}^{\infty} \left[\sum_{l=0}^{\infty} (-1)^l \hat{N}(l, w, y, \xi) \right] \cos \frac{w\pi v}{W} \end{aligned} \right\} \quad (12)$$

Thus, the boundary conditions on the lateral sides of the box must be treated separately for the incident and emergent radiances.

The lateral radiance terms in (12) are essentially one index modal amplitudes of the w and l type. Thus if we write

$$\left. \begin{aligned} N_{.w}(u, y, \xi) &\text{ for } \frac{2}{W_w} \int_0^W N(u, v, y, \xi) \\ &\times \cos \frac{w\pi v}{W} dv, \quad w = 0, 1, 2, \dots \\ N_{.l}(v, y, \xi) &\text{ for } \frac{2}{L_l} \int_0^L N(u, v, y, \xi) \\ &\times \cos \frac{l\pi u}{L} du, \quad l = 0, 1, 2, \dots \end{aligned} \right\} \quad (13)$$

then

$$\begin{aligned} \frac{4\xi_u}{L_l W_w} \int_0^W [N(0, v, y, \xi) - (-1)^l N(L, v, y, \xi)] \\ \times \cos \frac{w\pi v}{W} dv = \frac{2\xi_u}{L_l} N_{.w}(0, y, \xi) \\ - \frac{2\xi_u}{L_l} (-1)^l N_{.w}(L, y, \xi), \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{4\xi_v}{L_l W_w} \int_0^W [N(u, 0, y, \xi) - (-1)^w N(u, W, y, \xi)] \\ \times \cos \frac{l\pi u}{L} du = \frac{2\xi_v}{W_w} N_{.l}(0, y, \xi) \\ - \frac{2\xi_v}{W_w} N_{.l}(W, y, \xi). \end{aligned} \quad (15)$$

1) THE INCIDENT LATERAL RADIANCE AMPLITUDES

The incident amplitudes may be defined as functions of azimuth angle ϕ for each fixed set of variables l, w, y and μ . To facilitate these definitions, the quadrant characteristic function is introduced for $j = 1, 2, 3$ and 4 as

$$\chi_j(\phi) = \begin{cases} 1 & \text{if } (j-1)\frac{\pi}{2} \leq \phi < \frac{j\pi}{2} \\ 0 & \text{if } \phi \text{ lies elsewhere in } [0, 2\pi]. \end{cases} \quad (16)$$

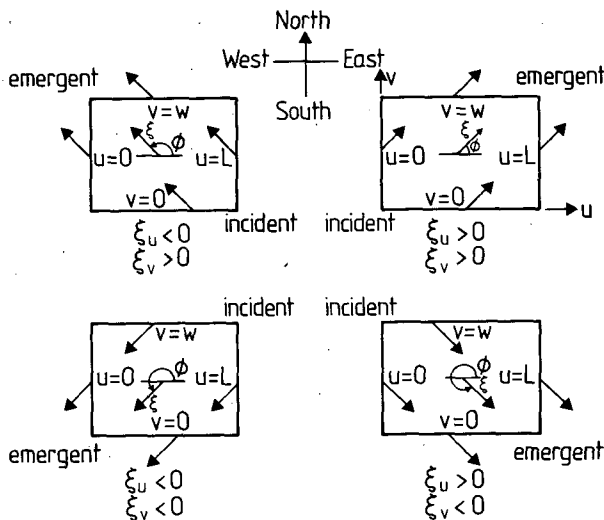


FIG. 2. Plan view of the parameter surface X_y with projections of directional vector ξ on X_y .

By employing this function and introducing for brevity

$$\left. \begin{aligned} \text{"}N_l\text{"} & \text{ for } N_l(0, y, \xi) \\ \text{"}N_l^W\text{"} & \text{ for } N_l(W, y, \xi) \\ \text{"}N_w\text{"} & \text{ for } N_w(0, y, \xi) \\ \text{"}N_w^L\text{"} & \text{ for } N_w(L, y, \xi) \end{aligned} \right\}$$

$$N_{.w}(0, y, \xi) = \sum_{l=0}^{\infty} \hat{N}(l, w, y, \xi) \quad (19)$$

$$= \sum_{w'}^{\infty} \sum_{l'}^{\infty} \hat{N}(l', w', y, \xi) \delta_{w-w'},$$

$$N_{.w}(L, y, \xi) = \sum_{l'=0}^{\infty} (-1)^{l'} \hat{N}(l, w, y, \xi), \quad (20)$$

and combining the magnitudes of the incident amplitudes as read off from Fig. 2, and using (14) and (15), then the incident lateral radiance amplitudes reduce to

$$\hat{N}_{si}(l, w, y, \mu, \phi) = \eta \left\{ N_l \frac{2}{W_w} \sin\phi [\chi_1(\phi) + \chi_2(\phi)] \right. \\ - N_l^W \frac{2}{W_w} (-1)^w \sin\phi [\chi_3(\phi) + \chi_4(\phi)] \\ + N_w \frac{2}{L_l} \cos\phi [\chi_1(\phi) + \chi_4(\phi)] \\ \left. - N_w^L \frac{2}{L_l} (-1)^l \cos\phi [\chi_2(\phi) + \chi_3(\phi)] \right\} \quad (17)$$

for $l, w = 0, 1, 2, \dots, a < y < b, \eta = \sin\theta, 0 \leq \theta \leq \pi$ and $0 < \eta < 2\pi$. In practice, the incident radiance distributions (i.e., N_l, N_l^W, N_w, N_w^L) are specified over an incident hemisphere at each point of the sides of the box. More specifically, $N(u, 0, y, \xi)$ is specified for all u, y over the south face of the box and all ξ such that $\xi_v > 0$ (the incident hemisphere), and we set $N(u, 0, y, \xi) = 0$ for all $\xi_v < 0$ (the emergent hemisphere) on the south face of the box.

2) THE EMERGENT LATERAL RADIANCE AMPLITUDES

The general form of the lateral radiance amplitude appears in (14) and (15). For the values of u and v required in (14) and (15), we first find for general u

$$N_{.w}(u, y, \xi) = \frac{2}{W_w} \\ \times \int_0^W \sum_{w'=0}^{\infty} \left(\sum_{l'=0}^{\infty} \hat{N}(l', w', y, \xi) \cos \frac{l'\pi u}{L} \cos \frac{w'\pi v}{W} \right) \\ \times \cos \frac{w\pi v}{W} dv,$$

which after integration and observing the orthonormal property (10) becomes

$$N_{.w}(u, y, \xi) = \sum_{l'=0}^{\infty} \hat{N}(l', w', y, \xi) \cos \frac{l'\pi u}{L}. \quad (18)$$

For $u = 0$ and $u = L$ (the east and west sides of the box), the emergent lateral radiance amplitudes are

and similarly for the other emergent lateral radiance amplitudes over the south and north faces

$$\left. \begin{aligned} N_l(0, y, \xi) &= \sum_{w=0}^{\infty} \hat{N}(l, w, y, \xi) \\ N_l(W, y, \xi) &= \sum_{w=0}^{\infty} (-1)^w \hat{N}(l, w, y, \xi) \end{aligned} \right\}. \quad (21)$$

Hence the emergent radiance amplitudes over all four faces of $X(a, b)$ are representable as linear combinations of the 2 index modal amplitudes $\hat{N}(l, w, y, \xi)$. This fact makes the system (7) solvable as a set of familiar coupled integro-differential equations in the unknowns $\hat{N}(l, w, y, \xi)$.

