On the Three-Dimensional Propagation of Stationary Waves

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ABSTRACT

A locally applicable (nonzonally-averaged) conservation relation is derived for quasi-geostrophic stationary waves on a zonal flow, a generalization of the Eliassen-Palm relation. The flux which appears in this relation constitutes, it is argued, a useful diagnostic of the three-dimensional propagation of stationary wave activity. This is illustrated by application to a simple theoretical model of a forced Rossby wave train and to a Northern Hemisphere winter climatology. Results of the latter procedure suggest that the major forcing of the stationary wave field derives from the orographic effects of the Tibetan plateau and from nonorographic effects (diabatic heating and/or interaction with transient eddies) in the western North Atlantic and North Pacific Oceans and Siberia. No evidence is found in the data for wave trains of tropical origin; forcing by the orographic effects of the Rocky mountains seems to be of secondary importance.

1. Introduction

Since the work of Andrews and McIntyre (1976), the Eliassen-Palm (EP) flux has been increasingly used in analyses of the propagation of wave activity and of the interaction between waves and mean flows (e.g., Edmon *et al.*, 1980; Dunkerton *et al.*, 1981). As such, it is a much more revealing and less misleading diagnostic than the frequently-applied energetic arguments, the limitations of which have become increasingly apparent (e.g., Dickinson, 1969; McIntyre, 1980; Plumb, 1983). The EP flux has been applied with particular effect in some recent studies of stratospheric warmings (e.g., Dunkerton *et al.*, 1981; Palmer, 1981) where its use has led to the identification of hitherto unrecognized processes involved in these phenomena (see the reviews of McIntyre, 1982; Plumb, 1982).

The EP flux is a zonally-averaged quantity and therefore can provide insight only into zonally-averaged latitudinal and vertical wave propagation characteristics. While, as just noted, this does not detract from its usefulness in the winter stratosphere, nor indeed, does it preclude certain tropospheric applications (Edmon *et al.*, 1980), much of the interest in the analysis of the atmospheric circulation is lost if the data are longitudinally-averaged. For example, while it is recognized that much of the stationary wave activity in the atmosphere is generated by the effects of orographic and thermal forcing in middle latitudes, the possible contribution of planetary-scale disturbances propagating into middle latitudes from tropical heat sources (e.g., Horel and Wallace, 1981; Hoskins and Karoly, 1981; Webster, 1981; Simmons, 1982) is yet to be fully assessed. The (zonally-averaged) EP flux for tropospheric stationary waves in Northern Hemisphere winter (Edmon *et al.*, 1980) shows propagation out of middle latitudes into the tropics but it is not clear whether this means that essentially all stationary wave activity propagates in this way or whether what the EP flux is describing is the sum of strong equatorward propagation at some longitudes and weaker poleward propagation elsewhere. It therefore seems desirable to define a parallel to the EP flux which permits local diagnosis of the three-dimensional circulation and thereby helps to avoid such ambiguities.

In this paper, such a flux is derived for linear, quasi-geostrophic disturbances on a zonal flow. The restriction to zonal flows permits application to stationary waves. Indeed, the flux as defined here is specific to the stationary component of atmospheric wave motions; one can in fact do better for transient waves, removing the restriction of zonal basic states (Andrews, 1983; Hoskins *et al.*, 1983; Plumb, 1985a, b).

The derivation of this flux is outlined in Section 3, where it is shown that it is possible to generalize the Eliassen-Palm theorem to define a conservable three-dimensional measure of the flux of wave activity which is nondivergent for steady, conservative waves.[1] Despite the fact that this quantity is an unaveraged quadratic function of the wave field, it is independent of the wave phase in the limit of almost-plane waves on a slowly varying basic state and, indeed, it is shown by application to a theoretical example of a planetary-scale wavetrain generated by localized forc-

[1] Andrews (1983) cites McIntyre as having been aware of this generalization.

ing (Section 6) that this property is retained in more realistic situations.

With this basis, the flux has been evaluated for the stationary wave field of the winter Northern Hemisphere as described by a 10-year climatology. Results of this calculation, which are presented in Section 7, highlight the two major wavetrains propagating across the Pacific Ocean from eastern Asia and across the North Atlantic Ocean from the east coast of North America. Implications for the interpretations of this diagnostic procedure and for our understanding of the maintenance of the stationary wave field are discussed.

