Barotropic Equatorial Waves: The Nonuniformity of the Equatorial Beta-Plane

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ABSTRACT

Some equatorially trapped motions cannot be modeled by the equatorial beta-plane. Our proof is a counter-example: if the zonal wavenumber m is large, barotropic Rossby–Haurwitz waves decay with latitude outside a narrow band about the equator and can be approximated by Hermite functions. The rather subtle effects of spherical geometry which create these barotropic equatorial waves have important implications for short zonal wavelength motion in the tropics.

The equatorial beta-plane approximation is a widely used tool for studying motions which are confined to low latitudes. Holton (1975) and Moore and Philander (1977) provide good reviews for the atmosphere and ocean. The orthodox view is expressed by Holton (1975, p. 49): "For qualitative discussions, it is permissible to use the equatorial beta-plane for motions trapped within a distance of the equator as large as $|y| \approx a$ (where a is the radius of the earth)." Unfortunately, this is not true. It is possible for wave motions to be trapped within a very narrow band about the equator and yet be *completely missed* by the equatorial beta-plane.

The usual justification for the beta-plane is *a posteriori:* when the wave equations on the equatorial beta-plane are linearized about a resting state, the system can be reduced to a single equation for the north–south velocity v:

$$v_{\theta\theta} + [(-m/\sigma - m^2 + \epsilon\sigma^2) - \epsilon\theta^2]v = 0, \qquad (1)$$

where θ is latitude, the subscript denotes differentiation, m the zonal wavenumber, σ the frequency, and

$$\epsilon \equiv 4\Omega^2 a^2/(gh), \quad [\text{"Lamb's parameter"}], \qquad (2)$$

where Ω is the angular frequency of the earth's rotation, a the earth's radius, g the gravitational constant, and h the "equivalent depth" of the mode. As explained in Holton (1975), Lamb's parameter is proportional to the square of the vertical wavenumber; for the stratification typical of the lower stratosphere, $\epsilon = 100$ implies a vertical wavelength of about 28 km.

The eigensolutions of (2) that decay exponentially for large $|\theta|$ are given by

$$v_n = (\text{constant})e^{-(1/2)\epsilon^{1/2}\theta^2}H_n(\epsilon^{1/4}\theta), \qquad (3)$$

where $H_n(y)$ is the Nth Hermite polynomial. As ϵ increases, the waves become more and more closely trapped about the equator. This gives an (apparent!) *a posteriori* justification for Taylor-expanding the trigonometric functions about $\theta = 0$.

This argument that *equatorial confinement* is *sufficient* to justify the beta-plane is given by many sources. Lindzen (1967), for example, states that "any solution which decays sufficiently fast before . . . the value of y corresponding to the North Pole . . . is likely to be a valid approximation to solutions on a sphere." Inspecting (3), one sees that this belief is equivalent to the notion that $\epsilon \gg 1$ (equivalently, short vertical wavelength) is a *sufficient* condition for the validity of the equatorial beta-plane. It has been assumed that $\epsilon \gg 1$ is also *necessary;* barotropic solutions of (1) show no equatorial confinement. Longuet-Higgins' (1968) long and exhaustive treatment of Laplace's tidal equation, for example, discusses Hermite function solutions like (3), only as the limit $\epsilon \to \infty$.

Nonetheless, all these assertions are false if the zonal wavenumber m is sufficiently large. To show this, we offer a counter-example: Rossby–Haurwitz waves, which are the barotropic ($\epsilon = 0$) solutions of the linearized wave equations on the *sphere.* The streamfunction ψ is given without approximation by

$$\psi_n = (\text{constant})e^{im\lambda}P_n{}^m(\mu), \qquad (4)$$

where λ is longitude; $P_n{}^m(\mu)$ is the associated Legendre function; $\mu \equiv \sin(\theta)$; and ψ satisfies

$$[(1 - \mu^2)\psi_\mu]_\mu + \left[-\frac{m}{\sigma} - \frac{m^2}{(1 - \mu^2)}\right]\psi = 0. \qquad (5)$$