Similar to (17), by employing Fig. 2 and equations (14) and (15) describing emergent radiances and with some manipulation, the emergent lateral radiance amplitudes can be written

$$\hat{N}_{se}(l, w, y, \mu, \phi) = \eta \left\{ N_l \frac{2}{W_w} \sin\phi [\chi_3(\phi) + \chi_4(\phi)] \right. \\ - N_l^W \frac{2}{W_w} (-1)^w \sin\phi [\chi_1(\phi) + \chi_2(\phi)] \\ + N_w \frac{2}{L_l} \cos\phi [\chi_2(\phi) + \chi_3(\phi)] \\ \left. - N_w^L \frac{2}{L_l} (-1)^l \cos\phi [\chi_1(\phi) + \chi_4(\phi)] \right\} \quad (22)$$

for $l, w = 0, 1, 2, \dots, a < y < b, \eta = \sin\theta, 0 \leq \theta \leq \pi, 0 < \phi < 2\pi$. Notice that there is rotational symmetry between the incident amplitudes (17) and the emergent amplitudes (22) in the sense that we have the correspondences $\chi_1 \leftrightarrow \chi_3$ and $\chi_2 \leftrightarrow \chi_4$. These correspondences are clear from Fig. 2. For example, the incident radiances on the west and south faces in quadrant 1 become the emergent radiances over these faces in quadrant 3. There is, however, a fundamental distinction between the two lateral radiances which was emphasized above. This distinction can be made more explicit by using in (22) the double-sum version of (19) to (21); and by rearranging we obtain

$$\hat{N}_{se}(l, w, y, \mu, \phi) = -\eta \sum_{w'=0}^{\infty} \sum_{l'=0}^{\infty} \hat{N}(l', w', y, \mu, \phi) \\ \times S(l', w', l, w, \phi), \quad (23)$$

where we have introduced the *emergent lateral tensor*

$$S(l', w', l, w, \phi) = -\frac{2}{L_l} \cos\phi [\chi_2(\phi) + \chi_3(\phi) - (-1)^{l+l'}] \\ \times (\chi_1(\phi) + \chi_4(\phi)) \delta_{w-w'} - \frac{2}{W_w} \sin\phi [\chi_3(\phi) \\ + \chi_4(\phi) - (-1)^{w+w'} (\chi_1(\phi) + \chi_2(\phi))] \delta_{l-l'} \quad (24)$$

of the fourth order. It is clear by inspection that for each ϕ each component $S(l', w', l, w, \phi)$ of the tensor with $l + l'$ or $w + w'$ even is non negative. Hence for such indices, \hat{N}_{se} given by (24) is non positive for $\eta > 0$ and represents a loss term analogous to the more familiar loss term $-\alpha(y)\hat{N}(l, w, y, \mu, \phi)$. Since S does not involve radiance distributions, the tensor $S(l', w', l, w, \phi)$ is an inherent property—a geometric property belonging to the medium $X(a, b)$ and independent of the light field. As such, it belongs to all box like optical media.

c. *Boundary divergence term*

As with the emergent lateral radiance term, the boundary divergence term forms an integral part of the solution of the entire radiance field and cannot be specified arbitrarily. If we represent the radiance function in its general multimode form (4), then the first of the boundary divergence terms in (11) becomes

$$A_0 = \frac{4\xi_u}{L_l W_w} \int_0^W \left\{ \frac{l\pi}{L} \int_0^L N(u, v, y, \xi) \sin \frac{l\pi u}{L} du \right\} \\ \times \cos \frac{w\pi v}{W} dv \\ = \frac{4\xi_u}{L_l W_w} \int_0^W \left\{ \frac{l\pi}{L} \sum_{l'=0}^{\infty} \sum_{w'=0}^{\infty} \hat{N}(l', w', y, \xi) \cos \frac{w'\pi v}{W} \right. \\ \left. \times \left[\int_0^L \cos \frac{l'\pi u}{L} \sin \frac{l\pi u}{L} du \right] \right\} \cos \frac{w\pi v}{W} \quad (25)$$

It is a relatively straightforward task to carry out the remaining integrals in (25) to produce

$$A_0 = \sum_{l'=0}^{\infty} \sum_{w'=0}^{\infty} \hat{N}(l', w', y, \xi) \\ \times \frac{2\xi_u}{L_l} A(l', l) (1 - \delta_{l,l'}) \delta_{w-w'}, \quad (26)$$

where

$$A(l', l) = \frac{l}{2} \left[\frac{1}{l+l'} \{1 - (-1)^{l+l'}\} \right. \\ \left. + \frac{1}{l-l'} \{1 - (-1)^{l-l'}\} \right] \quad (27)$$

By symmetry, the second of the boundary divergence terms in (11) is

$$B_0 = \sum_{l'=0}^{\infty} \sum_{w'=0}^{\infty} \hat{N}(l', w', y, \xi) \\ \times \left[\frac{2\xi_v}{W_w} A(w', w) [1 - \delta_{w-w'}] \delta_{l-l'} \right] \quad (28)$$

The combined terms are now written as a single amplitude

$$\hat{N}_B(l, w, y, \mu, \phi) \\ = \eta \sum_{l'=0}^{\infty} \sum_{w'=0}^{\infty} \hat{N}(l', w', y, \mu, \phi) B(l', w', l, w, \phi), \quad (29)$$

where we introduce the *boundary divergence tensor*

$$B(l', w', l, w, \phi) = \frac{2 \cos\phi}{L_l} A(l', l) (1 - \delta_{l,l'}) \delta_{w-w'} \\ + \frac{2 \sin\phi}{W_w} A(w', w) (1 - \delta_{w-w'}) \delta_{l-l'}. \quad (30)$$

Whereas $S(l', w', l, w, \phi)$ characterizes the radiance output over the boundary of the box, $B(l', w', l, w, \phi)$ describes the net radiance output through the boundaries of the box. These two terms are kept conceptually separate for the time being but they can be simply grouped together (as will become evident below) to make up the total loss term.

d. *Equation of transfer for spatial modes*

The developments in the preceding paragraphs concentrated on reworking the lateral radiance terms and the boundary divergence terms in (11) into the more succinct form presented in (17) and (22) and (29). Thus the equation transfer for the general spatial mode $\hat{N}(l, w, y, \xi)$ becomes

$$\mu \frac{\partial \hat{N}}{\partial y}(l, w, y, \mu, \phi) = -\alpha(y) \hat{N}(l, w, y, \mu, \phi) \\ + \hat{N}_B(l, w, y, \mu, \phi) + s(y) \int_{\Xi} \hat{N}(l, w, y, \xi') p(y, \xi', \xi) \\ \times d\Omega(\xi') + \hat{N}_{se}(l, w, y, \mu, \phi) + \hat{N}_{si}(l, w, y, \mu, \phi) \\ + \hat{N}_i(l, w, y, \mu, \phi). \quad (31)$$

This equation can be written in matrix form to emphasize its formal similarity to the plane parallel equation of transfer by writing

“ $\hat{N}(y, \xi)$ ” or “ $\hat{N}(y, \mu, \phi)$ ”

for

$$\begin{bmatrix} \hat{N}(1, 1, y, \mu, \phi) & \hat{N}(1, 2, y, \mu, \phi) & \cdots \\ \hat{N}(2, 1, y, \mu, \phi) & \hat{N}(2, 2, y, \mu, \phi) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

and similarly for $N_{si}(y, \xi)$, $N_{se}(y, \xi)$ and $N_e(y, \xi)$. Moreover we write “ $S(\xi)$ ” for $\{S(l', w', l, w, \xi): l', w', l, w = 0, 1, 2 \dots\}$ and “ $B(\xi)$ ” for $B(l', w', l, w, \xi)$ and observe that by (23) $N_{se} = -N(y, \xi)\eta S(\xi)$ and by (29) $N_B = \eta N(y, \xi)B(\xi)$. Then (31) becomes

$$\begin{aligned} \mu \frac{\partial N}{\partial y}(y, \xi) &= -N(y, \xi)[\alpha(y)I + \eta S(\xi) - \eta B(\xi)] \\ &+ s(y) \int_{\Xi} N(y, \xi') p(y, \xi', \xi) d\Omega(\xi') \\ &+ N_{si}(y, \xi) + N_e(y, \xi) \end{aligned} \quad (32)$$

for $-1 < \mu, 0 \leq \eta \leq 1, \xi \in \Xi$, and $a < y < b$. The effect of the lateral finiteness of the optical medium $X(a, b)$ is now readily apparent with the presence of a new loss term $-N(y, \xi)\eta[S(\xi) - B(\xi)]$ and the presence of a new source term $N_{si}(y, \xi)$. These terms explicitly relate to the loss of energy from the sides of the box and the gain of energy from the incident radiation on the sides of the box. Here I is an identity tensor of order 4 where we have written

$$“I” \text{ for } \{\delta_{l-l'}\delta_{w-w'}; l', w', l, w = 0, 1, 2, \dots\}.$$

Both the $S(\xi)$ and $B(\xi)$ tensors and the source vector $N_{si}(y, \xi)$ tend to zero as the lateral dimensions L and W of $X(a, b)$ approach infinity. Thus, subject to the lateral homogeneity and stratification assumptions stated at the outset, the equation of transfer for the finite medium $X(a, b)$ reverts to the classical for the plane-parallel setting, as the box becomes a plane-parallel medium.

