Experiments with a Spectral Tropical Cyclone Model

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Abstract

The three-layer balanced axisymmetric tropical cyclone model presented by Ooyama is generalized to three dimensions and the resultant primitive equations are solved using the spectral (Galerkin) method with Fourier basis functions on a doubly-periodic midlatitude \( \beta \)-plane. The nonlinear terms are evaluated using the transform method where the necessary transforms are performed using FFT algorithms. The spectral equations are transformed so that the dependent variables represent the normal modes of the linearized equations. For the three-layer model, the normal modes correspond to internal or external gravity or rotational modes or to inertial oscillations associated with the constant depth boundary layer. When the governing equations are written in terms of the normal modes, the linear terms can be evaluated exactly and the application of the nonlinear normal mode initialization scheme proposed by Machenhauer is straightforward.

Results from a simulation with an axisymmetric initial condition on an \( \beta \)-plane are presented and it is shown that the model can produce a vortex similar to tropical cyclones observed in nature. The energy of the gravity modes and rotational modes are calculated for this simulation and it is shown that the gravity mode energy is more than an order of magnitude smaller than the rotational mode energy. The model is then run on the \( \beta \)-plane and it is shown that the variation of the Coriolis parameter with latitude causes the tropical cyclone to move toward the northwest at about \( 2 \text{ m s}^{-1} \), in agreement with several other studies. It is also shown that the dispersion of the rotational modes causes the tropical cyclone to elongate toward the west and develop sharper geopotential gradients toward the east. The model is also run with a basic state wind profile and it is shown that the motion of the storm produces asymmetries in the boundary layer convergence field.

The effect of initialization procedures on a tropical cyclone simulation is also studied. The results from linear and nonlinear normal mode initialization procedures and results from applying an initialization procedure based on the nonlinear balance equation are compared. It is shown that the nonlinear normal mode initialization procedure results in much smaller track and intensity forecast errors, and prevents the excitation of spurious gravity waves.

1. Introduction

Since the early 1970s, spectral methods have been used to solve a wide variety of problems in meteorology and fluid dynamics and have been shown to have several advantages over finite difference methods. When an appropriate spectral method is used, the approximate solution to a differential equation can converge exponentially to the true solution as the number of degrees of freedom increases (Gottlieb and Orszag, 1977). For finite difference methods, the approximate solution converges algebraically to the true solution. Spectral methods can also reduce the effect of computational dispersion which results from an inaccurate prediction of the phases of the shorter wavelengths in numerical models (Orszag, 1971). With careful evaluation of variable coefficient and nonlinear terms, the type of instability described by Phillips (1959) can be eliminated with the use of spectral methods (Orszag, 1972). In some cases, semi-implicit time differencing methods can be applied with very little extra work than required for explicit schemes (Hoskins and Simmons, 1975) and at times, linear terms can be computed exactly (Platzman, 1960).

Parallel with the development of spectral models, many advances have been made in the numerical modeling of tropical cyclones since the early work of Ooyama (1969a,b), Yamasaki (1968) and others. For example, Ooyama's tropical cyclone model was axisymmetric and balanced, and treated the large-scale environment as three incompressible fluid layers. The effect of cumulus convection was included as a mass transport between two of the fluid layers and was parameterized in terms of the large-scale variables. Since this time, three-dimensional primitive equations have been developed (e.g. Jones, 1977a,b; Kurihara and Tuleya, 1981) and the treatment of cumulus convection has become more detailed. For example, Rosenthal (1978) presented a tropical cyclone simulation with latent heat release by the resolvable scales of the model, and Hack and Schubert (1981a) used a parameterization scheme which considered an ensemble of
cumulus clouds. A comprehensive review of the development of tropical cyclone models can be found in Anthes (1982).

Although the physics and geometry used in tropical cyclone models have become more sophisticated, all tropical cyclone models have been solved using finite difference methods. In this paper, a three-dimensional primitive equation tropical cyclone model is developed and solved using a spectral method. Since part of the emphasis of this work was to test the applicability of the spectral method to the tropical cyclone problem, it was felt that the physical system should be fairly simple. On the other hand, the physical system should be complicated enough to contain the basic dynamics of tropical cyclone development. These two factors led us to the tropical cyclone model developed by Ooyama (1969a,b). Ooyama's model has a fairly simple vertical structure and cumulus parameterization scheme, but can still reproduce many aspects of tropical cyclones found in nature. Thus, the vertical structure and cumulus parameterization scheme introduced by Ooyama were used in this study. However, in order to develop a tropical cyclone model with a variable Coriolis parameter, which could be used to study storm motion, it was necessary to relax the axisymmetric and balance approximations used by Ooyama.

The basic idea of spectral methods is to expand the spatially dependent part of the dependent variables in truncated series and then use the governing equations to determine the time dependent series amplitudes. As discussed by Gottlieb and Orszag (1977), there are two basic types of spectral approximations. The first of these is the Galerkin method where the basis functions of the series expansions are orthogonal with respect to some inner product and satisfy the same boundary conditions as the dependent variables. The second method is the tau method where the basis functions do not satisfy the boundary conditions individually, but rather extra degrees of freedom are added in such a way that the series as a whole satisfies the boundary conditions. The Galerkin method is generally used for problems with fairly simple boundary conditions, while the tau method can include more complicated boundary conditions.

In meteorological applications the Galerkin method has been used extensively in global or hemispheric models using spherical harmonics (e.g., Bourke, 1974) or Hough functions (Kasahara, 1977, 1978) as basis functions. For a tropical cyclone simulation it is necessary to consider a limited area domain in order to resolve the scales of motion involved. For this case, if radiation boundary conditions are included it is necessary to use the tau method where the appropriate basis functions are Chebyshev polynomials. A difficulty of this method is that the Chebyshev polynomials oscillate rapidly near the boundaries. This extra resolution makes it necessary to apply semi-implicit time differencing methods in order to avoid the need for a restrictively small time step. When semi-implicit methods are applied in primitive equation models, it is necessary to solve an elliptic equation at each time step (Kwizak and Robert, 1971). In finite difference models this equation can be solved efficiently using relaxation methods since the derivative operators which appear couple values of the dependent variables at only a limited number of grid points. For the tau method with Chebyshev basis functions, however, derivative operators couple all of the spectral amplitudes so that the resulting linear system contains a full matrix. It may be possible to overcome this difficulty by solving the elliptic equation using an iterative procedure. Haidvogel et al. (1980) have used a form of the tau method in a balanced barotropic ocean model where the advection terms at the boundaries were treated implicitly. They showed that the necessary elliptic equation could be solved efficiently using the alternating direction implicit (ADI) method. In a primitive equation model this procedure would be somewhat more involved since the gravity wave terms of the governing equations would also have to be treated implicitly. For this case, it might be necessary to apply a multigrid method which can be used for problems with spectral discretization (Zang et al., 1982).

In order to bypass the above difficulties, a somewhat simpler approach was taken. Instead of using a fairly small domain with a radiation boundary condition, an infinite domain was simulated by using a fairly large, periodic domain. For this case, the Galerkin method can be applied with Fourier components as basis functions. With Fourier basis functions, the problems with the time differencing discussed above do not occur since the Fourier components have uniform resolution over the domain. Another advantage to this approach is that the Fourier components are the eigenfunctions of the linear operators which appear in the governing equations. This makes it possible to write the governing equations in terms of the linear normal modes of the model. In this form, the linear terms can be computed exactly, and the nonlinear normal mode initialization introduced by Machenhauer (1977) can be applied straightforwardly.

In Section 2, a brief derivation of the three-dimensional primitive equation version of Ooyama's model is presented. In Section 3, the solution of the governing equations using the Galerkin method with Fourier basis functions is discussed. In Section 4, the governing equations are transformed so that the dependent variables are the amplitudes of the model normal modes. The application of nonlinear normal mode initialization is also discussed. In Section 5, results are presented from a tropical cyclone simulation with an axisymmetric initial condition on an f-plane. For this simulation, the amount of energy in various modes (gravity and rotational) of the solution is calculated. In Section
6, the previous simulation is repeated on the β-plane and the results are compared. In Section 7, a tropical cyclone simulation with a basic state wind is presented. Nonlinear normal mode initialization is applied in the middle of this simulation to see how the subsequent evolution of the tropical cyclone is changed. These results are discussed in Section 8.

2. Governing equations

In this section, a brief description of the governing equations is presented. The physical model is based on the work of Ooyama (1969a) and the reader is referred to the original paper for further details. Ooyama’s model treats the large-scale atmosphere as three layers of incompressible fluid with the effects of cumulus convection included as a mass flux between two of the fluid layers. The magnitude of the mass flux is parameterized in terms of the large-scale variables. Ooyama’s model is generalized by relaxing the axisymmetric and balance approximations and by solving the governing equations on a midlatitude β-plane. Further details on the derivation of the three-dimensional version of Ooyama’s model can be found in Bliss (1980) and DeMaria (1983).

Consider a fluid which consists of three stably stratified homogeneous layers as shown in Fig. 1. The equations which govern the motion of this fluid are the momentum and continuity equations for each layer which are given by

$$\frac{\partial \mathbf{v}_0}{\partial t} + f \mathbf{k} \times \mathbf{v}_0 + \nabla \phi_0 = -\frac{w^-}{H_0} (\mathbf{v}_0 - \mathbf{v}_1) + \mathbf{F}_0, \quad (2.1)$$

$$\frac{\partial \mathbf{v}_1}{\partial t} + f \mathbf{k} \times \mathbf{v}_1 + \nabla \phi_1 = -\frac{w^+}{H_0} (\mathbf{v}_0 - \mathbf{v}_1) + \mathbf{F}_1, \quad (2.2)$$

$$\frac{\partial \mathbf{v}_2}{\partial t} + f \mathbf{k} \times \mathbf{v}_2 + \nabla \phi_2 = -\frac{w^+}{H_0} (\mathbf{v}_0 - \mathbf{v}_1) + \mathbf{F}_1, \quad (2.3)$$

$$H_0 \nabla \cdot \mathbf{v}_0 + w = 0, \quad (2.4)$$

$$\frac{\partial h_1}{\partial t} + H_1 \nabla \cdot \mathbf{v}_1 - w = -\nabla \cdot (h_1 \mathbf{v}_1) - Q, \quad (2.5)$$

$$\frac{\partial h_2}{\partial t} + H_2 \nabla \cdot \mathbf{v}_2 = -\nabla \cdot (h_2 \mathbf{v}_2) + \frac{1}{\epsilon} Q, \quad (2.6)$$

where

- \( \mathbf{v}_i \) is the horizontal velocity of layer \( i \) \( (i = 0, 1, 2) \)
- \( u_i \) is the eastward component of horizontal velocity of layer \( i \)
- \( v_i \) is the northward component of horizontal velocity of layer \( i \)
- \( w \) is the vertical velocity at the top of layer 0
- \( H_i \) is the mean thickness of layer \( i \)
- \( h_i \) is the deviation of thickness of layer \( i \) from the mean thickness
- \( f \) is the Coriolis parameter
- \( g \) is the acceleration of gravity
- \( F_i \) is the friction term of layer \( i \)
- \( Q \) is the mass per unit area and time transported from layer 1 to layer 2
- \( \rho_i \) is the density of layer \( i \)
- \( \epsilon = \rho_2/\rho_1 \)
- \( \phi_0 = g(h_1 + \epsilon h_2) \)
- \( \phi_1 = g(h_1 + h_2) \)
- \( w^+ = \frac{1}{2}(|w| + w) \)
- \( w^- = \frac{1}{2}(|w| - w) \)
- \( \nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \)

As can be seen in Fig. 1, variables with subscripts 0, 1 or 2 correspond to the lower, middle or upper layers respectively. Following Ooyama, layer 0 is assumed to have a constant thickness and the same density as layer 1. Layer 0 is included so that atmospheric boundary layer processes can be simulated. Since the thickness of layer 0 is constant, the continuity equation (2.4) is simplified and the pressure gradient terms are the same for layers 0 and 1 in (2.1) and (2.2). Since the density of layers 0 and 1 is the same, it is possible for mass to travel across the interface between these layers when \( w \neq 0 \). The second term from the right in (2.1) and (2.2) must be included so that the total momentum of layers 0 and 1 is conserved when this transport occurs.