2. Quasi-geostrophic waves on a zonal flow

In the first instance, the derivation that follows is restricted, for simplicity of exposition, to midlatitude beta-plane geometry. (Extension of the major results to spherical geometry will be presented in Section 4.) In quasi-geostrophic flow the eastward and northward velocities (u, v) are defined by a streamfunction ψ, with

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}. \tag{2.1}$$

The vertical gradient of ψ is related to potential temperature θ by the hydrostatic relation for a perfect gas

$$\frac{\partial\psi}{\partial z} = \frac{Rp^\kappa\theta}{fH}, \tag{2.2}$$

where R is the gas constant, f the Coriolis parameter, p = pressure/1000 mb and $z = -H \ln p$ where H is a constant scaleheight. Then, defining the quasi-geostrophic potential vorticity

$$q = f + \beta y + \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{f^2}{p}\frac{\partial}{\partial z}\left(\frac{p}{N^2}\frac{\partial\psi}{\partial z}\right), \tag{2.3}$$

where N is the buoyancy frequency, the quasi-geostrophic potential vorticity equation is

$$\frac{dq}{dt} \equiv \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = s, \tag{2.4}$$

where s represents the sources and/or sinks of potential vorticity and is zero for conservative motion.

Consider small perturbations to a steady zonal flow for which

$$u = U(y, z), \quad v = 0, \quad \psi = \Psi(y, z),$$
$$q = Q(y, z), \quad \theta = \Theta(y, z).$$

The perturbations are taken to be formally $O(\epsilon)$ in amplitude, where $\epsilon \ll 1$, so that we may expand

$$\psi = \Psi + \epsilon\psi' + O(\epsilon^2), \tag{2.5}$$

etc. Then, at $O(\epsilon)$, the perturbation potential vorticity equation becomes, from (2.4)

$$\frac{Dq'}{Dt} + v'\frac{\partial Q}{\partial y} = s', \tag{2.6}$$

where from (2.3),

$$q' = \frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2} + \frac{f^2}{p}\frac{\partial}{\partial z}\left(\frac{p}{N^2}\frac{\partial\psi'}{\partial z}\right) \tag{2.7}$$

and where

$$D/Dt = \partial/\partial t + U\partial/\partial x.$$

(Note that we may formally render (2.6) valid at finite amplitude by incorporating nonlinear effects into s'). Multiplying (2.6) by q' immediately gives the perturbation enstrophy equation

$$\frac{1}{2}\frac{Dq'^2}{Dt} + v'q'\frac{\partial Q}{\partial y} = s'q'. \tag{2.8}$$

3. The conservation relation for waves on a zonal flow

The quasi-geostrophic, zonally-averaged Eliassen-Palm theorem follows simply from (2.8) and the fact that $\overline{v'q'}$ (where the overbar denotes zonal average) may be written as a divergence (e.g., Edmon *et al.*, 1980). Generalization of this result to the nonzonally-averaged situation is a straightforward extension of the same procedure. From (2.1), and (2.7),

$$v'q' = \frac{\partial\psi'}{\partial x}\left[\frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2} + \frac{f^2}{p}\frac{\partial}{\partial z}\left(\frac{p}{N^2}\frac{\partial\psi'}{\partial z}\right)\right]$$
$$= \frac{1}{p}\nabla\cdot\mathbf{B}^{(R)} \tag{3.1}$$

where $\nabla\cdot$ is the three-dimensional divergence operator and

$$\mathbf{B}^{(R)} = \begin{pmatrix} B_x^{(R)} \\ B_y^{(R)} \\ B_z^{(R)} \end{pmatrix}$$

$$= p\begin{pmatrix} \frac{1}{2}\left(\frac{\partial\psi'}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial\psi'}{\partial y}\right)^2 - \frac{f^2}{2N^2}\left(\frac{\partial\psi'}{\partial z}\right)^2 \\ \frac{\partial\psi'}{\partial x}\frac{\partial\psi'}{\partial y} \\ \frac{f^2}{N^2}\frac{\partial\psi'}{\partial x}\frac{\partial\psi'}{\partial z} \end{pmatrix} \tag{3.2}$$

or, using (2.1) and (2.2),

$$\mathbf{B}^{(R)} = p\begin{pmatrix} v'^2 - E \\ -u'v' \\ fv'\theta'/(\partial\Theta/\partial z) \end{pmatrix} \tag{3.3}$$

since $N^2 = (Rp^\kappa/H)\partial\Theta/\partial z$, where

$$E = \frac{1}{2}\left(u'^2 + v'^2 + \frac{Rp^\kappa\theta'^2}{H\partial\Theta/\partial z}\right) \quad (3.4)$$

is the wave energy density.

Since $\partial Q/\partial y$ is independent of x and t, (2.8) may therefore be written in the form

$$\frac{DA}{Dt} + \nabla\cdot\mathbf{B}^{(R)} = C, \quad (3.5)$$

where

$$A = \frac{1}{2}p\frac{q'^2}{\partial Q/\partial y} \quad (3.6)$$

and

$$C = p\frac{q's'}{\partial Q/\partial y}. \quad (3.7)$$

Equation (3.5) is a conservation relation for wave activity, relating the rate of change following the mean flow of the density A of wave activity to the divergence of the radiative (i.e., nonadvective) flux $\mathbf{B}^{(R)}$ and sources and sinks represented by C. The flux $\mathbf{B}^{(R)}$ differs from the EP flux only by the addition of a zonal component and thus its divergence reduces to the EP flux divergence when zonal averages are taken.