5. The azimuthal mode form of the equation of transfer

The equation of transfer presented in (31) or (32) for the amplitudes $\hat{N}(l, w, y, \mu, \phi)$ is now reduced one step further. By using the Fourier decomposition summarized in (6) we obtain a set of coupled equations in the unknown amplitudes $\hat{N}_j(l, w, y, \mu, a)$ where l, w and a are respectively length, width and azimuthal integer indexes ranging from 0 to ∞ for $j = 1, 2$. Thus for a fixed l, w, a we work with a function of one depth variable y and one direction variable μ as in the case for plane parallel solutions.

The procedure to obtain the equation for $\hat{N}_j(l, w, y, \mu, a)$ is, in principle, straightforward and thus we only outline the derivation. We multiply each side of (25) by $[\cos(a\phi)]/(1 + \delta_a)\pi$ and integrate over the interval $[0, 2\pi]$ for $a = 0, 1, 2 \dots$ and for $j = 1$. For $\hat{N}_2(l, w, y, \mu, a)$, each term of (31) is multiplied by $[\sin(a\phi)]/\pi$ and integrated over $[0, 2\pi]$. Here we utilize the fact that for a periodic function f of period 2π represented as

$$f(\phi) = \sum_{a=0}^{\infty} [f_1(a) \cos a\phi + f_2(a) \sin a\phi], \quad (33)$$

we have

$$\left. \begin{aligned} f_1(a) &= \frac{1}{(1 + \delta_a)\pi} \\ &\times \int_0^{2\pi} f(\phi) \cos a\phi d\phi \\ f_2(a) &= \frac{1}{\pi} \int_0^{2\pi} f(\phi) \sin a\phi d\phi \end{aligned} \right\}, \quad a = 0, 1, 2, \dots \quad (34)$$

It is straightforward to carry out these operations on each term in (31). The manipulations involved however are lengthy especially when the various side radiance terms (17), (22) and (29) are integrated over ϕ . Only the final derivation is presented below and in more detail in Appendix B.

a. Lateral radiance amplitudes

$$\begin{aligned} \hat{N}_{se,k}(l, w, y, \eta, a) &= \frac{-\eta}{(1 + \delta_a\delta_{k-1})\pi} \sum_{a'=0}^{\infty} \sum_{w'=0}^{\infty} \sum_{l'=0}^{\infty} \sum_{j=1}^2 [\sum \hat{N}_j(l', w', y, \mu, a')] \\ &\times S_{jk}(l', w', a', l, w, a) \end{aligned} \quad (35)$$

for the two azimuthal modes $k = 1, 2$. In this case the emergent lateral tensor

$$\begin{aligned} S_{jk}(l', w', a', l, w, a) &= [(-1)^{a'+a} + (-1)^{w'+w}] \frac{F_{L,jk}}{W_w}(a', a)\delta_{l-l'} \\ &+ [(-1)^{a'+a} + (-1)^{l'+l}] \frac{F_{W,jk}}{L_l}(a', a)\delta_{w-w'}, \end{aligned} \quad (36)$$

where the functions $F_{L,jk}$ and $F_{W,jk}$ are defined in Appendices B and C.

b. Boundary divergence amplitudes

$$\begin{aligned} \hat{N}_{B,k}(l, w, y, \mu, a) &= -\eta \sum_{a'=0}^{\infty} \sum_{w'=0}^{\infty} \sum_{l'=0}^{\infty} \sum_{j=1}^2 [\sum \hat{N}_j(l', w', y, \mu, a')] \\ &\times B_{jk}(l', w', a', l, w, a) \end{aligned} \quad (37)$$

for the two azimuthal modes $k = 1, 2$. The boundary divergent tensor is defined as

$$\left. \begin{aligned}
 B_{11}(l', w', a', l, w, a) &= -\frac{1}{L_l} A(l', l)(1 - \delta_{l-l'})\delta_{w-w'}\gamma_{11}(a', a) \\
 B_{21}(l', w', a', l, w, a) &= \frac{1}{W_w} A(w, w')(1 - \delta_{w-w'})\delta_{l-l'}\gamma_{21}(a', a) \\
 B_{12}(l', w', a', l, w, a) &= -\frac{1}{W_w} A(w, w')(1 - \delta_{w-w'})\delta_{l-l'}\gamma_{12}(a', a) \\
 B_{22}(l', w', a', l, w, a) &= -\frac{1}{L_l} A(l', l)(1 - \delta_{l-l'})\delta_{w-w'}\gamma_{22}(a', a)
 \end{aligned} \right\} \tag{38}$$

where $\gamma_{ij}(a', a)$ is defined in Appendix B.

c. The incident lateral radiance amplitudes

$$\begin{aligned}
 \hat{N}_{sl,k}(l, w, y, \mu, a) &= \frac{\eta}{(1 + \delta_a \delta_{k-1})\pi} \sum_{a'=0}^{\infty} \left\{ [N_{l,1} + N_{l,1}^W(-1)^{w+a'+a}] \frac{F_{L,1k}}{W_w} \right. \\
 &+ [N_{w,1} + N_{w,1}^L(-1)^{w+a'+a}] \frac{F_{W,1k}}{L_l} \\
 &+ [N_{l,2} + N_{l,2}^W(-1)^{w+a'+a}] \frac{F_{L,2k}}{W_w} \\
 &\left. + [N_{w,2} + N_{w,2}^L(-1)^{w+a'+a}] \frac{F_{W,2k}}{L_l} \right\} \tag{39}
 \end{aligned}$$

for $k = 1, 2$ and $l, w, a = 0, 1, 2, \dots, a \leq y \leq b$ and $-1 < \mu < 1, 0 \leq \eta \leq 1$. The incident amplitudes $N_{l,j}, N_{l,j}^W$ have arbitrary structure and no *a priori* connections with the amplitudes $\hat{N}(l, w, y, \mu, a)$. These incident amplitudes are formally

$$\left. \begin{aligned}
 N_{l,1} &= \frac{1}{(1 + \delta_a)\pi} \int_0^\pi N_l(0, y, \mu, \phi) \cos a\phi d\phi \\
 N_{l,1}^W &= \frac{1}{(1 + \delta_a)\pi} \int_\pi^{2\pi} N_l(W, y, \mu, \phi) \cos a\phi d\phi \\
 N_{w,1} &= \frac{1}{(1 + \delta_a)\pi} \int_{-\pi/2}^{\pi/2} N_w(0, y, \mu, \phi) \cos a\phi d\phi \\
 N_{w,1}^L &= \frac{1}{(1 + \delta_a)\pi} \int_{\pi/2}^{3\pi/2} N_w(L, y, \mu, \phi) \cos a\phi d\phi
 \end{aligned} \right\} \tag{40}$$

and

$$\left. \begin{aligned}
 N_{l,2} &= \frac{1}{\pi} \int_0^\pi N_l(0, y, \mu, \phi) \sin a\phi d\phi \\
 N_{l,2}^W &= \frac{1}{\pi} \int_\pi^{2\pi} N_l(W, y, \mu, \phi) \sin a\phi d\phi \\
 N_{w,2} &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} N_w(0, y, \mu, \phi) \sin a\phi d\phi \\
 N_{w,2}^L &= \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} N_w(L, y, \mu, \phi) \sin a\phi d\phi
 \end{aligned} \right\} \tag{41}$$

The ranges of ϕ integration for the incident radiance are shown in Fig. 3. The incident radiances are zero outside the indicated π sweeps of ϕ . Over the π sweeps, the incident radiances are specified by the investigator (e.g., see Fig. 2).