The mass transport term \( Q \) which appears in (2.5) and (2.6) allows the inclusion of diabatic effects in the incompressible fluid system. In a compressible fluid,
diabatic processes allow fluid parcels to move across surfaces of constant potential temperature. In the in-

\[ F_0 = -\frac{C_D}{H_0} |V_0| V_0 + \lambda \nabla^2 V_0, \]  
\[ F_1 = \lambda \nabla^2 V_1 - \mu \frac{(V_1 - V_2)}{(H_1 + h_1)}, \]  
\[ F_2 = \lambda \nabla^2 V_2 + \mu \frac{(V_1 - V_2)}{\epsilon (H_2 + h_2)} + \frac{Q (V_1 - V_2)}{\epsilon (H_2 + h_2)}, \]  

(A) \hspace{1cm} (B) \hspace{1cm} (C) \hspace{1cm} (D)

where \( |V_0| \) is the magnitude of the boundary layer horizontal velocity. Term (A) represents the surface drag calculated from the bulk aerodynamic formula. The drag coefficient \( C_D \) is assumed to have a constant value of 0.0015. The terms labeled (B) represent horizontal eddy diffusion of momentum where the horizontal eddy diffusion coefficient \( \lambda \) is assumed to have a constant value of \( 10^2 \text{ m}^2 \text{s}^{-1} \). The terms labeled (C) represent the effect of vertical diffusion due to the velocity shear across the interface between layers 1 and 2. These terms act in the usual downgradient sense (i.e., to reduce the vertical shear), and the shear stress coefficient \( \mu \) has a constant value of \( 5 \times 10^{-4} \text{ m s}^{-1} \). Term (D) represents the effect of the mixing of momentum when mass is transported from layer 1 to layer 2. This term can be derived by assuming that the fluid from layer 1 conserves its momentum as it is transported to layer 2.

The diabatic term \( Q \), which represents the collective effects of cumulus convection, is parameterized in terms of the large-scale variables. Following Ooyama, \( Q \) is given by

\[ Q = \begin{cases} \eta w & \text{if } w > 0 \\ 0 & \text{if } w \leq 0, \end{cases} \]  

(2.10)

where

\[ \eta = 1 + \frac{\Delta_0 - \Delta_2}{\Delta_2 - \Delta_1}, \]  
\[ \Delta_0 = (\theta_{e0} - \Theta) \]  
\[ \Delta_1 = (\theta_{e1} - \Theta) \]  
\[ \Delta_2 = (\theta_{e2} - \Theta) \]  

In (2.12) \( \theta_{e0} \) and \( \theta_{e1} \) are the equivalent potential temperatures of layers 0 and 1 respectively, \( \theta_{e2} \) is the saturated equivalent potential temperature of layer 2 and \( \Theta \) is a constant reference temperature. Equation (2.10) shows that the diabatic forcing will be nonzero only when the vertical velocity at the top of the boundary layer is positive. The magnitude of \( Q \) depends on the proportionality factor \( \eta \) which depends on the vertical distribution of equivalent potential temperature, \( \lambda \), and the compressible system, diabatic processes allow fluid parcels to move between layers of different density. The friction terms in (2.1)–(2.3) are given by

as can be seen in (2.11). Eq. (2.11) can be derived by considering the moist static energy budget of the air inside the cumulus clouds.

The equivalent potential temperature deviations \( \Delta_2 \), \( \Delta_1 \), and \( \Delta_0 \) are predicted from the following equations:

\[ \Delta_2 = \Delta_2 + \frac{g \gamma}{c_p} (1 - \epsilon) h_2, \]  
\[ \Delta_1 = -10 \text{ K}, \]  
\[ \frac{\partial \Delta_0}{\partial t} = -u_0 \frac{\partial \Delta_0}{\partial x} - v_0 \frac{\partial \Delta_0}{\partial y} + \frac{w}{H_0} (\Delta_0 - \Delta_1) \]  
\[ + \lambda \nabla^2 \Delta_0 + \frac{C_E}{H_0} |V_0| (\Delta_2 - \Delta_0), \]  

(2.15)

where

\[ \Delta_s = \Delta_s + \frac{g \alpha}{c_p} (h_1 + \epsilon h_2). \]  

(2.16)

Eq. (2.13) is derived by analogy with a compressible fluid and shows that \( \Delta_2 \) increases as the thickness of layer 2 increases. This then decreases \( \eta \) in (2.11) and the magnitude of the diabatic term \( Q \) in (2.10). This decrease in \( Q \) with increasing \( h_2 \) simulates the effect of warming in upper levels using the analogy between increasing temperature in a compressible fluid and the increase in thickness of an incompressible fluid layer.

As can be seen from Eq. (2.14), \( \Delta_1 \) is set to a constant value of \( -10 \text{ K} \). Ooyama argued that the variation of the midlevel equivalent potential temperature was not of critical importance in the basic dynamics of tropical cyclones and the success of his model tends to confirm this assumption.

The boundary layer equivalent potential temperature deviation \( \Delta_0 \) is treated as a prognostic variable as is predicted by the conservation equation (2.15). The first two terms on the right side of (2.15) represent horizontal advection and the third term represents vertical advection. The fourth term represents diffusion where the horizontal eddy diffusion coefficient \( \lambda \) is assumed to be the same as the coefficient which appears in the momentum diffusion terms in (2.7)–(2.9).
last term on the right side of (2.15) represents the surface flux of equivalent potential temperature as calculated from the bulk aerodynamic formula, where \( \Lambda_s \) is the sea-surface saturation equivalent potential temperature deviation and \( C_E \) is the air-sea exchange coefficient. The value of \( C_E \) is assumed to be equal to the exchange coefficient for momentum \( C_D \) that appears in (2.7). As can be seen in (2.16), \( \Lambda_s \) is assumed to vary linearly with the surface pressure, which allows for a larger surface flux of \( \Lambda_0 \) when the surface pressure is low. The value of \( \Lambda_s \) in (2.16) depends on the sea surface temperature. In all of the model simulations, \( \Lambda_s \) was set to 30 K which corresponds to a sea surface temperature of about 28°C.

As shown by several authors (e.g., Madala and Piacsek, 1975; Kitade, 1980), the inclusion of a variable Coriolis parameter causes a model tropical cyclone to drift toward the northwest at speeds between 1 and 3 m s\(^{-1}\). In order to include a variable Coriolis parameter and still use periodic boundary conditions, it was necessary to use the midlatitude \( \beta \)-plane approximation. This is possible because \( f \) is approximated by \( f_0 + \beta y \) and \( \beta y \) is always neglected compared to \( f_0 \), except when \( f \) is differentiated (Lindzen, 1967).

In order to apply the midlatitude \( \beta \)-plane approximation, the differentiated momentum equations must be used. Thus, the momentum equations (2.1)–(2.3) are replaced by the vorticity and divergence equations for each layer (\( i = 0, 1, 2 \)) which can be written as

\[
\frac{\partial \zeta_i}{\partial t} + f_0 \delta_i + \beta \frac{\partial \psi_i}{\partial x} = -\frac{\partial}{\partial x} (u_i \zeta_i) - \frac{\partial}{\partial y} (v_i \zeta_i)
+ \frac{\partial}{\partial x} (V_{yi} + F_{yi}) - \frac{\partial}{\partial y} (V_{xi} + F_{xi}), \tag{2.17}
\]

\[
\frac{\partial \delta_i}{\partial t} - f_0 \delta_i + \beta \frac{\partial \chi_i}{\partial x} + \nabla^2 \phi_i = -\frac{\partial}{\partial y} (u_i \zeta_i) + \frac{\partial}{\partial x} (v_i \zeta_i)
+ \frac{\partial}{\partial y} (V_{yi} + F_{yi}) + \frac{\partial}{\partial x} (V_{xi} + F_{xi}) - \nabla^2 \left( \frac{u_i^2 + v_i^2}{2} \right), \tag{2.18}
\]

where

\[
V_{xi} = \frac{w^-}{H_0} (u_0 - u_i), \quad V_{yi} = \frac{w^-}{H_0} (v_0 - v_i),
\]

\[
V_{x0} = \frac{w^-}{H_0} (u_0 - u_1), \quad V_{y0} = \frac{w^-}{H_0} (v_0 - v_1),
\]

\[
V_{x1} = \frac{w^+}{(H_1 + h_1)} (u_0 - u_1), \quad V_{y1} = \frac{w^+}{(H_1 + h_1)} (v_0 - v_1),
\]

\[
V_{x2} = 0, \quad V_{y2} = 0,
\]

and \( F_{yi} \) and \( F_{xi} \) are the \( x \) and \( y \) components of \( F_i \) defined by (2.7)–(2.9). In (2.17)–(2.18), the horizontal velocity components \( u_i \) and \( v_i \) can be evaluated from the vorticity and divergence \( \zeta_i \) and \( \delta_i \) by defining a streamfunction \( \psi_i \) and velocity potential \( \chi_i \) as follows:

\[
u_i = \left\{ \begin{array}{l}
\frac{\partial \psi_i}{\partial y} + \frac{\partial \chi_i}{\partial x} \\
\frac{\partial \psi_i}{\partial x} - \frac{\partial \chi_i}{\partial y}
\end{array} \right\} \tag{2.19}
\]

Then,

\[
\zeta_i = \frac{\partial v_i}{\partial x} - \frac{\partial u_i}{\partial y} = \nabla^2 \psi_i \tag{2.20}
\]

In (2.17), the velocity potential contribution to the \( \beta u_i \) term has been neglected, and in (2.18) the streamfunction contribution to the \( \beta u_i \) term has been neglected. Stevens et al. (1977) have shown that this additional approximation must be made to avoid non-oscillatory solutions to the linearized equations.