It follows from (3.5) that $\mathbf{B}^{(R)}$ is nondivergent for conservative waves ($C = 0$) if the wave amplitude as measured by A is constant following the mean flow. However, most interesting applications will be with time-averaged data for which A will be constant *locally* (at a fixed point; i.e. $\langle\partial A/\partial t\rangle = 0$ when $\langle\ \rangle$ denotes a time-average). It would appear more useful, therefore, to rewrite (3.5) as

$$\frac{\partial A}{\partial t} + \nabla\cdot\mathbf{B}^{(T)} = C, \quad (3.8)$$

where

$$\mathbf{B}^{(T)} = \mathbf{B}^{(R)} + \mathbf{U}A \quad (3.9)$$

is the *total* (radiative + advective) flux of wave activity, and $\mathbf{U} = (U, 0, 0)$. Then $\mathbf{B}^{(T)}$ is nondivergent for *locally* steady, conservative waves.

4. Stationary waves

Since $\mathbf{B}^{(T)}$ is an unaveraged quadratic function of wave variables, it will in general be phase-dependent, i.e., it will include an oscillatory component on a scale of one-half the wavelength of the O(ϵ) wave field. This phase dependence can be removed for transient eddies simply by time-averaging $\mathbf{B}^{(T)}$; for quasi-stationary eddies, however, time-averaging is not equivalent to phase-averaging. Nevertheless, as will now be shown, it is possible to define the flux in such a way as to ensure that it is independent of these undesirable small-scale components, at least in the limit of almost-plane waves in a slowly varying basic state.

First note, following Edmon *et al.* (1980) and Andrews (1983) that only the divergence of $\mathbf{B}^{(T)}$ appears in (3.8) and therefore $\mathbf{B}^{(T)}$ may be replaced by any vector

$$\mathbf{F}_s = \mathbf{B}^{(T)} + \mathbf{G}, \quad (4.1)$$

where $\nabla\cdot\mathbf{G}$ is a nonconservative quantity (i.e. one which vanishes in conservative flow) since (3.8) may then be written

$$\frac{\partial A}{\partial t} + \nabla\cdot\mathbf{F}_s = C_s, \quad (4.2)$$

where

$$C_s = C + \nabla\cdot\mathbf{G} \quad (4.3)$$

is a nonconservative term. It turns out that a suitable choice for $\mathbf{G}$ for stationary waves is

$$\mathbf{G} = \frac{1}{4}p\begin{pmatrix} \dfrac{\partial^2(\psi'^2)}{\partial y^2} + \dfrac{f^2}{p}\dfrac{\partial}{\partial z}\left(\dfrac{p}{N^2}\dfrac{\partial(\psi'^2)}{\partial z}\right) - \dfrac{2r'q'}{\partial Q/\partial y} \\ -\dfrac{\partial^2(\psi'^2)}{\partial x\partial y} \\ -\dfrac{f^2}{N^2}\dfrac{\partial^2(\psi'^2)}{\partial x\partial z} \end{pmatrix}, \quad (4.4)$$

where r' is a nonconservative perturbation quantity defined by

$$\frac{\partial r'}{\partial x} = s'. \quad (4.5)$$

Note that

$$\nabla\cdot\mathbf{G} = -\frac{1}{2}p\frac{\partial(r'q')/\partial x}{\partial Q/\partial y}. \quad (4.6)$$

Now, with some manipulation using (2.7) and (3.6), substitution of (3.2), (3.9) and (4.4) into (4.1) gives

$$\mathbf{F}_s = \frac{1}{2}p\begin{pmatrix} \left(\dfrac{\partial\psi'}{\partial x}\right)^2 - \psi'\dfrac{\partial^2\psi'}{\partial x^2} \\ \quad + \left[\psi'q' + (Uq'^2 - r'q')\Big/\dfrac{\partial Q}{\partial y}\right] \\ \dfrac{\partial\psi'}{\partial x}\dfrac{\partial\psi'}{\partial y} - \psi'\dfrac{\partial^2\psi'}{\partial x\partial y} \\ \dfrac{f^2}{N^2}\dfrac{\partial\psi'}{\partial x}\dfrac{\partial\psi'}{\partial z} - \dfrac{f^2}{N^2}\psi'\dfrac{\partial^2\psi'}{\partial x\partial z} \end{pmatrix}. \quad (4.7)$$