We now proceed with the main goal of this section. By collecting the various amplitudes a single equation of transfer can be obtained. This is carried out mainly to show the connection with the plane parallel medium formalism. This is done in two stages. First we fix l, w, a and collect together the cosine and sine azimuthal amplitudes. Thus the amplitude pair $[\hat{N}_1(l, w, y, \mu, a), \hat{N}_2(l, w, y, \mu, a)]$ is considered a single entity. Then (32) can be written

$$\begin{aligned}
 \mu \frac{\partial}{\partial y} (N_1, N_2) &= -\alpha(y)(N_1, N_2) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \pi S(y) \int_{-1}^{-1} (N_1, N_2) \begin{vmatrix} 1 + \delta_a & 0 \\ 0 & 1 \end{vmatrix} p(y, \mu', \mu, a) d\mu' \\
 &- \frac{\eta}{\pi} \sum_{a'=0}^{\infty} \sum_{w'=0}^{\infty} \sum_{l'=0}^{\infty} (N'_1, N'_2) \begin{vmatrix} \frac{S_{11}(l', w', a', l, w, a)}{1 + \delta_a} & S_{12}(l', w', a', l, w, a) \\ \frac{S_{21}(l', w', a', l, w, a)}{1 + \delta_a} & S_{22}(l', w', a', l, w, a) \end{vmatrix} \\
 &- \frac{\eta}{\pi} \sum_{a'=0}^{\infty} \sum_{w'=0}^{\infty} \sum_{l'=0}^{\infty} (N'_1, N'_2) \begin{vmatrix} B_{11}(l', w', a', l, w, a) & B_{12}(l', w', a', l, w, a) \\ B_{21}(l', w', a', l, w, a) & B_{22}(l', w', a', l, w, a) \end{vmatrix} + (N_{s1,1}, N_{s1,2}) + (N_{e,1}N_{e,2}), \tag{42}
 \end{aligned}$$

where we write for brevity “ N_k ” for $\hat{N}_k(l, w, y, \mu, a)$ and “ N'_k ” for $\hat{N}_k(l', w', y, \mu, a')$. Further simplifications are possible if we write

$$\begin{aligned}
 S_{jk} & \text{ for } \left\{ \frac{S_{jk}(l', w', a', l, w, a)}{(1 + \delta_a \delta_{k-1})}; l', w', a', l, w, a = 0, 1, \dots \right\}, \\
 B_{jk} & \text{ for } \{B_{jk}(l', w', a', l, w, a); l', w', a', l, w, a = 0, 1, \dots\}, \\
 \text{“}I\text{”} & \text{ for } \{\delta_{l-l'} \delta_{w-w'} \delta_{a-a'}; l', w', a', l, w, a = 0, 1, 2 \dots\}, \\
 \text{“}P_{jk}\text{”} & \text{ for } \{1 + \delta_a \delta_{k-1}\} \delta_{j-k} \delta_{l-l'} \delta_{w-w'} \delta_{a-a'} p(\mu', \mu, a'); l', w', a', l, w, a = 0, 1, 2 \dots\}.
 \end{aligned}$$

Then (42) can be written

$$\begin{aligned}
 \mu \frac{\partial}{\partial y} (N_1, N_2) = & -\alpha(y)(N_1, N_2) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \pi s(y) \int_{-1}^{+1} (N'_1, N'_2) \begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} d\mu' - \frac{\eta}{\pi} (N_1, N_2) \begin{vmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{vmatrix} \\
 & - \frac{\eta}{\pi} (N_1, N_2) \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} + (N_{s1,1}, N_{s1,2}) + (N_{\epsilon,1}, N_{\epsilon,2}). \quad (43)
 \end{aligned}$$

By setting $\mathbb{N} = (N_1, N_2)$ and making other obvious notational definitions, the radiative transfer equation for spatial and azimuthal mode matrix form becomes

$$\begin{aligned}
 \mu \frac{\partial \mathbb{N}}{\partial y} = & -\mathbb{N} \left(\alpha \mathbb{I} + \frac{\eta}{\pi} \mathbb{S} + \frac{\eta}{\pi} \mathbb{B} \right) \\
 & + \pi s \int_{-1}^{+1} \mathbb{N} P d\mu' + \mathbb{N}_{S1} + \mathbb{N}_{\epsilon}. \quad (44)
 \end{aligned}$$

The π factors in (44) on comparison to (32) are reminders of the azimuthal parameterization; \mathbb{N} is now a function only of depth and zenith angle. The derivation of this form of the equation of transfer represents the attainment of the goal stated at the outset of this paper, namely to reduce the radiative transfer equation for a box shaped medium $X(a, b)$ to plane parallel form.

6. Solutions

The solution of (44) follows closely the classical solution techniques adopted for the plane-parallel transfer

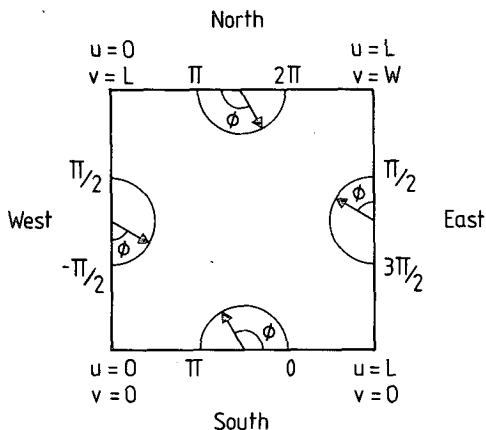


FIG. 3. The ranges of the ϕ integration for incident radiances on X_y .

problem which derive a system of coupled differential equations with the integral term replaced by a suitable quadrature formula. For the finite transfer problem, one again establishes a system of coupled differential equations. This coupling arises not only through the discretization of the angular integral but also through interactions of the various geometric modes. This geometric coupling is indicated by the presence of the \mathbb{S} and \mathbb{B} terms in (44).

The techniques for solving the plane-parallel form of the radiative transfer equation have been described at length elsewhere, and only a broad outline is discussed here. The starting point is to recast the radiative transfer equation into its local interaction form involving the introduction of local reflection, transmission and source terms. This is discussed in some detail below since the development of the present local interaction principle is slightly different from its plane parallel counterpart due to the presence of the \mathbb{S} and \mathbb{B} tensors, and the additional coupling that results with the introduction of geometric modes. Once the local interaction form of the radiative transfer equation is derived, the response radiances at and within the boundaries of the medium can be determined by applying the global interaction principles. Thus the radiance solutions for an optically thick medium can then be determined by applying suitable “composition” rules (in particular that of doubling and adding), as originally described in Preisdorfer (1965).

The local interaction principle

The radiance “vector” \mathbb{N} in (44) is a function of 3 modes, one of length, one of width and one of azimuth, as well as being a function of zenith angle and depth in the medium. By bringing the modal dependence of the various terms in (44) to view, we can write

$$\text{“}N(j, \mu, y)\text{”}$$

for

$$\text{“}N\text{” for } [N_1(l, w, a, \mu, y), N_2(l, w, a, \mu, y)]$$

where the l , w , and a modes have been suitably condensed to a single mode j . Thus for each zenith angle μ , and if there are totally m such l , w , and a modes, then $N(j, \mu)$ is a vector of $2m$ elements. We can similarly write

“ $S(j', j)$ ” for $S(l', w', a', l, w, a)$,

“ $I(j', j)$ ” for $I(l', w', a', l, w, a)$,

“ $P(y, j', j, \mu, \mu')$ ” for $P(y, l', w', a', l, w, a, \mu', \mu)$

and similarly for the other terms in (43).

1) INTRODUCTION OF QUADRATURE

Quadrature is introduced to replace the integral term in (44) with a summation term. Thus (44) can be explicitly written as a pair of equations which describe upward and downward streams of flux

$$\begin{aligned} \pm \mu_i \frac{dN}{dy}(j, \pm \mu_i, y) &= \sum_{j'=0}^{2m-1} \left\{ - \left[\alpha(y) I(j', j) + \frac{\eta_i}{\pi} S(j', j) + \frac{\eta_i}{\pi} B(j', j) \right] \right. \\ &\quad \times N(j', \pm \mu_i, y) + \pi S(y) \\ &\quad \times \left[\sum_{i'=1}^n c_{i'} N(j', +\mu_{i'}, y) P(y, j', j, +\mu_{i'} \pm \mu_i) \right. \\ &\quad \left. \left. + \sum_{i'=1}^n c_{i'} N(j', -\mu_{i'}, y) P(y, j', j, -\mu_{i'}, \pm \mu_i) \right] \right\} \\ &\quad + N_s(j, \pm \mu_i, y) + N_d(j, \pm \mu_i, y) \end{aligned} \quad (45)$$

where $\eta_i = (1 - \mu_i^2)^{1/2}$ and μ_i 's and c_i 's are the abscissas and weights of the chosen quadrature.