As a tropical cyclone develops from a weak vortex, the gradients of the dependent variables become very sharp near the center of the storm. This indicates that the spectral representations of the dependent variables will have larger and larger amounts of energy in the higher wavenumbers. Since the spectral representations are truncated, there is a tendency for the energy of the last few modes to become too large. This phenomenon, known as spectral blocking, is discussed in some detail by Machenhauer (1979) in the context of the one-dimensional nonlinear advection equation. Machenhauer indicates that a scale selective energy dissipation can be applied to reduce the errors caused by the blocking. In global models, linear diffusion terms of the form \( K_H \nabla^2 \) or \( K_H \nabla^4 \) have been added to the prognostic equations to inhibit blocking (Bourke, 1974; Simmons and Hoskins, 1978). In the current model it was found that linear diffusion terms of the form \( K_H \nabla^6 \) effectively inhibited spectral blocking. Terms of this form with \( K_H = 10^{-22} \) m\(^2\) s\(^{-2}\) were added to all of the prognostic governing equations. In a simple one-dimensional diffusion problem, the amplitude of the highest mode in a Fourier series with 36 terms on a 3600 km periodic domain decreases by a factor of \( e \) in about 30 minutes using this form of diffusion.

In summary, (2.4)–(2.6) and (2.17)–(2.18) govern the motion of the large-scale atmosphere, and equations (2.10)–(2.16) relate the diabatic term \( Q \) to the large-scale variables. Table 1 lists the prognostic and diagnostic variables which appear in the above equations and gives the values of the specified parameters which were used in all the model simulations.

3. Solution using the Fourier-Galerkin method

In order to solve the governing equations developed in the previous section using the Galerkin method with Fourier basis functions, it first must be assumed that all of the dependent variables are periodic on the \( x \) interval \([0, L_x]\) and the \( y \) interval \([0, L_y]\). Each of
<table>
<thead>
<tr>
<th>Prognostic variables</th>
<th>Diagnostic variables</th>
<th>Specified parameters</th>
<th>Equations where parameters appear</th>
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<tbody>
<tr>
<td>$\psi_i$</td>
<td>$\xi_i$</td>
<td>$H_0 = 1 \text{ km}$</td>
<td>several</td>
</tr>
<tr>
<td>$\chi_i$</td>
<td>$\beta_i$</td>
<td>$H_1 = H_2 = 5 \text{ km}$</td>
<td>several</td>
</tr>
<tr>
<td>$h_i$</td>
<td>$\eta_i$</td>
<td>$\epsilon = 0.9$</td>
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</tr>
<tr>
<td>$p_i$</td>
<td>$f_{o,1}$</td>
<td>${}$</td>
<td>(2.17), (2.18)</td>
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<tr>
<td>$\Lambda_0$</td>
<td>$\eta$</td>
<td>$g = 9.8 \text{ m s}^{-2}$</td>
<td>several</td>
</tr>
<tr>
<td>$\Lambda_1$</td>
<td>$\Lambda_2$</td>
<td>$= 0$</td>
<td>(2.13)</td>
</tr>
<tr>
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<td>$\Lambda_4$</td>
<td>$= 0$</td>
<td>(2.12)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$= 10^3 \text{ m}^2 \text{ s}^{-1}$</td>
<td></td>
<td>several</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$= 5 \times 10^{-7} \text{ m} \text{ s}^{-1}$</td>
<td></td>
<td>(2.8), (2.9)</td>
</tr>
</tbody>
</table>

The dependent variables can then be expanded in truncated double Fourier series given by

$$F(x, y, t) = \sum_k \sum_l \hat{F}_{kl}(t) \xi_{kl}(x, y), \quad (3.1)$$

$$\hat{F}_{kl}(t) = \langle F(x, y, t), \xi_{kl}(x, y) \rangle, \quad (3.2)$$

where

$$\xi_{kl} = e^{i(kx+ly)}, \quad (3.3)$$

$$\frac{1}{L_x L_y} \int_0^{L_x} \int_0^{L_y} \varphi \psi dxdy, \quad (3.4)$$

$$k = \frac{2\pi m}{L_x}, m = 0, \pm 1, \ldots, \pm M, \quad (3.5)$$

$$l = \frac{2\pi n}{L_y}, n = 0, \pm 1, \ldots, \pm N, \quad (3.5)$$

$$\{ \xi_{kl}, \xi_{k'l'} \} = \begin{cases} 0 & \text{if } k \neq k' \text{ or } l \neq l' \\ 1 & \text{if } k = k' \text{ and } l = l' \end{cases} \quad (3.6)$$

the asterisk denotes complex conjugate, and $F(x, y, t)$ represents any of the dependent variables.

To apply the Galerkin method, the series expansions of the form (3.1) are substituted into the governing equations and the inner product defined by (3.4) is taken with each basis function $\xi_{kl}$. When this procedure is applied, the diagnostic equations (2.19)–(2.20) become

$$\hat{\xi}_{kl,t} = -(k^2 + l^2) \hat{\psi}_{kl,t} - (k^2 + l^2) \hat{\chi}_{kl,t} - g(k^2 + l^2) (h_{1kl} + \epsilon h_{2kl}) = \hat{P}_{kl}, \quad (3.7)$$

$$\hat{u}_{kl,t} = -i l \hat{\psi}_{kl,t} + i k \hat{\chi}_{kl,t} \quad \hat{v}_{kl,t} = i k \hat{\psi}_{kl,t} + i l \hat{\chi}_{kl,t}. \quad (3.8)$$

Applying the Galerkin procedure to (2.4)–(2.6) and (2.17)–(2.18) for each layer gives

$$-(k^2 + l^2) \frac{d\hat{\psi}_{okl}}{dt} - (k^2 + l^2) f_0 \hat{\chi}_{okl} + i k \hat{\beta} \hat{\psi}_{okl} = \hat{P}_{okl}, \quad (3.9)$$

$$-g(k^2 + l^2) (h_{1kl} + \epsilon h_{2kl}) = \hat{P}_{kl}, \quad (3.10)$$

$$-(k^2 + l^2) \frac{d\hat{\chi}_{okl}}{dt} - (k^2 + l^2) f_0 \hat{\psi}_{okl} + i k \hat{\beta} \hat{\chi}_{okl} = \hat{P}_{okl}, \quad (3.11)$$
iasing error is introduced which prevents the nonlinear instability of the type described by Phillips (1959).

Some of the nonlinear terms which appear in the governing equations, such as the diabatic term \( Q \) and the surface flux terms, are not quadratic in the dependent variables. For these terms, the trapezoidal quadrature will not be exact, and some aliasing error will occur. Bourke et al. (1977) have shown that in a 5-level hemispheric spectral model with moist physics, the aliasing error introduced by the higher order nonlinear terms, is negligible. Hoskins and Simmons (1975) have presented a similar result in a global spectral model without moist physics. This was verified in the current model by running several tropical cyclone simulations with varying resolution on the transform grid. It was found that increasing the resolution beyond what is needed to calculate quadratic nonlinear terms exactly results in only small changes in the solution. (The relative changes in the prognostic variables at a fixed point were on the order of 2–3\% after 4 days of integration.) Thus, all the nonlinear terms are calculated using the minimum number of grid points necessary to resolve the quadratic nonlinear terms.

For the trapezoidal quadrature, (3.1)–(3.2) can be written as

\[
F(x_p, y_q, t) = \sum_k \sum_j \hat{F}_{kl}(t) e^{ikx_p + jy_q},
\]

(3.17)

\[
\hat{F}_{kl}(t) = \frac{1}{(3N + 1)(3M + 1)} \times \sum_{q=0}^{3N} \sum_{p=0}^{3M} F(x_p, y_q, t) e^{-ikx_p - jy_q},
\]

(3.18)

where

\[
x_p = \left[ \frac{L_x}{(3M + 1)} \right] p,
\]

\[
y_q = \left[ \frac{L_y}{(3N + 1)} \right] q.
\]

(3.19)

All of the series in (3.17)–(3.18) can be written in terms of discrete Fourier transforms (see DeMaria, 1983 for details) so that efficient Fast Fourier Transform (FFT) algorithms can be used. The current model uses FFT routines which were written in vector code for a Cray-1 computer. For the case when \( M = N = 35 \), which was used in most of the model simulations, the transforms can be evaluated approximately 13 times faster when the FFT algorithms are used. Since most of the model computing time is used for the transforms, the use of the FFT algorithm greatly increases the efficiency of the model.

4. Normal mode form of the governing equations

The linear terms which appear in (3.9)–(3.16) can be simplified by transforming the governing equations so that the dependent variables are the normal modes of the linearized equations. The first step in this procedure is to determine the vertical normal modes of the linear terms in (3.9)–(3.16). This can be accomplished using a method described by Veronis and Stommel (1956) for an oceanographic problem. As can be seen in (3.9)–(3.16), the linear terms for each layer are coupled. To find the vertical modes, the layer 0 vorticity equation (3.9) is added to arbitrary constants multiplied by the layer 1 and layer 2 vorticity equations; similar procedures are carried out for the divergence and continuity equations. This procedure results in three equations that contain various linear combinations of the dependent variables. The arbitrary constants are determined by requiring that the linear combinations of the dependent variables be proportional. When this procedure is applied, (3.9)–(3.16) become

\[
\begin{align*}
\frac{d\hat{G}_{kl}}{dt} = - (k^2 + l^2) H_0 \hat{\psi}_{kl} = \hat{P}_{3kl} + c_3 \hat{P}_{6kl}, \\
-(k^2 + l^2) \frac{d\hat{S}_{kl}}{dt} = -(k^2 + l^2) \hat{S}_{kl} \hat{\psi}_{kl} + i\beta k \hat{S}_{kl} \\
&= \hat{P}_{1kl} + \frac{H_1}{H_0} \hat{P}_{4kl} + \frac{H_2}{H_0} c_3 \hat{P}_{7kl}, \\
-(k^2 + l^2) \frac{d\hat{V}_{kl}}{dt} = -(k^2 + l^2) \hat{V}_{kl} \hat{\psi}_{kl} + i\beta k \hat{V}_{kl} \\
&= -g_l(k^2 + l^2) \hat{G}_{kl} = \hat{P}_{2kl} + \frac{H_1}{H_0} \hat{P}_{5kl} + \frac{H_2}{H_0} c_3 \hat{P}_{8kl},
\end{align*}
\]

(4.1)–(4.3)

where

\[
\begin{align*}
\hat{G}_{kl} &= \hat{h}_{1kl} + c_3 \hat{h}_{2kl} \\
\hat{S}_{kl} &= \hat{\psi}_{0kl} + \frac{H_1}{H_0} \hat{\psi}_{1kl} + \frac{H_2}{H_0} c_3 \hat{\psi}_{2kl} \\
\hat{V}_{kl} &= \hat{\chi}_{0kl} + \frac{H_1}{H_0} \hat{\chi}_{1kl} + \frac{H_2}{H_0} c_3 \hat{\chi}_{2kl}
\end{align*}
\]

(4.4)

\[
g_l = g \left( 1 + \frac{H_1}{H_0} + \frac{H_2}{H_0} c_3 \right)
\]

(4.5)

and \( c_3, j = 1, 2 \), are the roots of

\[
c^4 + c \left( \frac{H_0 + H_1}{H_2} - 1 \right) - \epsilon \left( \frac{H_0 + H_1}{H_2} \right) = 0.
\]

(4.6)

For \( j = 1 \) or 2, (4.1)–(4.3) represent two systems of three equations where the new dependent variables defined by (4.4) are decoupled for each \( j \) in the linear case. It will be shown that these two systems govern the motion of the external and internal vertical modes of the model.