Now, for stationary waves (2.6) gives, using (4.5),

$$U\frac{\partial q'}{\partial x} + \frac{\partial\psi'}{\partial x}\frac{\partial Q}{\partial y} = \frac{\partial r'}{\partial x},$$

whence, on integration (demanding that r', like q' and ψ' is a perturbation quantity whose integral over x vanishes) and multiplication by q' we obtain

$$\psi'q'\frac{\partial Q}{\partial y} + Uq'^2 - r'q' = 0. \tag{4.8}$$

Therefore the final term in the x-component of $\mathbf{F}_s$ in (4.7) vanishes whence

$$\mathbf{F}_s = \frac{1}{2}p \begin{pmatrix} \left(\frac{\partial\psi'}{\partial x}\right)^2 - \psi'\frac{\partial^2\psi'}{\partial x^2} \\ \frac{\partial\psi'}{\partial x}\frac{\partial\psi'}{\partial y} - \psi'\frac{\partial^2\psi'}{\partial x\partial y} \\ \frac{f^2}{N^2}\left(\frac{\partial\psi'}{\partial x}\frac{\partial\psi'}{\partial z} - \psi'\frac{\partial^2\psi'}{\partial x\partial z}\right) \end{pmatrix} \tag{4.9}$$

and the conservation relation is given by (4.2) with, from (3.7), (4.3) and (4.6),

$$C_s = \frac{p}{\partial Q/\partial y}\left(q's' - \frac{1}{2}\frac{\partial}{\partial x}(r'q')\right). \tag{4.10}$$

Now we consider the WKB limit of weakly dissipated, slowly varying waves of the form

$$\psi'(x, y, z, t) = p^{-1/2}\chi \sin\Gamma, \tag{4.11}$$

where the phase function is

$$\Gamma = kx + ly + mz - \sigma. \tag{4.12}$$

and χ, k, l, m and σ are slowly varying functions of space and time. Then $\mathbf{F}_s$ has the phase-independent form

$$\mathbf{F}_s \approx \chi^2 \begin{pmatrix} k^2 \\ kl \\ \frac{f^2km}{N^2} \end{pmatrix}. \tag{4.13}$$

Since, for stationary waves, the group velocity is

$$\mathbf{c}_g = \frac{2}{K^4}\frac{\partial Q}{\partial y}\begin{pmatrix} k^2 \\ kl \\ \frac{f^2km}{N^2} \end{pmatrix}, \tag{4.14}$$

where $K^2 = k^2 + l^2 + f^2m^2/N^2$,

$$\mathbf{F}_s = \frac{1}{2}\frac{K^2\chi^2}{\partial Q/\partial y}\mathbf{c}_g, \tag{4.15}$$

i.e. $\mathbf{F}_s$ is parallel (or opposite, if $\partial Q/\partial y < 0$) to the group velocity, in agreement with the general arguments of Hayes (1977). Note, however, that the constant of proportionality in (4.15) does not, in the WKB limit, equal the density A defined by (3.6); A is in fact phase-dependent in this limit. However it is possible to define a density[2]

$$A_s = A + p\frac{E}{U}, \tag{4.16}$$

which does reduce to the phase independent quantity $\frac{1}{2}K^2\chi^2/(\partial Q/\partial y)$ in this limit, and thereby rewrite (4.2), since $\partial A_s/\partial t = 0$ for stationary waves, in the form

$$\frac{\partial A_s}{\partial t} + \nabla\cdot\mathbf{F}_s = C_s \tag{4.17}$$

or

$$\frac{DA_s}{Dt} + \nabla\cdot\mathbf{F}_s^{(R)} = C_s, \tag{4.18}$$

where

$$\mathbf{F}_s^{(R)} = \mathbf{F}_s - \mathbf{U}A_s$$

might be regarded as the advective flux of a conservable quantity whose density is A_s. Then

$$\left.\begin{aligned} \mathbf{F}_s^{(R)} &= (\mathbf{c}_g - \mathbf{U})A_s \\ \mathbf{F}_s &= \mathbf{c}_g A_s \end{aligned}\right\}$$

in the WKB limit.

In general, then, the flux $\mathbf{F}_s$ exhibits all the advantages of the EP flux as an indicator of the propagation of wave activity. To summarize these properties, they are:

(i) $\mathbf{F}_s$ is a conservable measure of the flux of wave activity, appearing in the conservation relation (4.17), where A_s and C_s are defined by (4.16) and (4.10). For steady ($\partial A_s/\partial t = 0$) conservative ($C_s = 0$) waves, $\mathbf{F}_s$ is nondivergent.

(ii) For westerly flows, $U > 0$, A_s is positive definite. Then convergence of $\mathbf{F}_s$ indicates the piling-up of wave activity, while divergence of $\mathbf{F}_s$ indicates its export.

(iii) In the limit of almost-plane waves, $\mathbf{F}_s$ is a phase-independent quantity which is parallel to the group velocity.

(iv) If zonal averages are taken, $\mathbf{F}_s$ reduces to the EP flux, except for the addition of a zonal component which is of no particular consequence for the zonally-averaged case. This may be seen directly from (3.3), (3.9) and (4.1), noting from (4.4) that the y- and z-components of $\mathbf{G}$ vanish on zonal averaging.