2) CONSTRUCTION OF VECTORS AND MATRICES

We now write “ $N^\pm(k)$ ” for “ $N(j \pm \mu_i, y)$ ” for $j = 1, \dots, 2m$ and $i = 1, \dots, n$ by suitably linearizing j and i to k . Thus the radiance vectors are considerably strung out and are now of length $2mn$. We next write (45) in its matrix form

$$\begin{aligned} \frac{\pm \mathbf{M}}{\alpha(y)} \frac{d\mathbf{N}^\pm}{dy} &= -\mathbf{N}^\pm \left[\mathbf{I} + \frac{\mathbf{L}}{\pi \alpha(y)} (\mathbf{S} + \mathbf{B}) \right] \\ &\quad + \pi \omega_0(y) \mathbf{N}^\pm \mathbf{C} \mathbf{P} + \pi \omega_0(y) \mathbf{N}^\pm \mathbf{C} \mathbf{P} \\ &\quad + \frac{\mathbf{N}_s^\pm(y)}{\alpha(y)} + \frac{\mathbf{N}_d^\pm(y)}{\alpha(y)}, \end{aligned} \quad (46)$$

where \mathbf{M} , \mathbf{L} , \mathbf{S} , \mathbf{C} , \mathbf{P} and \mathbf{B} are $2mn \times 2mn$ matrices and \mathbf{I} is the $2mn \times 2mn$ identity matrix. \mathbf{M} , \mathbf{C} and \mathbf{L} are the diagonal matrices

$$\begin{aligned} \mathbf{M} &= \{ \delta_{j'-j} \delta_{i'-i} \mu_i \}, \quad \mathbf{L} = \{ \delta_{j'-j} \delta_{i'-i} \eta_i \}, \\ \mathbf{C} &= \{ \delta_{j'-j} \delta_{i'-i} c_i \} \end{aligned}$$

for $j, j' = 0, 1, \dots, 2m - 1$ and $i, i' = 1, \dots, 2n$. Here the modal series have been truncated in (6) to NL, NW and NA terms, that is $m = NL \times NW \times NA$. The \mathbf{S} and \mathbf{B} matrices are defined as

$$\begin{aligned} \mathbf{S}(\mathbf{B}) &= \{ \delta_{i'-i} S(\mathbf{B})(j, j'); i, i' = 1, \dots, n \}, \\ &\quad j, j' = 0, \dots, 2m - 1 \end{aligned}$$

and the \mathbf{P}^\pm matrices are of the form

$$\begin{aligned} \mathbf{P}^+ &= \{ \delta_{j'-j} P(j, j', \mu_i, \mu_{i'}); j, j' = 0, 1, \dots, 2m - 1 \}, \\ &\quad i, i' = 1, \dots, n \end{aligned}$$

$$\begin{aligned} \mathbf{P}^- &= \{ \delta_{j'-j} P(j, j', -\mu_i, \mu_{i'}); j, j' = 0, 1, \dots, 2m - 1 \}, \\ &\quad i, i' = 1, \dots, n \end{aligned}$$

and for $0 \leq \mu \leq 1$. We thus define the $2mn$ by $2mn$ matrices

$$\begin{aligned} \tau(y) &= -\{ \mathbf{I} + \mathbf{L}(\mathbf{S} + \mathbf{B}) / \pi \alpha(y) \\ &\quad - \pi \omega_0(y) \mathbf{C} \mathbf{P}^+ \} \mathbf{M}^{-1}, \end{aligned} \quad (47)$$

$$\rho(y) = \{ \pi \omega_0(y) \mathbf{C} \mathbf{P}^- \} \mathbf{M}^{-1}, \quad (48)$$

$$\epsilon(y) = \frac{1}{\alpha(y)} \{ \mathbf{N}_s^\pm(y) + \mathbf{N}_d^\pm(y) \} \mathbf{M}^{-1}. \quad (49)$$

The local interaction form of (44) then becomes

$$\left. \begin{aligned} \frac{d\mathbf{N}^+}{dy}(y) &= \mathbf{N}^+(y) \tau(y) + \mathbf{N}^-(y) \rho(y) + \epsilon^+(y) \\ - \frac{d\mathbf{N}^-}{dy}(y) &= \mathbf{N}^-(y) \tau(y) + \mathbf{N}^+(y) \rho(y) + \epsilon^-(y) \end{aligned} \right\} \quad (50)$$

which is exactly analogous to the plane parallel statement of local interaction principle (cf. Preisendorfer, 1965) and therefore the solution now follows the procedure of doubling and adding with the inclusion of sources as used previously for plane-parallel setting by, e.g., Stephens (1980).

7. Results

The multimode radiative transfer equation (44) was solved for a box-like optical medium vertically illuminated by a unit irradiance only at its uppermost face. The scattering phase function was employed in the form of a Legendre series (e.g., Van de Hulst, 1980) with an asymmetry parameter of 0.85 (perhaps typical of cumulus clouds in the atmosphere). Only the conservative scattering case is presented in this paper.

a. Modal truncation

The feasibility of solving (44) in matrix form depends, naturally, on the size of the matrices and ultimately on the number of modes chosen to truncate the series (6). It is therefore important to set some error estimate on this truncation. Fig. 4 illustrates the

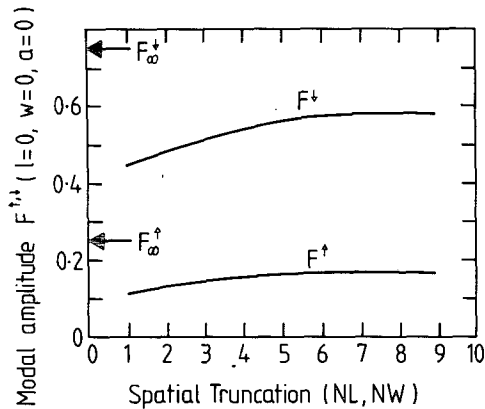


FIG. 4. The emergent flux amplitudes $\hat{F}(0, 0, 0)$ from the top and base of the box as a function of the number of modes chosen to truncate the series (6). $\tau_w = 20$, $\tau_H = 5$.

variation of the flux amplitude for $l = 0$, $w = 0$, $a = 0$, for a cube of optical dimension 5 as a function of the total number of spatial modes chosen to represent the spatial distribution of radiation. The quantity shown in Fig. 4 can be interpreted as the total upwelling or downwelling radiant flux $\hat{F}(0, 0, 0)$ through the upper or lower faces of the cube. Convergence of the quantity $\hat{F}(0, 0, 0)$ is apparent for $NL, NW \geq 3$ but even for $NL, NW = 1$ [i.e., $j, j' = 0$ in the matrices of (38)] the solutions are a significant improvement on the plane-parallel solutions (shown on the diagram by the arrows).

Figure 5 further illustrates this point by depicting the distribution of azimuthally averaged (i.e. $a = 0$) flux across the upper and lower faces of the box. The distribution corresponds to the variations of flux from the west to east side of the box at $v = W/2$ (refer to Fig. 1 and the inset in Fig. 5). The flux distribution, shown for 5 different truncations of the spatial modes, is sensitive to the number of such modes chosen to truncate the series in (6), especially for $NL < 5$ but the total energy passing through the faces of the box

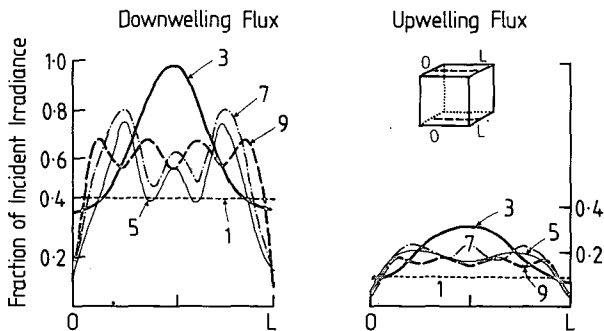


FIG. 5. The distribution of emergent flux across the top and base of the box as shown in the inset. These spatial distributions of flux are shown for the various modal truncations indicated.

is fairly independent of the truncation for $NL, NW \geq 3$.

b. Comparison with Monte Carlo solutions

The multimode radiative transfer equation was solved assuming a modal truncation of 3 (i.e., $NL, NW = 3$). The calculations were performed for a parallelepiped box of varying width-to-depth ratios, and the results are shown in Figs. 6a and 6b as a function of this ratio. These diagrams show the total flux (as a fraction of the incident flux) leaving the upper and lower faces of the box (but not including the energy that escapes through the sides of the box). The Monte