In addition to (4.1)–(4.3), two other equations result from the vertical mode procedure which are given by
\(- (k^2 + l^2) \frac{dS_{okl}}{dt} - (k^2 + l^2) \psi_{okl} + ik \beta S_{okl} = \hat{p}_{1kl} - \hat{p}_{4kl}, \quad (4.7)\)
\(- (k^2 + l^2) \frac{d\psi_{okl}}{dt} + (k^2 + l^2) \psi_{okl} + ik \beta \psi_{okl} = \hat{p}_{2kl} - \hat{p}_{5kl}, \quad (4.8)\)

where
\[\begin{align*}
S_{okl} &= \hat{\psi}_{okl} - \hat{\psi}_{1kl} \\
\psi_{okl} &= \hat{\psi}_{okl} - \hat{\psi}_{1kl},
\end{align*}\]
(4.9)

\[\mathbf{p}_{jkl} = \begin{pmatrix} [\hat{p}_{3kl} + c_j \hat{p}_{6kl}] \\ - \left( \frac{1}{k^2 + l^2} \right) \left[ \hat{p}_{1kl} + \frac{H_1}{H_0} \hat{p}_{4kl} + \frac{H_2}{H_0} c_j \hat{p}_{7kl} \right] \\ - \left( \frac{1}{k^2 + l^2} \right) \left[ \hat{p}_{2kl} + \frac{H_1}{H_0} \hat{p}_{5kl} + \frac{H_2}{H_0} c_j \hat{p}_{8kl} \right] \end{pmatrix}_{j=1,2}, \quad (4.10)\]

\[\mathbf{p}_{0kl} = \begin{pmatrix} - \left( \frac{1}{k^2 + l^2} \right) \left[ \hat{p}_{1kl} - \hat{p}_{4kl} \right] \\ - \left( \frac{1}{k^2 + l^2} \right) \left[ \hat{p}_{2kl} - \hat{p}_{5kl} \right] \end{pmatrix}, \quad (4.11)\]

\[\mathbf{A}_{jkl} = \begin{pmatrix} 0 & 0 & -H_0(k^2 + l^2) \\ 0 & -i \beta k & \frac{f_0}{(k^2 + l^2)} \\ g_j & -f_0 & -i \beta k \end{pmatrix}_{j=1,2}, \quad (4.12)\]

For \(j = 1 \text{ or } 2\), the terms on the left side of Eq. (4.10) represent a linear system of three coupled equations in three unknowns. These equations can be transformed into three scalar equations where the linear terms are decoupled, as shown by Cane and Sarachik (1976) for an equatorial \(\beta\)-plane model. To do this, let the inner product of two column vectors \(u_{jkl}\) and \(v_{jkl}\) with the dimensions of \(W_{jkl}\) be defined by
\[\langle u_{jkl}, v_{jkl} \rangle = d_{jkl}^2 u_{1j} v_{1j}^* + u_{2j} v_{2j}^* + u_{3j} v_{3j}^*, \quad (4.13)\]

where \(u_{1j}, v_{1j}\), etc. are the components of \(u_{jkl}\) and \(v_{jkl}\), the asterisk denotes complex conjugate and \(d_{jkl}^2\) will be determined later to make the inner product dimensionally correct. Using the above inner product, a scalar transform can be defined as follows:
\[W_{jkl} = \frac{1}{E_{jkl}} (\mathbf{W}_{jkl}, \mathbf{K}_{jkl}), \quad (4.14)\]

where \(\mathbf{K}_{jkl}\) is a three component vector which is the kernel of the transform and \(E_{jkl}\) is a normalization factor.

When (4.10) is transformed using (4.15), the linear terms will decouple provided that the kernel of the transform has the following property:
\[\mathbf{A}_{jkl} \mathbf{K}_{jkl} + i \nu_{jkl} \mathbf{K}_{jkl} = 0. \quad (4.16)\]
The above equation implies that \(\mathbf{K}_{jkl}\) are the eigenvectors of the matrix \(\mathbf{A}_{jkl}\) with corresponding eigenvalues \(-i \nu_{jkl}\). If \(\mathbf{A}_{jkl}\) has the property
\[\mathbf{A}_{jkl} \mathbf{u}_{jkl} = (\mathbf{u}_{jkl}, \mathbf{v}_{jkl}), \quad (4.17)\]
then \(\mathbf{A}_{jkl}\) is skew-Hermitian with respect to the inner product (4.14) and the eigenvectors \(\mathbf{K}_{jkl}\) will be orthogonal. It is straightforward to show that (4.17) is satisfied provided that \(d_{jkl}^2\) is given by
\[d_{jkl}^2 = \frac{g_j}{H_0(k^2 + l^2)}, \quad (4.18)\]
and that (4.16) implies
\[\mathbf{K}_{jkl} = \begin{pmatrix} -i \nu_{jkl} \\ \frac{\beta k v_{jkl} - \nu_{jkl} k^2 + l^2}{\nu_{jkl}} \right) \end{pmatrix}_{j=1,2}, \quad (4.19)\]

where \(\nu_{jkl}\) for \(r = 1, 2, 3\) are the three roots of
\[\nu_{jkl}(\nu_{jkl}^2 - \beta) = g_j H_0(k^2 + l^2), \quad (4.20)\]

and \(\nu_{jkl} = \nu_{jkl} - \frac{\beta}{k^2 + l^2}\). For the case when \(j = 0,\)
the scalar transform in (4.15) is defined by the inner product (4.14) except that the first term on the right does not appear. For this case, the kernel is given by

\[ K_{jklr} = \begin{pmatrix} \pm i \\ 1 \end{pmatrix}, \quad (4.21) \]

with

\[ \nu_{jklr} = \frac{\beta k}{(k^2 + l^2)} \pm f_0, \quad (4.22) \]

where either the plus or minus sign is chosen for \( r = 1 \) or 2.

Now, taking the inner product of (4.10) with \( K_{jklr} \) and multiplying by \( E_{jklr}^{-1} \) gives

\[ \frac{dW_{jklr}}{dt} + \frac{1}{E_{jklr}} (A_{jklr} W_{jklr}, K_{jklr}) = P_{jklr}, \quad (4.23) \]

where \( W_{jklr} \) is defined by (4.15) and \( P_{jklr} \) is given by

\[ P_{jklr} = \frac{1}{E_{jklr}} (P_{jklr}, K_{jklr}). \quad (4.24) \]

Using (4.16) and (4.17), (4.23) becomes

\[ \frac{dW_{jklr}}{dt} - i \nu_{jklr} W_{jklr} = P_{jklr}. \quad (4.25) \]

The above equation is the transformed version of (4.10) which must be solved for \( j = 0, 1, 2; \, r = 1, 2, 3 \) and for each wavenumber \( k \) and \( l \).

Since \( A_{jklr} \) is skew-Hermitian, the eigenvectors \( K_{jklr} \) are orthogonal and the inverse of the scalar transform shown in (4.15) is given by

\[ W_{jklr} = \sum_r W_{jklr} K_{jklr} \begin{cases} r = 1, 2, 3; \, j = 1, 2 \\ r = 1, 2; \, j = 0, \end{cases} \quad (4.26) \]

provided that the normalization factor \( E_{jklr} = (K_{jklr}, K_{jklr}) \) is given by

\[ E_{jklr} = (K_{jklr}, K_{jklr}). \quad (4.27) \]

For the linear case, \( P_{jklr} = 0 \) and the solution to (4.25) is given by

\[ W_{jklr}(t) = W_{jklr}(0)e^{i\nu_{jklr}t}, \quad (4.28) \]

where \( W_{jklr}(0) \) is determined from initial conditions. Thus, \( W_{jklr} \) are the amplitudes of normal modes of the linear equations, which oscillate with the frequencies \( \nu_{jklr} \).

For \( j = 1 \) or 2, the frequencies \( \nu_{jklr} \) are given by (4.20). If it is assumed that \( \nu_{jklr} \) is approximately equal to \( \nu_{jklr}^{\text{st}} \), then (4.20) becomes

\[ \nu_{jklr}^{\text{st}} = g_1 H_0 (k^2 + l^2) + f_0^3. \quad (4.29) \]

In the above form, it can be seen that two of the frequencies for \( j = 1 \) or 2 correspond to gravity-inertia waves. Using the values of the parameters listed in Table 1, \( (g_1 H_0)^{\frac{1}{2}} \) takes the values of 324 m s\(^{-1}\) and 52 m s\(^{-1}\) for \( j = 1 \) or 2. These are similar to the pure gravity wave speeds for the external and the first internal vertical modes of a fully stratified model (Fulton and Schubert, 1980). If it is assumed that \( \nu_{jklr} \) is much smaller than \( f_0 \), then (4.20) becomes

\[ \nu_{jklr}^{\text{st}} = \frac{\beta k}{k^2 + l^2 + f_0^3 g_1 H_0}. \quad (4.30) \]

From the above equation it can be seen that the third frequency for \( j = 1 \) or 2 corresponds to a Rossby wave.

When \( j = 0 \), the frequencies are given by (4.22). For this case, the frequencies correspond to inertial oscillations which are slightly modified by \( \beta \). There are only two frequencies for this case because the height of the lowest layer was assumed to be constant.

Figure 2 shows the frequencies for each \( j \) and \( r \) as a continuous function of \( k \) for several meridional wavenumbers \( n \) (\( l = 2\pi n/L_0 \)) for a 4000 by 4000 km domain. Fig. 2a shows the external and internal gravity wave frequencies and Fig. 2b shows the internal and external Rossby wave frequencies and the frequencies associated with the constant depth boundary layer. Comparing Fig. 2a with Fig. 2b it can be seen that there is a large frequency separation between the gravity and Rossby waves with the frequencies of the inertial oscillations about halfway between.