(v) From (4.17) the divergence of $\mathbf{F}_s$, and therefore the generation or dissipation of wave activity, is directly related to nonconservative effects (or nonlinearity). In addition, boundaries may also be sources or sinks. For example it is possible to show that, even for conservative flow, the upward flux of $\mathbf{F}_s$ from a horizontal boundary is generally zero only if $w' = 0$ there. Therefore topography may of course also be a

[2] Note the difference between this density and that defined by Andrews (1983) and Plumb (1985a) as an Eulerian analogue of "pseudoenergy." The latter is approximated by $pE - AU$ for conservative flow, and in fact vanishes for stationary waves in the WKB limit. More generally the true pseudoenergy of generalized Lagrangian-mean theory (Andrews and McIntyre, 1978) vanishes for any stationary wave.

source of stationary wave activity, although such effects do not appear explicitly in (4.2).

5. Extension to spherical geometry

On the sphere, the nondivergent geostrophic wind velocities are

$$\left.\begin{aligned} u &= -\frac{1}{a}\frac{\partial\psi}{\partial\phi} \\ v &= \frac{1}{a\cos\phi}\frac{\partial\psi}{\partial\lambda} \end{aligned}\right\}, \tag{5.1}$$

where the streamfunction $\psi = \Phi/2\Omega\sin\phi$, where Φ is the geopotential and Ω the Earth's rotation rate, a the Earth's radius and (ϕ, λ) are respectively latitude and longitude. As always with quasi-geostrophic theory, it is implicitly assumed that the latitudinal scale of the motion is much smaller than a. The quasi-geostrophic potential vorticity is, in terms of ψ,

$$q = \frac{1}{a^2\cos^2\phi}\frac{\partial^2\psi}{\partial\lambda^2} + \frac{1}{a^2\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial\psi}{\partial\phi}\right) + \frac{4\Omega^2}{p}\sin^2\phi\frac{\partial}{\partial z}\left(\frac{p}{N^2}\frac{\partial\psi}{\partial z}\right); \tag{5.2}$$

this satisfies the equation $dq/dt = s$.

The extension of the nonzonally averaged results to spherical geometry proceeds, as in Sections 3 and 4, by multiplying (5.2) by v' to give

$$pv'q' = \frac{1}{\cos\phi}\nabla\cdot\mathbf{B}^{(R)}, \tag{5.3}$$

where now

$$\mathbf{B}^{(R)} = \begin{pmatrix} B_\lambda \\ B_\phi \\ B_z \end{pmatrix} = p\cos\phi\begin{pmatrix} v'^2 - E \\ -u'v' \\ 2\Omega\sin\phi v'\theta'/(\partial\Theta/\partial z) \end{pmatrix}, \tag{5.4}$$

where E is defined by (3.4). Then, multiplying (2.6) by q' leads once again to the conservation relation (3.5) with

$$A = \tfrac{1}{2}pq'^2 a\cos^2\phi/(\partial Q/\partial\phi), \tag{5.5}$$

$$C = ps'q'a\cos^2\phi/(\partial Q/\partial\phi). \tag{5.6}$$

The total flux $\mathbf{B}^{(T)}$ is still given by (3.9) and we may once again use (4.1) to define a modified flux $\mathbf{F}_s$ which is phase-independent and parallel to $\mathbf{c}_g$ for all waves in the almost-plane-wave limit.

Following the procedures of Section 4, the flux $\mathbf{F}_s$ now takes the form

$$\mathbf{F}_s = p\cos\phi\begin{pmatrix} \dfrac{1}{2a^2\cos^2\phi}\left[\left(\dfrac{\partial\psi'}{\partial\lambda}\right)^2 - \psi'\dfrac{\partial^2\psi'}{\partial\lambda^2}\right] \\ \dfrac{1}{2a^2\cos\phi}\left(\dfrac{\partial\psi'}{\partial\lambda}\dfrac{\partial\psi'}{\partial\phi} - \psi'\dfrac{\partial^2\psi'}{\partial\lambda\partial\phi}\right) \\ \dfrac{2\Omega^2\sin^2\phi}{N^2 a\cos\phi}\left(\dfrac{\partial\psi'}{\partial\lambda}\dfrac{\partial\psi'}{\partial z} - \psi'\dfrac{\partial^2\psi'}{\partial\lambda\partial z}\right) \end{pmatrix}. \tag{5.7}$$

Thus defined, $\mathbf{F}_s$ has the properties (i)–(v) of Section 4. Demonstration of this fact is a straightforward application of the same arguments as used there and will not be repeated here.