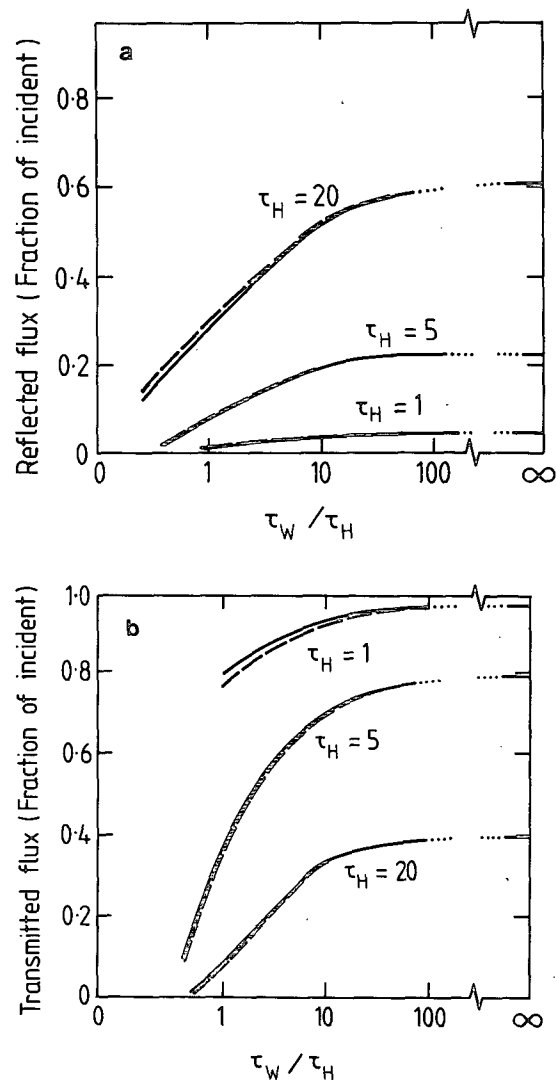


FIG. 6. The upward and downward flux through (a) the top and (b) the base of the box as a function of width to depth ratio for the three optical depths indicated. Shown are the Monte Carlo calculations of Davies (solid curves) and the multimode solutions (dashed curves). The fluxes are expressed as fractions of the incident irradiance.

Carlo simulations (solid curves) of Davies (1978) are compared with the multimode solutions (dashed curves) for three optical depths indicated.

Figure 7 shows the relationship between the energy that escapes through the sides of the box, (again in terms of the fraction of the incident irradiance) as a function of the width to depth ratio (solid curve). Also shown is a simple hyperbolic width relationship ($1/W$) which resembles closely the variation of lateral radiation escaping through the sides of the box. This is not surprising in view of the hyperbolic dependence of the emergent lateral tensor and the boundary divergent tensor on the width (or length) of the box as indicated in (36) and (38).

c. Flux profiles through the box

The two tensors S and B manifest themselves in the radiative transfer equation in the form of an attenuation of radiance. This attenuation, as mentioned previously, is a function only of the geometric properties of the box. Thus the decay of radiation through a medium will be far more enhanced in a medium which is constrained laterally than in a laterally infinite (or plane parallel) medium. This is apparent in Fig. 8 which illustrates the vertical profiles of total (fractional) upwelling and downwelling flux through the parameter surfaces X_y (i.e. horizontal surfaces) within the medium. Three sets of profiles are illustrated, the infinite (plane parallel) flux profiles, the profiles within a cube and the profiles in a parallelepiped medium with square cross sections X_y with a width to depth ratio of 0.4 for an optical depth of 5. The influence of the "attenuation" of radiation by lateral escape of energy is clearly evident and, as can be inferred from Fig. 7, is more pronounced for the smaller width to depth ratios.

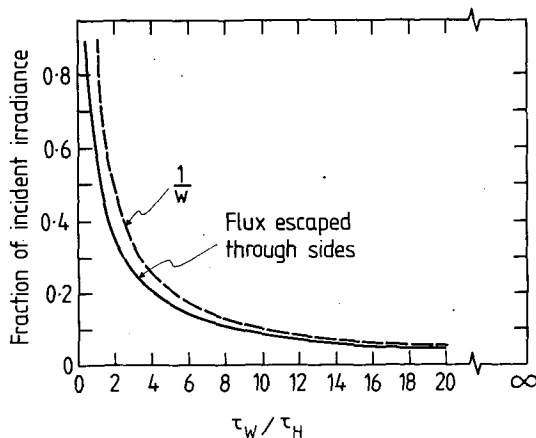


FIG. 7. The amount of energy that escapes the sides of the box (a fraction of the incident irradiance) as a function of width-to-depth ratio. Also shown is a simple hyperbolic relationship with the box width (dashed curve).

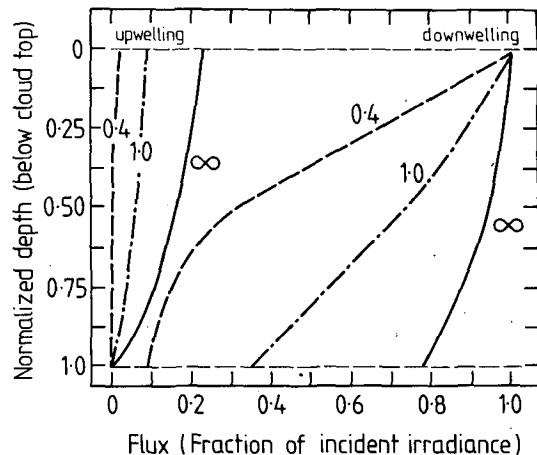


FIG. 8. The vertical profiles of total upwelling and downwelling radiant flux (fraction of the incident irradiance) through the parameter surfaces X_y for three width-to-depth ratios, 0.4, 1.0 and ∞ .

8. Conclusions

In this study we develop a new method for solving the radiative transfer equation for which the setting is a medium that is laterally finite. The paper introduces the multimode approach, and a new radiative transfer equation (44) is developed which is strongly analogous to its more familiar plane-parallel counterpart. The new equation accounts for the lateral boundaries of the medium in terms of source and sink effects.

The procedures for solving the multimode radiative transfer are outlined and solutions are presented. The results show that the multimode solutions agree well with Monte Carlo simulations in terms of total flux through the horizontal parameter surfaces of the box. Truncation of the multimode series (6) reveals that the total flux through these surfaces is relatively insensitive to the number of modes selected (especially for $NW, NL > 3$) whereas the spatial distribution of the flux is more sensitive to the modal truncation.

The results also illustrate the behavior of the energy that escapes laterally through the sides of the box revealing an approximately hyperbolic dependence on the width of the box. It was also illustrated how the presence of a "new optical property", the emergent radiance tensors, provides for the decay of radiation through the faces of the medium. Thus it may be possible to detect this optical property of cumulus clouds experimentally from measurements of the vertical flux or radiance profile within the clouds. One could then check these empirical estimates against a formal estimate derived from the S and B tensors, in terms of the mean geometric width of the cloud.

While the derivation of (44) has been based on the specification of a box shaped medium, we believe that similar radiative transfer equations can be derived for other shaped media (e.g. cylindrical, spherical or gen-

erally polyhedral solids, for which case the emergent lateral tensors may assume a y dependence). We expect that the radiative transfer equation for a generally shaped finite optical medium has the form

$$\mu \frac{\partial \mathbf{N}}{\partial y} = -\mathbf{N}\mathbf{A} + \pi_s \int_{-1}^{+1} \mathbf{N}P d\mu' + \mathbf{N}_{si} + \mathbf{N}_e$$

where \mathbf{A} is an appropriately determined (net boundary divergence + lateral + standard) attenuation matrix which together with an incident radiance term \mathbf{N}_{si} carries the effects of the lateral finiteness of the medium.