One of the simulations which will be presented was run on an \( f \)-plane (\( \beta = 0 \)) rather than the \( \beta \)-plane. For this case, (4.29) and (4.30) are exact for \( j = 1 \) or 2. The gravity wave frequencies given by (4.29) are approximately the same as for the \( \beta \)-plane case, but the Rossby wave frequencies given by (4.30) are identical. For the \( f \)-plane case, these modes correspond to the part of the solution which is in exact geostrophic balance. In the linear case, these modes do not change with time. For convenience, the modes which do not correspond to gravity waves for \( j = 1 \) or 2 will be referred to as rotational modes for the \( f \)-plane case and Rossby modes for the \( \beta \)-plane case.

There are several advantages to writing the governing equations in the form of (4.25). This equation can be multiplied by an integrating factor which gives

\[ \frac{d}{dt} (W_{jklr} e^{-i\nu_{jklr}t}) = P_{jklr} e^{-i\nu_{jklr}t}. \quad (4.31) \]

When written in this form, the above equation can be solved using time differencing where the linear terms of the model are computed exactly. Since the motion of the external gravity waves in the model is governed by the linear terms, this allows the use of an explicit time differencing scheme where the time step is not restricted by these waves.

Letting \( t = \tau \Delta t \) and using forward time differencing for the first time step, and the second-order Adams-Bashforth scheme for all subsequent time steps gives

\[ W_{jklr}^{(1)} = (W_{jklr}^{(0)} + \Delta t P_{jklr}^{(0)}) e^{i\nu_{jklr} \Delta t}, \quad (4.32) \]
\[ W_{jkr}^{(r+1)} = W_{jkr}^{(r)} e^{i\omega_k \Delta t} \]
\[ + \Delta t \left[ \frac{3}{2} P_{jkr}^{(r)} e^{i\omega_k \Delta t} - \frac{1}{2} P_{jkr}^{(r-1)} e^{i\omega_k 2\Delta t} \right] \], \hspace{1cm} (4.33)

where the superscript indicates the time level. The Adams-Bashforth scheme was chosen over the leapfrog scheme because it can be applied to both the oscillation and decay equations. Since the leapfrog scheme is unstable for the decay equation, it would be necessary to treat the damping terms which appear in \( P_{jkr} \) by some other method. Another advantage to using the Adams-Bashforth scheme is that the computational mode which appears in any three-level scheme is damped for both the oscillation and the decay equation. A disadvantage of the Adams-Bashforth scheme is that although the computational mode is damped, the physical mode is slightly unstable. This is not a major problem, however, since the amplification factor is proportional to \( (\Delta t)^4 \) so that for small time steps, the growth rate becomes negligible. For a typical model simulation there was no sign of instability by the end of a four-day integration.

Since (4.31) is nonlinear, the stability properties of (4.32)–(4.33) are difficult to determine. From numerical experiments with wavenumber truncations of \( M = N = 35 \) on a 3600 km by 3600 km domain, it was found that a 90 second time step was adequate for all of the tropical cyclone simulations to be presented. If the linear terms were treated explicitly, the stability criterion of the form \( \sigma \Delta t \approx 1 \) (with \( \sigma \) the maximum frequency of the external gravity waves) would require a time step of about 36 seconds. Thus, the exact treatment of the linear terms increases the model efficiency by a factor of about 2.5. As shown by Robert et al. (1972), the use of a semi-implicit time integration scheme in a baroclinic model allowed the use of a time step up to six times larger than the explicit limit. It may then be possible to develop a time integration scheme for (4.25) that further increases the model efficiency.

Another advantage of writing the governing equations in the form of (4.25) is that the nonlinear normal mode initialization scheme introduced by Machenhauer (1977) can be applied straightforwardly. The first step in applying the procedure is to divide \( W_{jkr} \) into a slow (low frequency) mode part and a fast (high frequency) mode part as follows:

\[ W_{jkr} = W_{jkr}^s + W_{jkr}^f. \] \hspace{1cm} (4.34)

As can be seen in Fig. 2, the Rossby waves can be considered the slow modes and the gravity waves can be considered the fast modes. The frequencies of the boundary layer modes, however, lie about halfway between the Rossby and gravity wave frequencies. In practice it was necessary to consider the boundary layer
modes to be slow to obtain convergence of the initialization scheme.

Now, suppose the initial values of the dependent variables are given so that $W^{f}_{jkl}$ and $W^{f}_{jkl}$ can be computed. To apply Machenhauer’s initialization scheme, the slow mode contribution is retained and the fast mode contribution is discarded. The fast mode contribution is then recomputed assuming that the time derivative of $W_{jkl}^{f}$ is small enough so that (4.25) gives

$$W_{jkl}^{f} = \frac{i}{w_{jkl}^{f}} P_{jkl}^{f}. \quad (4.35)$$

Since the nonlinear terms couple all of the normal modes, it is necessary to solve (4.35) using an iterative procedure described below:

1. Calculate $W_{jkl}^{f}$ and $W_{jkl}^{f}$ from initial conditions.
2. Set $W_{jkl}^{f}$ to zero and calculate $P_{jkl}^{f}$.
3. Calculate $W_{jkl}^{f}$ using (4.35).
4. Repeat step 3) with $P_{jkl}^{f}$ calculated using new values of $W_{jkl}^{f}$ until the iteration converges.

5. Tropical cyclone simulation with an axisymmetric initial vortex on an f-plane

As a first test of the model, an initial condition with an axisymmetric vortex in layers 0 and 1 was used. The tangential wind $V$ as a function of radius is given by

$$V(r) = V_{m} \left[ 1 - \frac{r}{r_{m}} \right] \exp \left( 1 - \frac{r}{r_{m}} \right), \quad (5.1)$$

where the maximum tangential wind $V_{m}$ was specified to be 10 m s$^{-1}$ at a radius $r_{m}$ of 100 km.

The vorticity profile corresponding to (5.1) is given by

$$\zeta(r) = \frac{2V_{m}}{r_{m}} \left[ 1 - \frac{1}{2} \left( \frac{r}{r_{m}} \right)^{2} \right] \exp \left( 1 - \frac{r}{r_{m}} \right). \quad (5.2)$$

Using the relation

$$r = \sqrt{(x - x_{0})^{2} + (y - y_{0})^{2}}^{1/2}, \quad (5.3)$$

where $(x_{0}, y_{0})$ are the coordinates of the vortex center, the vorticity at the physical space grid points defined by (3.19) was calculated. Trapezoidal quadrature of the form of (3.18) was then used to calculate $\tilde{\zeta}_{l}$ so that (3.7) could be used to find $\tilde{\psi}_{kl}$. The initial vortex was assumed to be nondivergent so that the spectral coefficients of the velocity potential were set to zero.

Once the wind field was specified, the mass field was determined from the nonlinear balance equation. Neglecting the divergence, friction and $V_{xi}$, $V_{yi}$ terms in (2.18) gives

$$\nabla^{2}\phi_{l} = f_{0}\zeta_{l} - \frac{\partial}{\partial y} (u_{l}\zeta_{l}) + \frac{\partial}{\partial x} (v_{l}\zeta_{l})$$

$$- \nabla^{2} \left( \frac{u_{l}^{2} + v_{l}^{2}}{2} \right). \quad (5.4)$$

The above equation was solved using the Galerkin method to determine $\tilde{\phi}_{kl}$ where the terms on the right side were determined from $\tilde{\psi}_{kl}$. The spectral coefficients of $\phi_{l}$ were set to zero initially since layer 2 was assumed to be at rest. The boundary layer equivalent potential temperature deviation $\Delta_{0}$ was initially set to a constant value of 10 K which corresponds to a $\theta_{c}$ of approximately 350 K.

The model was run for 96 hours with the above vortex centered on a 3600 by 3600 km domain. The Coriolis parameter $f_{0}$ was evaluated at 20°N and the simulation was run on the f-plane ($\beta = 0$). As described previously, the value of $\Delta_{b}$ in (2.16) was set to 30 K which corresponds to a sea surface temperature of $\sim$28°C. The above simulation was run using spectral truncations of $M = N = 35$ with a 90 s time step and required about 20 minutes of computing time on NCAR’s Cray-1. As can be seen in (3.19) with $M = 35$ and $L_{c} = 3600$ km, the spacing of transform grid points is about 34 km.

Since the model predicts the amplitudes of the Fourier series it is possible to calculate the dependent variables at any given point in the domain. For the axisymmetric initial vortex, the dependent variables were calculated on a cylindrical grid centered on the storm. The tangential wind (VT) relative to the cylindrical coordinate system was calculated and the dependent variables were azimuthally averaged using eight values at each radial point.

In all of the simulations to be presented, the storm center is defined by the layer 1 streamfunction minimum. It is also possible to define the vortex center as the position of the surface pressure minimum or the layer 1 vorticity maximum. In all the simulations to be presented, these three estimates of the storm center were all within $\sim$10 km. The only exception to this was during the first 24 hours of a simulation with a weak vortex imbedded in a zonal flow where initially the streamfunction minimum was about 40 km from the vorticity maximum.

Figure 3 shows the azimuthal averages of the layer 1 and 2 tangential winds VT1 and VT2, the layer 1 and 2 geopotentials $\phi_{1}$ and $\phi_{2}$, the vertical velocity at the top of the boundary layer $w$ and the convective stability parameter $\eta$ at 0 and 96 h for the f-plane simulation. During the four-day integration, the maximum layer 1 tangential wind increases from 10 m s$^{-1}$ at a radius of 100 km to 43 m s$^{-1}$ at a radius of $\sim$60 km. The layer 2 tangential circulation which develops is cyclonic inside a radius of about 300 km with a much larger anticyclonic circulation outside of 300 km. The cyclonic circulation which develops in layer 2 is largely due to the transport of the layer 1 momentum by the diabatic term $Q$.

In the incompressible fluid system, the deviation of the surface pressure from its mean value ($P_{s}$) is given by

$$P_{s} = gp(h_{1} + e\zeta_{2}) = \rho \phi_{1}. \quad (5.5)$$
Assuming that $\rho$ is approximately 1 kg m$^{-2}$, the layer 1 geopotential can be interpreted as a surface pressure deviation. Fig. 3 shows that $P_1$ decreases from about $-3$ to $-40$ mb during the four day integration. Assuming a mean surface pressure of about 1010 mb, the model tropical cyclone would have a minimum surface pressure of approximately 970 mb at 96 hours. Thus, after four days, the model has produced a tropical cyclone with a maximum tangential wind of 43 m s$^{-1}$ at a radius of 60 km with a minimum surface pressure of about 970 mb. These values are consistent with some of the larger tropical cyclones observed in nature (Shea and Gray, 1973).

The basic structure of the wind and geopotential fields in Fig. 3 is similar to the results presented by Ooyama (1969a,b). It is difficult, however, to make a direct quantitative comparison with Ooyama’s model results because of the resolution limitations of the current model. Since Ooyama’s model was axisymmetric it was possible to use a very fine radial grid (Ooyama used a radial grid spacing of 5 km). Because the current model is three-dimensional, the resolution on the transform grid is restricted to about 35 km. An axisymmetric version of the current model is being developed so that a direct comparison with Ooyama’s results can be made.