The usual equatorial beta-plane treatment of (5) would simply neglect all the factors of μ^2. Suppose, however, that we approximate (5) under the assumption that $\mu \ll 1$ but $m^2\mu^2$ is O(1). Then (5) reduces to

$$\psi_{\theta\theta} + [(-m/\sigma - m^2) - m^2\theta^2]\psi = 0, \qquad (6)$$

where we have set $\mu \approx \theta$. Equation (6) has the same equatorial trapping term, proportional to θ^2, as its beta-plane equivalent, Eq. (1). It follows that the solutions of (6) (in this limit) are identical to those of (1) except that ϵ is replaced by a "spherical Lamb's parameter,"

$$\epsilon_{sph} \equiv m^2. \quad (7)$$

Abramowitz and Segun (1965) confirm this expectation by giving the asymptotic approximation

$$P_{m+n'}{}^{m}[\sin(\theta)] \sim q e^{-(1/2)m\theta^2} H_n(\theta/m^{1/2}),$$
$$m \rightarrow \infty, \quad n' \text{ fixed}, \quad \theta \sim O(m^{-1/2}), \quad (8)$$

where q is a constant. For $m \gg 1$ & $n' \ll m$, not only are Rossby–Haurwitz waves equatorially trapped—despite being barotropic with $\epsilon = 0$—but they can be accurately approximated by the same Hermite functions that give the structure of the usual $\epsilon \gg 1$ equatorial waves.

When we compare $P_{10}{}^{10}$ with its Hermite approximation, the maximum error is less than 2% of the maximum of the wave. We omit the graph of this wavenumber 10 comparison because the error is only a little larger than the thickness of the curve. Figure 1 compares the structure of the lowest latitudinal mode for three different zonal wavenumbers: 10, 20, and 40. The last has decayed to half its maximum amplitude less than 11° of latitude away from the equator. Most emphatically, the $m = 40$ Rossby–Haurwitz wave is equatorially trapped, so one would expect that the equatorial beta-plane approximation would be extremely accurate for it. One would also infer, according to the usual arguments, that such a mode would have a vertical wavelength of about 7 km. In actual fact, the equatorial beta-plane has missed this mode completely, and its vertical wavelength is ∞.

Although the author discovered that spherical harmonics of large zonal wavenumber were equatorially trapped while he was teaching a course on spectral methods, it is hardly surprising that a literature search turned up previous observations of the same phenomenon. The most complete discussion is Longuet-Higgins (1964), but Malkus (1968) makes the same approximation. The surprise is rather that this class of motions has been forgotten.

Longuet–Higgins (1964) was written before the development of the equatorial beta-plane approximation by Rattray, Matsuno, Lindzen, and others, so his "beta-plane" comparisons are entirely with the usual mid-latitude beta-plane. In his monumental treatise on the solutions of Laplace's tidal equation, published four years later, equatorial solutions are only the limit $\epsilon \rightarrow \infty$. The possibility that there could also be equatorial waves of Hermite function form even for $\epsilon = 0$ seems to have been lost in the rubble of time.

Conversely, Lindzen (1967) claimed, as quoted earlier, that all equatorially trapped motions could be accurately approximated by the beta-plane. As we have seen, this is true only when m is small; Hermite form alone does not guarantee a good approximation to the spherical solution because large zonal wavenumber can drastically alter the width of the Hermite function from what is predicted by the beta-plane.