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APPENDIX A

Definition of Symbols

α	volume attenuation coefficient
δ_k	Kronecker's delta
ϵ	source vector
ξ	directional unit vector
θ	zenith angle
μ	$\cos\theta$, quadrature abscissa
η	$\sin\theta$
Ξ	unit sphere
ρ	local reflection matrix
τ	optical dimension (e.g., τ_H optical depth)
ϕ	azimuth angle
χ	quadrant characteristic function
ω_0	single scattering albedo
\mathbb{A}	attenuation matrix
a	azimuthal mode
B, \mathbb{B}, B_{jk}	boundary divergent tensor
CC, CS	integral of cosine-cosine, cosine-sine product
c	quadrature weight
$d\Omega$	differential of solid angle
\hat{F}	flux amplitude
$F_{L,ij}, F_{w,ij}$	an abbreviated form for a sum of four trigonometric integrals
H	depth of box
I	identity matrix
L	length dimension of box
l	length mode
N	radiance
\hat{N}	radiance amplitude
N^\pm, \mathbb{N}	radiance vector
N_e	radiance source by emission
$\hat{N}_{se}, \hat{N}_{si}$	radiance amplitudes associated respectively with lateral emergence and lateral incidence

n	quadrature truncation
\mathbf{P}, \mathbf{P}	phase matrix
p	scattering phase function
S, \mathbf{S}	tensor for emergent radiances through cloud sides
SS, SC	integral of sine-sine and sine-cosine products
\mathbf{t}	local transmission matrix
s	volume scattering coefficient
u	horizontal (east-west) location variable
v	horizontal (north-south) location variable
W	width of box
w	width mode
$X(a, b)$	optical medium with upper surface at level a , lower surface at level b
X_y	horizontal surface within the optical medium at a general depth y , $a \leq y \leq b$
y	vertical variable (measured positive in upward direction)

APPENDIX B

Radiance Conditions

1. Emergent lateral radiance

This quantity is given in (22) as a function of N_l, N_l^W, N_w, N_w^L . By operating over $[0, 2\pi]$ in the usual Fourier way and by letting " $N_{l,j}$ ", $j = 1, 2$, denote the Fourier amplitudes of N_l , etc., we find

$$\begin{aligned} \hat{N}_{se}(l, w, y, \mu, \phi) &= \eta \left\{ \left(\sum_{a'} N_{l,1} \cos a' \phi + N_{l,2} \sin a' \phi \right) \sin \phi \frac{2}{W_w} \right. \\ &\quad \times [\chi_3(\phi) + \chi_4(\phi)] - \left(\sum_{a'} N_{l,1}^W \cos a' \phi + N_{l,2}^W \sin a' \phi \right) \\ &\quad \times \sin \phi \frac{2}{W_w} (-1)^w [\chi_1(\phi) + \chi_2(\phi)] \\ &\quad + \left(\sum_{a'} N_{w,1} \cos a' \phi + N_{w,2} \sin a' \phi \right) \cos \phi \frac{2}{L} \\ &\quad \times [\chi_2(\phi) + \chi_3(\phi)] - \left(\sum_{a'} N_{w,1}^L \cos a' \phi + N_{w,2}^L \sin a' \phi \right) \\ &\quad \left. \times \cos \phi \frac{2}{L} (-1)^l [\chi_1(\phi) + \chi_4(\phi)] \right\}, \quad (B1) \end{aligned}$$

where here and below the summations range from $a' = 0$ to $a' = \infty$. For the cosine amplitudes, from the procedure outlined in (28), we therefore encounter integrals of the form (for $k = 1, 2, 3$ and 4):

$$\left. \begin{aligned} \int_0^{2\pi} \chi_k(\phi) \sin(a' \pm 1)\phi \cos a\phi d\phi &\equiv \text{SC}(a' \pm 1, a, k) \\ \int_0^{2\pi} \chi_k(\phi) \cos(a' \pm 1)\phi \cos a\phi d\phi &\equiv \text{CC}(a' \pm 1, a, k) \end{aligned} \right\}, \quad (B2)$$

where terms like $\cos a'\phi \sin \phi$ in (B1) have been linearized via common trigonometric identities

$$\cos a'\phi \sin \phi = \frac{1}{2} [\sin(a' + 1)\phi - \sin(a' - 1)\phi]. \quad (B3)$$

Similarly, for the sine amplitude procedure we encounter integrals of the form

$$\left. \begin{aligned} \int_0^{2\pi} \chi_k(\phi) \sin(a' \pm 1)\phi \sin a\phi d\phi &\equiv \text{SS}(a' \pm 1, a, k) \\ \int_0^{2\pi} \chi_k(\phi) \cos(a' \pm 1)\phi \sin a\phi d\phi &\equiv \text{CS}(a' - 1, a, k) \end{aligned} \right\} \quad (B4)$$

It can be shown that from symmetry,

$$\left. \begin{aligned} F(a' \pm 1, a, 4) &= -(-1)^{a'+a} F(a' \pm 1, a, 2) \\ F(a' \pm 1, a, 3) &= -(-1)^{a'+a} F(a' \pm 1, a, 1) \end{aligned} \right\}, \quad (B5)$$

where $F = \text{SC}, \text{CC}, \text{SS}, \text{CS}$. These integrals are evaluated in Appendix B.

The main derivation step for the emergent radiances involves recasting (B1) using the definitions in (B2), (B4) and (B5) together with the trigonometric identities of the form of (B3). Eq. (B1) is then multiplied by the factor $(\cos a\phi)/[(1 + \delta_a)\pi]$, as in (28), to yield

$$\begin{aligned} \hat{N}_{se,1} &= \frac{\eta}{(1 + \delta_a)\pi} \sum_{a'=0}^{\infty} \left\{ [-N_{l,1}(-1)^{a'+a} - N_{l,1}^W(-1)^w] \right. \\ &\quad \times \frac{F_{L,11}}{W_w} + [-N_{l,2}(-1)^{a'+a} - N_{l,2}^W(-1)^w] \frac{F_{L,21}}{W_w} \\ &\quad + [-N_{w,1}(-1)^{a'+a} - N_{w,1}^L(-1)^l] \frac{F_{W,11}}{L_l} \\ &\quad \left. + [-N_{w,2}(-1)^{a'+a} - N_{w,2}^L(-1)^l] \frac{F_{W,21}}{L_l} \right\}, \quad (B6) \end{aligned}$$

where

$$\left. \begin{aligned} F_{L,11} &= [\text{SC}(a' + 1, a, 1) - \text{SC}(a' - 1, a, 1)] + [\text{SC}(a' - 1, a, 2) - \text{SS}(a' - 1, a, 2)] \\ F_{L,21} &= [\text{CC}(a' + 1, a, 1) - \text{CC}(a' - 1, a, 1)] + [\text{CC}(a' + 1, a, 2) - \text{CC}(a' - 1, a, 2)] \\ F_{w,11} &= [\text{CC}(a' + 1, a, 1) + \text{CC}(a' - 1, a, 1)] - (-1)^{a'+a} [\text{CC}(a' + 1, a, 2) - \text{CC}(a' - 1, a, 2)] \\ F_{w,21} &= [\text{SC}(a' + 1, a, 1) + \text{SC}(a' - 1, a, 1)] - (-1)^{a'+a} [\text{SC}(a' + 1, a, 2) + \text{SC}(a' - 1, a, 2)] \end{aligned} \right\} \quad (B7)$$

Similarly, the sine amplitudes, $F_{L,12}, F_{L,22}, F_{w,12}$, and $F_{w,22}$ have identical structure to the cosine amplitude functions but with SC and CC replaced in (B7) with SS and CS, respectively.

Now we use the double sum representation [as in (19)] for

$$\left. \begin{aligned} N_{l,j} &= \sum_{w'} \sum_{l'} \hat{N}_j(l', w', y, a') \delta_{l-l'} \\ N_{l,j}^W &= \sum_{w'} \sum_{l'} \hat{N}_j(l', w', y, a') (-1)^{w'} \delta_{l-l'} \\ N_{w,j} &= \sum_{w'} \sum_{l'} \hat{N}_j(l', w', y, a') \delta_{w-w'} \\ N_{w,j}^L &= \sum_{w'} \sum_{l'} \hat{N}_j(l', w', y, a') (-1)^{l'} \delta_{w-w'} \end{aligned} \right\} \quad (B8)$$

for $j = 1, 2$. Substituting (B8) into (B6), we find

$$\begin{aligned} \hat{N}_{se,1}(l, w, y, \mu, a) &= -\frac{\eta}{(1 + \delta_a)\pi} \\ &\quad \times \left[\sum_{a'} \sum_{w'} \sum_{l'} N_1(l', w', y, \mu, a') S_{11}(l', w', a', l, w, a) \right. \\ &\quad \left. - \sum_{a'} \sum_{w'} \sum_{l'} N_2(l', w', y, \mu, a') S_{21}(l', w', a', l, w, a) \right] \quad (B9) \end{aligned}$$

and a similar equation for $\hat{N}_{se,2}(l, w, y, \mu, a)$. These two equations can be condensed into the single equation by using $1/(1 + \delta_a \delta_{k-1})$ to replace $1/(1 + \delta_a)$ to

yield (29). Note that (30) in the text follows from the combination of (B6) and (B8).