Figure 3 also shows the radial structure of the boundary layer vertical velocity. At 96 h it can be seen that a large maximum in $w$ occurs just inside the radius of maximum wind with a region of negative $w$ at a radius of about 120 km. The region of negative $w$ and the oscillatory behavior which occurs at larger radii may be a reflection of the truncation error in the model. This indicates that it may be necessary to use more Fourier modes in the series expansions to eliminate this problem. Another possible solution would be to make a change of independent variables so that the resolution near the vortex center would be increased. Recent work with the model has indicated that the magnitude of the oscillation in $w$ can be reduced by making a more careful choice of the diffusion terms which were added to control spectral blocking. As discussed in Section 2, diffusion terms of the form $K_{pV}^6$ were added to the prognostic equations to control the blocking. Preliminary results indicate that terms of the form $K_{pV}^4$ may be more appropriate since they can still control the blocking, but reduce the magnitude of the $w$ oscillation.

All of the results shown in Fig. 3 are azimuthally averages of the dependent variables. Fig. 4 shows the standard deviation $\sigma$ from the eight point azimuthal average of the dependent variables at 96 h as a function of radius. Fig. 4 shows that the standard deviations of the tangential wind for layers 1 and 2 are less than about 0.2 m s$^{-1}$ and 0.3 m s$^{-1}$ respectively inside a radius of 700 km. From Fig. 3 it can then be seen that the azimuthally averaged tangential winds are about two orders of magnitude larger than the standard deviations. Similarly, the azimuthally averaged geopotentials for layers 1 and 2 are about two orders of
storm center, compared to $\sim 1800$ km in the current model. Thus, the influence of the lateral boundary in Anthes' model may have contributed to the asymmetric structure.

The lower portion of Fig. 4 shows the standard deviation of the vertical velocity field from the azimuthal average. From Fig. 3 it can be seen that the standard deviation is about a factor of 10 smaller than the mean inside 100 km, a factor of 3 smaller between 100 and 200 km and about the same size as the mean outside of 200 km. These asymmetries are probably due to the lack of enough Fourier modes to fully resolve the sharp vertical velocity peak which occurs near 60 km. There is also some contribution from gravity wave activity which propagates through the domain and reenters through the periodic boundary. Although not ideal, the asymmetries in the vertical velocity field are not a major problem since they do not appear to cause large asymmetries in the wind and geopotential fields.

The axisymmetry of the wind and geopotential fields at 96 hours in a primitive equation model with large diabatic heat sources on a doubly-periodic domain may be somewhat surprising. The linear theory of geostrophic adjustment (e.g., Schubert et al., 1980) indicates that an impulsive diabatic heat source with a length scale less than the Rossby radius of deformation will largely excite gravity wave motion and that only a small fraction of the total energy produces balanced flow. The gravity waves should then propagate away from the storm, pass through the periodic boundaries and eventually obscure the symmetric balanced flow. The reason this does not occur is because the arguments from the linear theory are largely based on initial value problems where all the heat is added instantaneously. As discussed by Schubert et al. (1980, Section 8), Hack and Schubert (1981b) and in detail by Silva Dias et al. (1983), the amount of gravity wave energy excited decreases as the time scale of the forcing becomes long compared to the period of the gravity waves. In the model, the diabatic forcing $Q$ is given by $nw$ where $w$ is positive. During the model simulation, $Q$ increased from its initial value of zero until about 48 hours when it began to slowly decrease. Thus, a rough estimate for a characteristic period for $Q$ is on the order of four days or about 100 hours. Fig. 2 shows the periods of the gravity waves as a function of wavelength for the midlatitude $\beta$-plane. For the case when $\beta = 0$, the periods are modified only slightly, since the gravity waves are not sensitive to the variation of the Coriolis force with latitude. For a wavelength of about 200 km (the approximate length scale of the $w$ maximum in Fig. 3), the periods of the gravity waves are on the order of 1 h for the internal mode and 0.2 h for the external mode. Thus, the time scale of the diabatic forcing is much longer than the periods of the gravity waves so that much less gravity wave energy than predicted from an initial value problem should be generated.
The above argument can be verified since the governing equations are transformed so that the dependent variables are the amplitudes of the normal modes of the linear equations. For the incompressible fluid system, the total kinetic energy KE and available potential energy APE can be defined by

\[
KE = \frac{\rho}{2} \int_{0}^{L_y} \int_{0}^{L_x} \left[ H_0 (u_0^2 + v_0^2) + (H_1 + h_1)(u_1^2 + v_1^2) \right] \, dx \, dy,
\]

\[
APE = \frac{\rho g}{2} \int_{0}^{L_y} \int_{0}^{L_x} \left[ [(1 - \epsilon)(H_0 + H_1 + h_1)^2 + \epsilon(H_0 + H_1 + h_2)^2 - \bar{P}] \right] \, dx \, dy,
\]

where

\[
\bar{P} = [(1 - \epsilon)(H_0 + H_1)^2 + \epsilon(H_0 + H_1 + H_2)],
\]

and the layer 1 density \( \rho \) is assumed to be 1.0 kg m\(^{-3}\). In (5.7), the available potential energy is defined as the total potential energy of the fluid minus the potential energy when \( h_1 \) and \( h_2 \) are zero. The sum of the kinetic and available potential energy was calculated using only certain modes of the solution. The basic procedure was to first set all the amplitudes \( W_{abr} \) equal to zero except, for example, those corresponding to the gravity waves. The physical space variables on the transform grid were then calculated from \( W_{abr} \) and the integrals in (5.6) and (5.7) were evaluated using trapezoidal quadrature. Using this procedure the energy of the internal rotational modes (IR), external rotational modes (ER), internal gravity waves (IG), external gravity waves (EG) and energy of the boundary layer modes (BL) were calculated for the symmetric tropical cyclone simulation.

In the linear case, the definition of kinetic energy given by (5.6) is modified by neglecting \( h_1 \) and \( h_2 \) compared to \( H_1 \) and \( H_2 \). For this case, only quadratic terms appear in the definitions of KE and APE. Assuming that the normal modes of the model are orthogonal, a Parseval relation could then be derived that related KE and APE to sums of the amplitudes of the normal modes. For this case, the sum of the modal energies described above would be equal to the total energy. In the nonlinear case, however, the cubic terms in the definition of KE must be included, so a Parseval relation cannot be used to determine KE. For the nonlinear case, then, the sum of the modal energies does not add up to the total energy. In practice, however, the contribution from the cubic terms in (5.6) was fairly small so that the sum of the modal energies was within a few percent of the total energy. Thus, the modal energies still give an indication of the amount of the total energy in various modes of the solution.

The energies of the modes of the model are shown in Fig. 5 as a function of time. Initially the amplitudes of the rotational modes are much larger than the gravity wave amplitudes since the mass and wind fields are in gradient wind balance. The gravity modes are not identically zero, however, since the modal decomposition is based on linear theory while the gradient wind equation (nonlinear balance equation) contains a nonlinear term. The amplitudes of the boundary layer modes are identically zero initially since it was assumed that the layer 0 and layer 1 wind fields were the same at \( t = 0 \). As time increases, the energy of all the modes increases, with the internal gravity waves and boundary layer modes increasing the most rapidly initially. After about 12 h, the energy of the internal gravity waves and boundary layer modes increase at a slower rate as the system appears to have adjusted to initial development of the boundary layer vertical velocity.

The most important feature which can be seen in Fig. 5 is that the energy of the rotational modes is much larger than the energy of the gravity modes. By 96 h the energy in either the external or internal rotational modes is more than an order of magnitude larger than the energy in the internal gravity waves and two orders of magnitude larger than the external gravity wave energy. This indicates that the time scale of the forcing modifies the arguments of the linear theory of geostrophic adjustment and also that the use of periodic boundary conditions in the current model is not as severe an approximation as might be thought.

The fact that the energy in the gravity modes is more than an order of magnitude less than the energy in
the rotational modes in Fig. 5 has some implications for primitive equation tropical cyclone models which include more general boundary conditions. Hack and Schubert (1981b) have developed a boundary condition for a grid-point model which minimizes the reflection of gravity waves. Their approach, which is based on the work of Bennett (1976), considers linearized versions of the primitive equations in cylindrical coordinates on an f-plane. In order to develop a boundary condition that can be used in practice, it is necessary to consider limiting cases of the linearized equations, which give radiation conditions for pure gravity waves or a condition appropriate for the balanced flow. Hack and Schubert investigated the radiation condition for gravity waves, but the results presented here indicate that at times, the boundary condition for the balanced flow may be more appropriate. The correct choice probably depends upon the length and time scales of the diabatic forcing in a particular model. For example, the gravity wave radiation condition might be appropriate for tropical cyclone models with explicit release of latent heat (e.g., Rosenthal, 1978) since the diabatic forcing varies on shorter time scales than in the current model, while the balanced flow condition might be appropriate for models with cumulus parameterization schemes with slowly varying diabatic forcing.

In primitive equation tropical cyclone models which use explicit time differencing schemes, the time step is limited by the speed of the external gravity waves. This restriction can be removed through the use of semi-implicit methods (Kwizak and Robert, 1971), at the expense of phase errors in the gravity waves. As shown in Fig. 5, the energy in the gravity wave part of the solution is more than an order of magnitude smaller than the energy of the rotational mode part of the solution. This indicates that the use of semi-implicit methods in primitive equation tropical cyclone models is probably justifiable.

6. Tropical cyclone simulation with an axisymmetric initial vortex on a β-plane

In this section, the f-plane simulation is repeated on the mid-latitude β-plane. The initial conditions and model parameters are the same as in the previous section with β evaluated at 20°N.

Many previous studies have shown that the inclusion of a variable Coriolis parameter will cause a model tropical cyclone to move toward the northwest at speeds between 1 and 3 m s⁻¹ (e.g., Madala and Piacsek, 1975; Kitade, 1980). The current model is in agreement with these results as can be seen in Fig. 6 which shows the track of the streamfunction minimum associated with the tropical cyclone in the β-plane simulation. Fig. 6 also shows the speed and direction of the tropical cyclone as a function of time. During the simulation, the model tropical cyclone accelerates until about 60 h after which time the storm maintains a constant speed of ~ 2.5 m s⁻¹. The motion is toward a direction slightly north of northwest with some indication of a more northwesterly motion after approximately 84 h. The cyclone track shown in Fig. 6 is quite similar to results presented by Jones (1977a). He found that when a variable Coriolis parameter is included in a numerical tropical cyclone simulation, the vortex moves north-northwestward in the developing stage and north-northwestward in the mature stage.