6. Example: linear, stationary, barotropic response to a localized vorticity forcing

To illustrate the application of the wave activity flux $\mathbf{F}_s$ consider a simple barotropic model of the atmospheric response to a localized vorticity forcing (e.g. Grose and Hoskins 1979). The perturbation vorticity is, from (4.2),

$$\zeta' = \frac{1}{a^2}\left(\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial\psi'}{\partial\phi}\right) + \frac{1}{\cos^2\phi}\frac{\partial^2\psi'}{\partial\lambda^2}\right) \tag{6.1}$$

for which the governing equation is, from (2.8),

$$\left(\frac{\partial}{\partial t} + \frac{U}{a\cos\phi}\frac{\partial}{\partial\lambda}\right)\zeta' + \frac{v'}{a}\frac{dQ}{d\phi} = S', \tag{6.2}$$

where

$$\frac{dQ}{d\phi} = 2\Omega\cos\phi - \frac{d}{d\phi}\left(\frac{1}{a\cos\phi}\frac{d}{d\phi}(U\cos\phi)\right). \tag{6.3}$$

The source/sink term is taken as

$$S' = X - \gamma\zeta', \tag{6.4}$$

where $\gamma = 1.0 \times 10^{-6}$ s^{-1} is the Rayleigh friction coefficient and X is the vorticity forcing. The latter is taken to be topography-like, with a dipolar structure in longitude, viz.,

$$X = \frac{\lambda}{\Delta}\exp[-(\phi - \phi_0)^2/2\Delta^2]\exp[-\lambda^2/2\Delta^2], \tag{6.5}$$

where $\Delta = 15°$ and $\phi_0 = 30°$. The assumed mean wind profile is a simple midlatitude jet

$$U(\phi) = U_0\sin^2 2\phi, \tag{6.6}$$

where $U_0 = 30$ m s^{-1}.

Equation (6.2) was solved by a spectral expansion in longitude

$$\psi'(\lambda, \phi) = \sum_{m=1}^{M}\chi_m(\phi)e^{im\lambda} + \chi_m^*(\phi)e^{-im\lambda}, \tag{6.7}$$

(where the asterisk denotes complex conjugate), truncating the series at $M = 15$. The resulting ordinary differential equation for $\chi_m(\phi)$ was then solved by finite differencing ($\Delta\phi = 6°$) using standard techniques subject to the boundary conditions, $\chi_m(0) = 0$ at the equator and the pole.

The geopotential field $\Phi'(\lambda, \phi) = 2\Omega\psi' \sin\phi$ thus calculated is shown in Fig. 1; the response is a decaying wavetrain trailing eastward from the forcing region and, as has been noted in the context of previous calculations of this type (e.g. Hoskins and Karoly, 1981; Webster, 1981) it is reminiscent of the "teleconnection" patterns which have been identified as a component of the Northern Hemisphere circulation (e.g., Wallace and Gutzler, 1981).

The vector $\mathbf{F}_s$ was evaluated from the calculated fields of ψ' and then mapped onto polar stereographic coordinates (r, α) by the transformation

$$\left.\begin{aligned} F_r &= -F_\phi/m(\phi) \\ F_\alpha &= F_\lambda/m(\phi) \end{aligned}\right\}, \tag{6.8}$$

where

$$m(\phi) = 2/(1 + \sin\phi) \tag{6.9}$$

is the map factor for the polar stereographic projection of the sphere. This mapping preserves the sign of the divergence since

$$(\nabla \cdot \mathbf{F})_G = m^2(\phi)(\nabla \cdot \mathbf{F})_P, \tag{6.10}$$

where the subscripts G and P refer to the spherical and polar coordinate systems respectively.

The flux (F_r, F_α) is plotted in Fig. 2. The pattern thus revealed is consistent with our interpretation of the response shown in Fig. 1 as a wavetrain trailing eastward from the wave source at $(\lambda, \phi) = (0°, 30°)$. It is also in accord with the ray tracing results of Hoskins and Karoly (1981; e.g. their Fig. 12), since in the slowly-varying limit $\mathbf{F}_s$ should be parallel to the local group velocity and therefore along their rays. The weakness of the flux close to the pole is probably a consequence of the weak zonal flow in that region, which prevents wave propagation into high latitudes.

Note that the strongest divergence of $\mathbf{F}_s$ occurs to the southeast of the center of the vorticity source, rather than at the peak of the effective topography. This is not altogether surprising, given the form of