2. Boundary divergence of radiance

We employ the definition of the radiance amplitude \hat{N}_B as given by (29). If we apply the integrations as defined in (34), then it is obvious that we encounter integrals of the form

$$\left. \begin{aligned} \frac{1}{(1 + \delta_a)\pi} \int_0^{2\pi} \cos \beta \phi \cos \beta' \phi d\phi &= \delta_{\beta-\beta'} \\ \frac{1}{\pi} \int_0^{2\pi} \sin \beta \phi \sin \beta' \phi d\phi &= \delta_{\beta-\beta'} - \delta_{\beta+\beta'} \end{aligned} \right\} \quad (B10)$$

If we employ, for brevity

$$\left\{ \begin{aligned} Z_l &= \frac{1}{L_l} A(l', l) (1 - \delta_{l-l'}) \delta_{w-w'} \\ Z_w &= \frac{1}{W_w} A(w', w) (1 - \delta_{w-w'}) \delta_{l-l'} \end{aligned} \right\}, \quad (B11)$$

then (30) substituted into (29) and integrated via (34) becomes (for the cosine amplitude)

$$\begin{aligned} \hat{N}_{B,1}(l, w, y, \mu, a) &= \eta \sum_{l'} \sum_{w'} \sum_{a'} [Z_l \hat{N}_1(l', w', y, \mu, a') \gamma_{11}(a', a) \\ &\quad - Z_w \hat{N}_2(l', w', y, \mu, a') \gamma_{21}(a', a)]. \quad (B12) \end{aligned}$$

It similarly follows for the sine amplitude

$$\begin{aligned} \hat{N}_{B,2}(l, w, y, \mu, a) &= \eta \sum_{l'} \sum_{w'} \sum_{a'} [Z_w \hat{N}_1(l', w', y, \mu, a) \gamma_{12}(a, a') \\ &\quad + Z_l \hat{N}_2(l', w', y, \mu, a) \gamma_{22}(a', a)], \end{aligned} \quad (B13)$$

where

$$\left. \begin{aligned} \gamma_{11}(a', a) &= \delta_{a'+1-a} + \delta_{a'-1-a} \\ \gamma_{21}(a', a) &= \delta_{a'+1-a} - \delta_{a'-1-a} \\ \gamma_{12}(a', a) &= \delta_{a'+1-a} - \delta_{a'+1+a} - \delta_{a'-1-a} + \delta_{a'-1+a} \\ \gamma_{22}(a', a) &= \delta_{a'+1-a} - \delta_{a'+1+a} + \delta_{a'-1-a} - \delta_{a'-1+a} \end{aligned} \right\}$$

Eqs. (37) and (38) therefore follow directly from (B11) to (B13) and the tensor B_{jk} can be readily derived. Note that we have introduced a negative sign in the definition of B_{jk} which makes it convenient to be combined with the other attenuation terms.

3. Incident radiance

The derivation of the incident radiance cosine and sine amplitudes follow closely that described above for the emergent radiance amplitudes. We return to (17) rather than (22). Notice the rotational symmetry between these equations (i.e., the correspondence $\chi_1 \leftrightarrow \chi_3$ and $\chi_2 \leftrightarrow \chi_4$). We can therefore immediately replace (B6) with

$$\begin{aligned} \hat{N}_{si,1}(l, w, y, \mu, a) &= \frac{\eta}{(1 + \delta_a)\pi} \sum_{a'} \left\{ [N_{l,1} + N_{l,1}^W (-1)^{w+a'+a}] \frac{F_{L,11}}{W_w} \right. \\ &\quad \left. + (N_{l,2} + N_{l,2}^W (-1)^{w+a'+a}) \frac{F_{L,21}}{W_w} \right\} \end{aligned}$$

$$\begin{aligned} &+ (N_{w,1} + N_{w,1}^L (-1)^{l+a'a}) \frac{F_{W,12}}{L_l} \\ &+ (N_{w,2} + N_{w,2}^L (-1)^{l+a'+a}) \frac{F_{W,22}}{L_l} \}. \end{aligned} \quad (B14)$$

Thus, in similar fashion to (B10), the cosine and sine amplitude terms can be grouped together by replacing $1/(1 + \delta_a)$ with $1/(1 + \delta_a \delta_{k-1})$. This yields (31). Note, in contrast to (B10), $N_{l,j}, N_{l,j}^W$ etc. have arbitrary structure and no a priori connections with the amplitudes $\hat{N}(l, w, y, \mu, a)$. These amplitudes are formally presented in (40) and (41).

APPENDIX C

Cosine-Sine Combinations

1. Definition of functions CC, SC, CS, SS

The functions of interest were introduced in Appendix B. Using the definition (16) of χ_j , we have

$$\begin{aligned} SC(a' \pm 1, a, k) &= \int_0^{2\pi} \chi_k(\phi) \sin(a' \pm 1)\phi \cos a\phi d\phi \\ &= \int_{(k-1)\pi/2}^{k\pi/2} \sin(a' \pm 1)\phi \cos a\phi d\phi, \end{aligned} \quad (C1)$$

$$\begin{aligned} CC(a' \pm 1, a, k) &= \int_0^{2\pi} \chi_k(\phi) \cos(a' \pm 1)\phi \cos a\phi d\phi \\ &= \int_{(k-1)\pi/2}^{k\pi/2} \cos(a' \pm 1)\phi \cos a\phi d\phi, \end{aligned} \quad (C2)$$

$$SS(a' \pm 1, a, k) = \int_{(k-1)\pi/2}^{k\pi/2} \sin(a' \pm 1)\phi \sin a\phi d\phi, \quad (C3)$$

$$CS(a' \pm 1, a, k) = \int_{(k-1)\pi/2}^{k\pi/2} \cos(a' \pm 1)\phi \sin a\phi d\phi. \quad (C4)$$

2. Computational forms for CC, SC, CS and SS

By direct evaluation, we obtain

$$\begin{aligned} CC(a' \pm 1, a, k) &= \begin{cases} \frac{S[(a' \pm 1 + a)k] - S[(a' \pm 1 + a)(k - 1)]}{2(a' \pm 1 + a)} + \frac{S[(a' \pm 1 - a)k] - S[(a' \pm 1 - a)(k - 1)]}{2(a' \pm 1 - a)} \\ \text{for } |a| \neq |a' \pm 1| \\ \frac{\pi}{4} \\ \text{for } |a| = |a' \pm 1| \neq 0 \\ \frac{\pi}{2} \\ \text{for } a = a' \pm 1 = 0, \end{cases} \\ SC(a' \pm 1, a, k) &= \begin{cases} \frac{-C[(a' \pm 1 + a)k] + C[(a' \pm 1 + a)(k - 1)]}{2(a' \pm 1 + a)} + \frac{-C[(a' \pm 1 - a)k] + C[(a' \pm 1 - a)(k - 1)]}{2(a' \pm 1 - a)} \\ \text{for } |a| \neq |a' \pm 1| \\ \frac{-C[2ak] + C[2a(k - 1)]}{4(a' \pm 1)} \\ \text{for } |a| = |a' \pm 1| \neq 0 \\ 0 \\ \text{for } a = a' \pm 1 = 0, \end{cases} \end{aligned}$$

$$\begin{aligned}
 SC(a' \pm 1, a, k) = & \begin{cases} \frac{-C[(a' \pm 1 + a)k] + C[(a' \pm 1 + a)(k - 1)]}{2(a' \pm 1 + a)} + \frac{C[(a' \pm 1 - a)k] - C[(a' \pm 1 - a)(k - 1)]}{2(a' \pm 1 - a)} & \text{for } |a| \neq |a' \pm 1| \\ \frac{-C[2ak] + C[2a(k - 1)]}{4(a' \pm 1)} & \text{for } |a| = |a' \pm 1| \neq 0 \\ 0 & \text{for } a = a' \pm 1 = 0, \end{cases} \\
 SS(a' \pm 1, a, k) = & \begin{cases} \frac{-S[(a' \pm 1 + a)k] + S[(a' \pm 1 + a)(k - 1)]}{2(a' \pm 1 + a)} + \frac{S[a' \pm 1 - a)k] - S[(a' \pm 1 - a)(k - 1)]}{2(a' \pm 1 - a)} & \text{for } |a| \neq |a' \pm 1| \\ \frac{\pi}{4} & \text{for } |a| = |a' \pm 1| \neq 0 \\ \frac{\pi}{2} & \text{for } a = a' \pm 1 = 0, \end{cases}
 \end{aligned}$$

where $s(a) = \sin a\pi/2$ and $c(a) = \cos a\pi/2$.

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