The β-plane simulation differed from the f-plane simulation in several ways other than storm movement. Fig. 7 shows the minimum surface pressure deviation P₁ [defined by (5.5)] and the maximum tangential wind VT₁ in layer 1 for each of these simulations. From Fig. 7 it can be seen that the intensification rate for each simulation is very similar until about 48 h. After this time, the intensity of the storm on the β-plane begins to level off, while the storm on the f-plane continues to intensify. By 96 h the f-plane storm is ~ 15 mb deeper with maximum winds about 10 m s⁻¹ stronger than the β-plane storm. Thus, the inclusion of a variable Coriolis parameter appears to inhibit intensification of the tropical cyclone.

The results presented here are somewhat different than the results from a three-layer tropical cyclone model on a β-plane presented by Madala and Piacsek (1975). They have shown that a tropical cyclone on the β-plane intensified at a slower rate before the storm stage and at the same rate afterwards when compared
to an \( f \)-plane simulation. Both results show that the intensification rate is reduced, but as can be seen in Fig. 7, the slower intensification rate occurs after the storm stage in the current simulation.

Another difference between \( f \)-plane and \( \beta \)-plane simulations was the structure of the upper-layer wind field. Fig. 8 shows the azimuthally averaged layer 1 and 2 tangential winds, layer 1 and 2 geopotentials, boundary layer vertical velocity and convective stability parameter for the \( \beta \)-plane simulation at 96 hours. These values were computed from a cylindrical coordinate system centered on the layer 1 streamfunction minimum associated with the storm. Comparing Fig. 8 with Fig. 3b it can be seen that the radial structure of the dependent variables is quite similar, except for the layer 2 tangential wind and corresponding geopotential. For the \( f \)-plane case there is a cyclonic vortex in the upper layer inside about 300 km which has a maximum tangential wind almost as large as for layer 1. For the \( \beta \)-plane case there is also a cyclonic vortex in layer 2 inside about 300 km, but the tangential wind speeds are much smaller than those of layer 1. This difference in structure is a result of the transport of momentum from layer 1 to layer 2 by the diabatic term \( Q \). As described in Section 2, when fluid is transported from layer 1 to layer 2 it conserves the momentum of layer 1 as it mixes with the fluid in layer 2. The cyclonic vortex which forms in layer 2 for the \( f \)-plane case is a result of this process. This was verified by repeating the \( f \)-plane experiment with the transport term set to zero. For the \( \beta \)-plane case, the tropical cyclone begins to move due to the differential advection of the earth’s vorticity by the storm circulation. When a vortex forms in the upper layer of the model, it also begins to move by this process. Since the radial structure of the upper and lower layer vortices are different, the motion induced by the \( \beta \)-effect will also be different. Thus, it is more difficult to establish a concentrated area of cyclonic rotation in the upper layer by momentum transport for the \( \beta \)-plane case since the center of the upper-layer vortex does not always remain directly over the lower-layer vortex.

The difference in the structure of the upper-layer tangential winds in the \( f \)-plane and \( \beta \)-plane simulations may help explain the difference in the intensification rates. The layer 2 tangential wind for the \( f \)-plane storm has anticyclonic shear between \( \sim 80 \) and \( 550 \) km, while the shear for the \( \beta \)-plane storm is much weaker. For the \( f \)-plane storm, the layer 2 shear vorticity is larger than the curvature vorticity outside of about 100 km so that the relative vorticity is negative outside this radius. This indicates that the inertial instability will be lower in this region for the \( f \)-plane storm than for the \( \beta \)-plane storm since inertial stability is related to absolute vorticity. This indicates that it may be easier for the \( f \)-plane storm to establish a radial circulation in the upper layer since the inertial stability is a measure of the resistance to horizontal motion. The enhanced radial circulation for the \( f \)-plane storm would allow mass to be removed from the inner regions and allow
the surface pressure to decrease, which would intensify the storm.

For the $f$-plane simulation, the azimuthally averaged dependent variables were representative of the total fields since the vortex remained axisymmetric. For the $\beta$-plane simulation, asymmetries developed as can be seen in Fig. 9 which shows the two-dimensional structure of the wind and geopotential fields for layers 1 and 2 at 96 h. In Fig. 9, only a portion of the domain is shown and the contoured fields are geopotential heights ($\phi/g$). The contour interval is 20 m for layer 1 and 10 m for layer 2.

In Fig. 9 the asymmetric structure of the wind and geopotential fields for layer 2 can be seen. There is some evidence of a cyclonic vortex near the storm center which is somewhat axisymmetric. Outside the cyclonic region, however, the flow is highly asymmetric with evidence of an outflow channel towards the south and southwest which begins on the eastern part of the storm. There is also some evidence of a small, closed anticyclone to the west of the storm center in layer 2. This rather complicated pattern is probably a result of the $\beta$-effect and also the momentum transport from layer 1 to layer 2.

In Fig. 9 it can be seen that the layer 1 fields are much more axisymmetric than the layer 2 fields, although some asymmetry exists. The vortex is elongated towards the west at large radii, with some indication of a sharper geopotential gradient towards the east at small radii. This structure can be explained by considering the dispersive properties of the Rossby waves. As can be seen from (4.28) for the linear case ($P_{\eta \theta r} = 0$), the analytic solution for each of the model normal modes is proportional to $e^{\nu t}$ where $\nu$ is the frequency of a given mode. For this case the phase speed is given by $-(\nu/k)$ so that a positive frequency corresponds to a westward propagating wave. Similarly, the zonal component of the group velocity for this case is given by $-\partial \nu/\partial k$. Fig. 2b shows the frequencies of the Rossby waves in the model. In Fig. 2b, the negative of the slope of the frequency curves determines the zonal component of the group velocity, so that the group velocity is negative for low zonal wavenumbers and positive for high zonal wavenumbers. This indicates that the long Rossby waves disperse energy towards the west, while the short Rossby waves disperse energy eastward. Thus, the elongation of the vortex to the west at large radii is due to dispersion of the long Rossby waves.

In Fig. 9, the sharper geopotential gradients in layer 1 are east of the storm center, although they are west of the initial position of the storm. Therefore, the eastward dispersion of the short Rossby wave part of the solution does not appear to explain this feature. The nonlinear interaction of the symmetric vortex and the advection of the earth's vorticity have resulted in a mean flow which causes the vortex to move towards the northwest. Thus, the sharper geopotential gradients may be interpreted in terms of the eastward dispersion of the short Rossby wave part of the solution where the entire pattern is advected by a mean flow.

The asymmetric structure of the layer 1 variables is similar to the structure of storms observed in nature. Holland (1983) has shown that the lower level cir-

![Fig. 9. The layer 1 and 2 horizontal wind and geopotential height fields for the $\beta$-plane simulation at 96 h. The height contour intervals are 20 and 10 m for layers 1 and 2 respectively.](image-url)
culation of developing hurricanes in the Australian region tends to be elongated towards the west. The structure of the lower layer circulation shown in Fig. 9 is also similar to results from a barotropic model on a β-plane presented by Anthes and Hoke (1975).

7. Tropical cyclone simulation with a basic state zonal wind

In the real atmosphere, the occurrence of an isolated symmetric tropical cyclone is quite rare. More commonly, storms are imbedded in some type of mean current which interacts with the storm circulation. In this section, a tropical cyclone simulation that includes a zonal wind is presented.

Since the model uses periodic boundary conditions in the north–south direction, the form of the zonal wind which can be included is somewhat restricted. For simplicity, the zonal wind for layer $i$ was specified to be

$$\vec{u}_i = U_i \sin \left( \frac{2\pi y}{L_y} \right),$$

(7.1)

where $U_i$ was chosen to be $-7.5$ m s$^{-1}$ for each layer. The zonal wind given by (7.1) corresponds to a single sine wave in the north–south direction, with easterlies in the southern half of the domain and westerlies in the northern half of the domain.

For this simulation, the vortex defined by (5.1) was added to the zonal wind in layers 0 and 1 and the mass field was determined using the nonlinear balance equation (5.4). The initial vortex for this experiment differs from the one used in the previous simulations in that the radius of maximum wind was 150 km and the vortex was centered at $x = 2200$ km and $y = 1400$ km on a 3600 by 3600 km domain. The initial position of the vortex was chosen to crudely simulate storms which form in an easterly current and eventually recurve due to the β-effect and the influence of mid-latitude westerlies. The remaining parameters and dependent variables are the same as those used in the β-plane simulation.

The upper portion of Fig. 10 shows the track of the tropical cyclone in this simulation. It can be seen that the storm moves westward initially but turns northward, and eventually towards the northeast after 96 h. This track appears to be a result of the steering current and the β-effect. The initial motion of the storm is simply a result of the advection by the easterly current. After a short time, the β-effect adds a northward component to the direction of motion, which carries the storm out of the easterly current and into the westerly current. This accounts for the turn towards the northeast near the end of the 5-day integration.

Figure 10 also shows the minimum surface pressure deviation and the maximum layer 1 wind speed for this simulation. For this case, the initial value of the layer 1 wind speed is greater than 10 m s$^{-1}$ since the initial vortex and basic state wind were added together. From this figure it can be seen that most of the development takes place during the first 60 hours of the integration. After this time, the storm maintains a relatively constant intensity equivalent to a minimal hurricane. Comparing Fig. 10 with Fig. 7, it can be seen that the intensity of this storm is quite similar to the storm in the β-plane simulation presented previously.

The two-dimensional structure of the wind and geopotential fields in this simulation is also similar to the β-plane simulation results shown in Fig. 9. The structure of the boundary layer vertical velocity field, however, is somewhat different than that in the previous simulations. Fig. 11 shows the boundary layer vertical velocity field in the region surrounding the tropical cyclone at 24, 48, 72 and 96 h. Initially the vertical velocity is zero since the initial condition is nondivergent. By 24 h a vertical velocity field has developed with a maximum of about 20 cm s$^{-1}$ located in the right front part of the storm in relation to the direction of motion (the arrow indicates the direction of motion of the storm). There is also a broad area of rising motion on the eastern side of the storm which spirals back towards the southwest. At 48 h the vertical motion pattern is similar to that at 24 h although the vertical
motion maximum in the right front part of the storm has increased to over 40 cm s$^{-1}$. This pattern continues at later times as the storm moves northward, and then northeastward, with the vertical motion maximum remaining in front and slightly to the right of the storm. The larger area of vertical motion on the east side of the storm also persists, although it is becoming less well defined by 72 h.

The spiral zone of rising motion to the east of the tropical cyclone in Fig. 11 appears to be related to the $\beta$-effect. This is similar to the result presented by Anthes and Hoke (1975) which showed that the inclusion of $\beta$ in a barotropic model causes a spiral shaped region of confluence to form on the eastern side of a cyclonic vortex.

The vertical motion maximum in the right-front part of the storm, however, does not appear to be caused by the $\beta$-effect, but rather is induced by the storm motion. The translation of the storm results in stronger winds relative to the earth to the right of the direction of motion. This causes the surface drag to be asymmetric which results in asymmetric convergence and vertical motion fields. Shapiro (1983) has presented examples of the steady state flow in a slab.
boundary layer forced by the translation of a symmetric vortex in gradient wind balance. His results show that the translation of the vortex causes the convergence to occur in a broad arc ahead of the storm, with the convergence maximum on the right side for faster moving storms, in good agreement with Fig. 11. Results similar to this were also presented by Miller (1958) who observed maximum low-level convergence values in the right-front part of a tropical cyclone. Thus, despite the coarse horizontal and vertical resolution, the current model is capable of representing some of the smaller scale features near the storm center found in other studies and observed in nature.