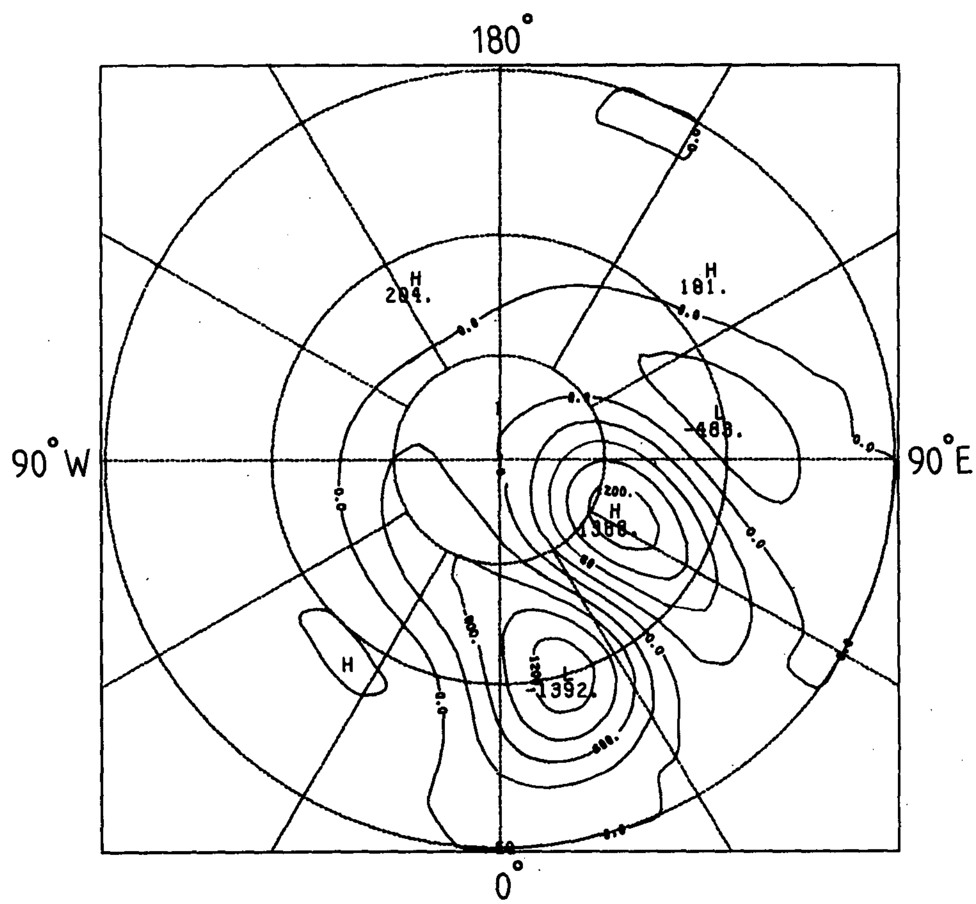

FIG. 1. Geopotential response to a localized vorticity forcing centered on the Greenwich meridian at latitude 30°N. Solution of (6.2) with the forcing (6.5). Amplitude is arbitrary.

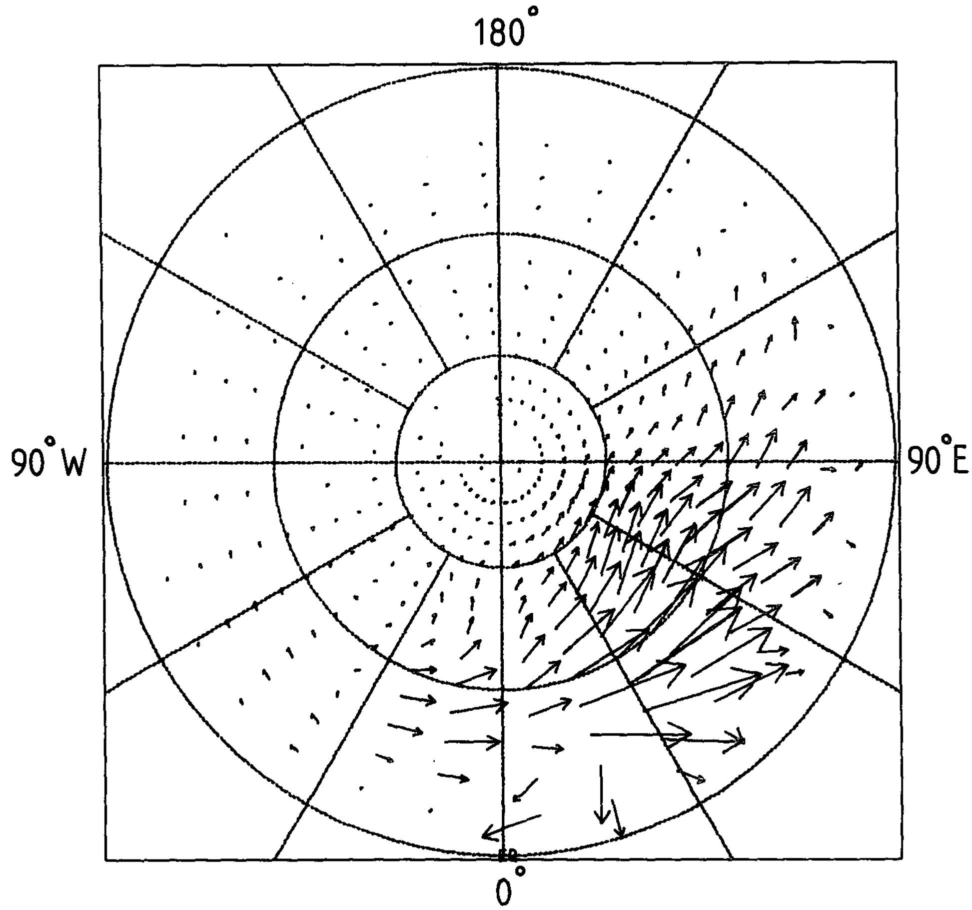

FIG. 2. Wave activity flux for the geopotential pattern of Fig. 1 (scale is arbitrary). See text for discussion.