The true degree of equatorial confinement is determined by the "effective Lamb's parameter"

$$\epsilon_{eff} \equiv \epsilon + \epsilon_{sph} = \epsilon + m^2, \quad (9)$$

as shown by comparison of (1) with (6), or by applying the limit that led to (6) to the more complicated spher-

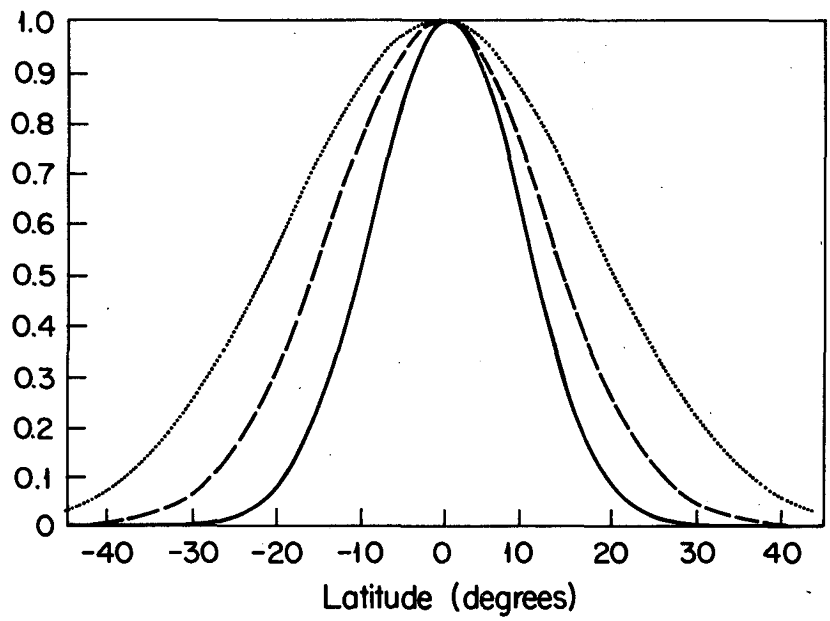

FIG. 1. The graphs of $P_{10}{}^{10}$ (dotted), $P_{20}{}^{20}$ (dashed), and $P_{40}{}^{40}$ (solid) as functions of latitude. The waves become progressively narrower, more and more closely confined about the equator, as the zonal wavenumber m increases.

ical equation for $\epsilon \neq 0$ which is given in Longuet–Higgins (1968). The conventional equatorial beta-plane is a legitimate approximation only when

$$\epsilon \gg \epsilon_{sph} \equiv m^2. \quad (10)$$

For simplicity, the analysis has been limited to free oscillations of a fluid linearized about a resting mean state, which is obviously quite unrealistic. However, the offending term $-m^2/(1-\mu^2)$ is *always* present with or without additional complications such as strong mean shear. Consequently, our major conclusion—that large zonal wavenumber may cause subtle and important effects in spherical geometry that are missed by the equatorial beta-plane—should remain true even for general circulation models.

Fortunately, most published studies of equatorial waves in the atmosphere have focused on such small m that the spherical effects of the $-m^2/(1-\mu^2)$ term are irrelevant. For high wavenumber motions, however, it is a different story. For a wave with a vertical wavelength of only 7 km, the sphericity corrections are as important as ϵ itself for $m \sim O(40)$. For motions with a wavelength as large as 30 km, the "spherical Lamb's parameter" is important even for zonal wavelengths as large as 4000 km (i.e., $m = 10$).

In the equatorial ocean, the *baroclinic* modes all have $\epsilon \sim O(100\,000)$ or larger, so the sphericity effect is completely negligible. However, $\epsilon \sim O(20)$ for the *barotropic* mode. Moore and Philander (1977) assert that "for the barotropic mode, the equatorial region cannot be treated as isolated from the rest of the ocean basin." Sphericity corrections will be very important for the barotropic mode, however, and there will be barotropic modes confined to within a few degrees of the equator that yet have zonal wavelengths of thousands of kilometers. This may mean that there is a much stronger coupling between these equatorially-trapped barotropic modes and the narrower baroclinic waves than had been previously suspected.

The moral is clear: the equatorial beta-plane approximation is not uniform in Lamb's parameter ϵ and zonal wavenumber m. When m is large, there will be waves of low latitudinal mode number which are equatorially trapped even if ϵ is small or 0. Conversely, applying the equatorial beta-plane even when $\epsilon \gg 1$ may give incorrect results if $m^2 \sim O(\epsilon)$.

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