8. The effect of nonlinear normal mode initialization on a tropical cyclone simulation

When using a numerical model which permits gravity wave oscillations, it is important to specify initial wind and mass fields which are consistent with each other in order to prevent the excitation of spurious gravity waves. Much progress has been made in this area since the introduction of nonlinear normal-mode initialization procedures by Machenhauer (1977) and Baer and Tribbia (1977). Since that time, these techniques have been used extensively in global prediction problems (e.g., Daley, 1979; Temperton and Williamson, 1979). In this section, the application of nonlinear normal mode initialization to the tropical cyclone problem is discussed. For this purpose, the previous simulation which considered the development of a tropical cyclone with a zonal wind on the β-plane is used as a control. This simulation is stopped after 48 hours and the dependent variables are used as initial conditions for the initialization experiments. It is then possible to determine the effect of the initialization by comparing the new simulation to the previous results.

In the nonlinear normal-mode initialization procedure, the amplitudes of the Rossby waves and the inertial oscillations associated with the constant depth boundary layer are not changed, while the gravity wave amplitudes are diagnosed as described in Section 4. Since the boundary layer modes are not changed, the boundary layer inflow is only slightly changed by the initialization procedure. This is a weakness of the current model which stems from the assumption of a constant depth boundary layer. If the depth of the boundary layer were allowed to vary, the two boundary layer modes would be replaced by an additional Rossby mode and two more gravity modes. Temperton and Williamson (1979) have shown that in a fully stratified model, Machenhauer's initialization scheme is capable of producing boundary layer inflow. Thus, although the boundary layer inflow in the current model is associated with the slow modes, the results are probably representative of what would occur in a more general model where the boundary layer inflow is associated with the fast modes.

Since the initial data to be used in this experiment comes from a previous simulation, the $\Delta_0$ field is known exactly at the initial time. In a simulation which used real data, this would not be the case since the initial moisture distribution would not be known. In order to keep the experiment as simple as possible, the $\Delta_0$ field was not changed in the initialization procedure.

As described in Section 4, the diagnostic relation for the gravity wave amplitudes is solved iteratively. Fig. 12 shows the amount of energy in the external and internal gravity waves after each iteration. In this figure, the $i$ on the $x$ axis indicates the amount of energy in these modes before the initialization procedure is applied. This figure shows that the iteration converges quite rapidly. After one iteration, the energies of the internal gravity and external gravity modes are at approximately 92.7% and 99.5% of their final values. Leith (1980) has shown that on the $f$-plane, the first pass through Machenhauer's initialization scheme is analogous to applying quasi-geostrophic theory. The fast mode part of the solution which is diagnosed in this case is then probably fairly close to the ageostrophic motion which would be diagnosed using quasi-geostrophic theory, except near the center of the vortex where the quasi-geostrophic theory breaks down.

Results from general circulation models indicate that nonlinear normal-mode initialization procedures do not always converge (e.g., Puri and Bourke, 1982). Daley (1981) has indicated that a lack of frequency separation can result in nonconvergence of the initialization procedure. Since the vertical structure of the current model is so crude, only the external and first internal modes are represented. For this case, there is a large frequency separation between the Rossby and gravity modes. If the vertical resolution were increased, higher vertical modes would be represented.

![Fig. 12. The energy of the internal and external gravity waves (IG) and (EG) after each iteration of nonlinear normal mode initialization applied to the tropical cyclone simulation with the zonal wind at 48 h. The point $i$ on the $x$-axis indicates the energy in the modes before the initialization is applied.](image-url)
For this case, the frequency separation would be reduced since the higher internal modes have lower gravity wave frequencies. Thus, the convergence properties of the normal-mode initialization procedure used here might be different when applied to a tropical cyclone model with increased vertical resolution.

In addition to the nonlinear normal mode initialization, two other initialization procedures were applied. The first of these was a linear normal-mode initialization procedure which simply sets the amplitudes of the fast modes to zero at the initial time. The second initialization procedure is based on the nonlinear balance equation. For this case it is assumed that the divergence and the time tendency of the divergence are initially zero. Thus, the velocity potential for each layer is set to zero and the mass field is calculated from the streamfunction using the nonlinear balance equation (5.4). For consistency it must also be assumed that the layer 0 and 1 streamfunctions are the same since the pressure gradient forces are the same for these two layers.

Figure 13 shows the time evolution of the external and internal gravity wave energy for each of the initialization procedures described above. The solid line in this figure shows the gravity wave energy for the control simulation. In this figure it can be seen that the linear initialization procedure causes a large amount of spurious gravity wave energy to be excited, even though the initial gravity wave amplitudes were set to zero (at $t = 48$ h). This is because the diabatic forcing and the nonlinear interaction of the Rossby modes rapidly produce gravity waves when the model integration begins. This is similar to results presented by Williamson (1976) which showed that linear normal mode initialization was not capable of suppressing gravity wave oscillations in a shallow water equation model.

In contrast to the linear normal mode initialization, the nonlinear initialization did not produce much spurious gravity wave energy. The energy of both internal and external gravity waves remained very close to the gravity wave energy in the control simulation. This indicates that the nonlinear normal-mode initialization procedure suppressed the artificial generation of gravity wave energy during the 72-hour simulation.

The initialization procedure based on the nonlinear balance equation did not generate as much gravity wave energy as the linear initialization, but did introduce some errors in the gravity wave amplitudes. As can be seen in Fig. 13, this initialization procedure reduced the internal and external gravity wave energy by 56% and 19% respectively, when applied at 48 h. After this time the internal gravity wave energy rapidly increases and after 9 hours (at 57 h), has exceeded the internal gravity wave energy in the control simulation.

Figure 14 shows the errors in tropical cyclone position, minimum surface pressure and maximum layer 1 wind speed introduced by each of the initialization procedures. This figure shows that the nonlinear normal mode initialization introduces the smallest errors in each of these parameters, especially during the first 48 hours after the procedure is applied. Fig. 14 also shows that the linear normal mode initialization introduces large errors in the minimum surface pressure at the initialization time. This is because near the storm center, the geostrophic approximation is not very accurate (at the initialization time, the maximum wind speed is about 30 m $s^{-1}$ at a radius of 85 km), while the Rossby modes in the model are near geostrophic balance. Thus, a vortex that is in approximate gradient balance cannot be represented by Rossby modes alone, but rather projects onto both gravity and Rossby modes. The linear initialization procedure then causes some distortion near the center of the vortex when the gravity modes are set to zero. As soon as the time integration begins, the surface pressure increases very rapidly which indicates that the vortex is adjusting back towards an approximate gradient balance. This
normal-mode initialization scheme (proposed by Machenhauer, 1977) straightforward. The nonlinear terms in the model were calculated using the transform method (Eliasen et al., 1970; Orszag, 1970) where the necessary transforms were evaluated using FFT algorithms.

The success of the model is related to use of the FFT and partially to the exact computation of the linear terms. In order to have the necessary horizontal resolution for a tropical cyclone simulation, 36 Fourier modes were used in the \( x \)- and \( y \)-directions on a 3600-km square domain. For this case, a four-day simulation required about 20 minutes of computing time on NCAR’s Cray-1 computer. It was shown that the FFT increased the model efficiency by about an order of magnitude and exact treatment of the linear terms saved an additional factor of \( \sim 2.5 \) (by allowing a larger time step than required by the external gravity waves). If the efficiency of the model was not increased by these two factors, the four-day simulation would have required about 8 hours of computing time. For this case, it is doubtful that the spectral model would be competitive with a finite difference version of the same model.

In Section 5, results were presented from a simulation on an \( f \)-plane which showed that the model can produce an axisymmetric vortex similar to tropical cyclones observed in nature. The results from the current model are also in qualitative agreement with the model results presented by Ooyama (1969a,b). For this simulation the energy in the gravity and rotational modes of the solution was calculated. It was shown that the gravity wave energy is more than an order of magnitude smaller than the rotational mode energy during the tropical cyclone simulation. This result seems to contradict results from the linear theory of geostrophic adjustment which indicate that small-scale diabatic heating will excite large amounts of gravity wave motion and only a small amount of balanced flow. These results, however, are largely based on initial value problems where all the heat is added instantaneously. When the time scale of the heating is long compared to the period of gravity waves, much less gravity wave motion will be produced. This result provides some justification for use of the periodic boundary conditions in the model and also has implications for models which include more general boundary conditions.

In Section 6, the simulation presented in Section 5 was repeated on the \( \beta \)-plane. For this case, the tropical cyclone moved towards the north-northwest at \( \sim 2.5 \) \( \text{m s}^{-1} \), in agreement with many other studies. The intensification rates of the tropical cyclones on the \( f \)-plane and \( \beta \)-plane were very similar during the first 48 hours. After this time, the \( \beta \)-plane storm leveled off with the intensity of a minimal hurricane, while the \( f \)-plane storm continued to intensify. This difference in the intensification rates appeared to be related to
the structure of the upper-layer tangential wind. On the $\beta$-plane, large asymmetries developed even though the initial vortex was symmetric. In the upper layer, the outflow from the storm tended to form a channel towards the equator. In the lower layers, the circulation tended to be elongated westwards, with sharper geopotential gradients to the east of the storm center. The structure of the lower-layer vortex was explained by considering the dispersive properties of the Rossby waves in the model.

A tropical cyclone simulation which included a zonal wind profile was presented in Section 7. The motion of the tropical cyclone was affected by the zonal wind and by the $\beta$-effect. The structure and intensity of the tropical cyclone in this simulation were very similar to those in the $\beta$-plane simulation presented in Section 6. Motion of the tropical cyclone caused the vertical velocity at the top of the boundary layer to be asymmetric. The maximum vertical velocity always occurred in the front and slightly to the right of the storm in relation to the direction of motion. This feature is very similar to results presented by Shapiro (1983).

The results from the simulation in Section 7 at 48 h were used as initial data to test the effect of initialization procedures on a tropical cyclone simulation. In Section 8, results from applying nonlinear normal mode initialization, linear normal mode initialization and an initialization procedure which uses the nonlinear balance equation were compared. It was shown that the iterative procedure used in the nonlinear normal mode initialization converged rapidly and that this procedure suppressed the excitation of gravity wave energy. This procedure also resulted in the smallest errors in the position and intensity of the storm when compared to the original simulation. It was also shown that the linear initialization procedure, which simply sets the gravity modes to zero, is inappropriate for tropical cyclone simulation. This is because both gravity and Rossby modes of the model are needed to represent a vortex in approximate gradient balance.

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