(4.10); the source term in (4.17) is not a simple linear function of the vorticity source. Note however that this term must vanish where the vorticity source vanishes. The fact that the divergence of $\mathbf{F}_s$ from a topographic source does not necessarily coincide with the highest topography needs to be borne in mind when interpreting atmospheric data, as will be seen in the following section.

7. Application to a Northern Hemisphere winter climatology

The results of the preceding example—which did not incorporate any assumption of a slowly-varying mean state or wave field—gives grounds to believe that the wave activity flux defined in this paper will be a useful diagnostic in the analysis of stationary waves in the atmospheric circulation.

In order to illustrate its use, $\mathbf{F}_s$ was evaluated from a climatological data set of 10 Northern Hemisphere (north of 20°N) winters (where "winter" here means the 120-day period beginning 15 November) based on daily NMC analyses from 1965 to 1975. This data set was kindly made available to the author by Dr. N. C. Lau; the data set itself, and the properties of the stationary waves it describes, are discussed more fully in Lau (1979).

The stationary flow was determined simply as the time-averaged flow over the entire 10 winters; the stationary waves were then obtained by removing the zonal averages of Φ, u, v and T at each latitude and height. In order to reduce the amplification of noise by successive differentiation, use was made of the geostrophic and thermal wind relations to rewrite (5.7) as

$$\mathbf{F}_s = p\cos\phi \times \begin{pmatrix} v'^2 - \dfrac{1}{2\Omega a \sin 2\phi}\dfrac{\partial(v'\Phi')}{\partial\lambda} \\ -u'v' + \dfrac{1}{2\Omega a \sin 2\phi}\dfrac{\partial(u'\Phi')}{\partial\lambda} \\ \dfrac{2\Omega \sin\phi}{S}\left[v'T' - \dfrac{1}{2\Omega a \sin 2\phi}\dfrac{\partial}{\partial\lambda}(T'\Phi')\right] \end{pmatrix} \quad (7.1)$$

where

$$S = \frac{\partial \hat{T}}{\partial z} + \frac{\kappa \hat{T}}{H} \quad (7.2)$$

is the static stability, the caret indicating an areal average over the area north of 20°N. In the form (7.1) calculation of $\mathbf{F}_s$ involves only first derivatives of observed quantities.

The stationary circulation, exemplified by the 250 mb height field shown in Fig. 3, shows the familiar climatological winter Northern Hemisphere flow; while the dominant features are the deep troughs over the east coast of Asia and eastern North America, the pattern of troughs and ridges pervades most of the Hemisphere. The activity flux $\mathbf{F}_s$, on the other hand, is much more localized, especially in the lower and middle troposphere; the flux is illustrated in Figs. 4–6.

The first point to note from Fig. 4 is that the vertical component of $\mathbf{F}_s$ is directed almost entirely upward, especially at upper levels; this is consistent with the conventional understanding of the stationary wave field propagating upward from below. Note, however, that this was not entirely true of the uncorrected flux $\mathbf{B}^{(T)}$ (not shown) which has more small-scale structure and some regions of substantial downward flux. The fact that these features are largely absent from $\mathbf{F}_s$ is an indication that the procedure of Section 4 to remove the phase dependence of the flux has been successful in practice.

The major features highlighted by Fig. 4 are two distinct wavetrains spreading upward, eastward and predominantly equatorward from eastern Asia across the North Pacific Ocean and from eastern North America across the North Atlantic. There is also the suggestion of a smaller and less intense wavetrain propagating upwards and eastwards from western North America. Smaller features apparent at low and middle tropospheric levels, but not higher, are located over central and eastern Europe. What is particularly remarkable about Fig. 4, in view of recent interest in the matter, is that there is no suggestion of any propagation of stationary waves out of the tropics into middle latitudes. Neither is there much support for suggestions (Held, 1983) that the major equatorward propagating wavetrains might be reflected poleward from low latitudes, although there may be a hint of this occurring weakly above 250 mb at about 110°W and 15°E (Fig. 4b).

Comparing the two major wavetrains, it is clear from Figs. 4 and 5 that the North Pacific wavetrain is both more intense and more extensive than that in

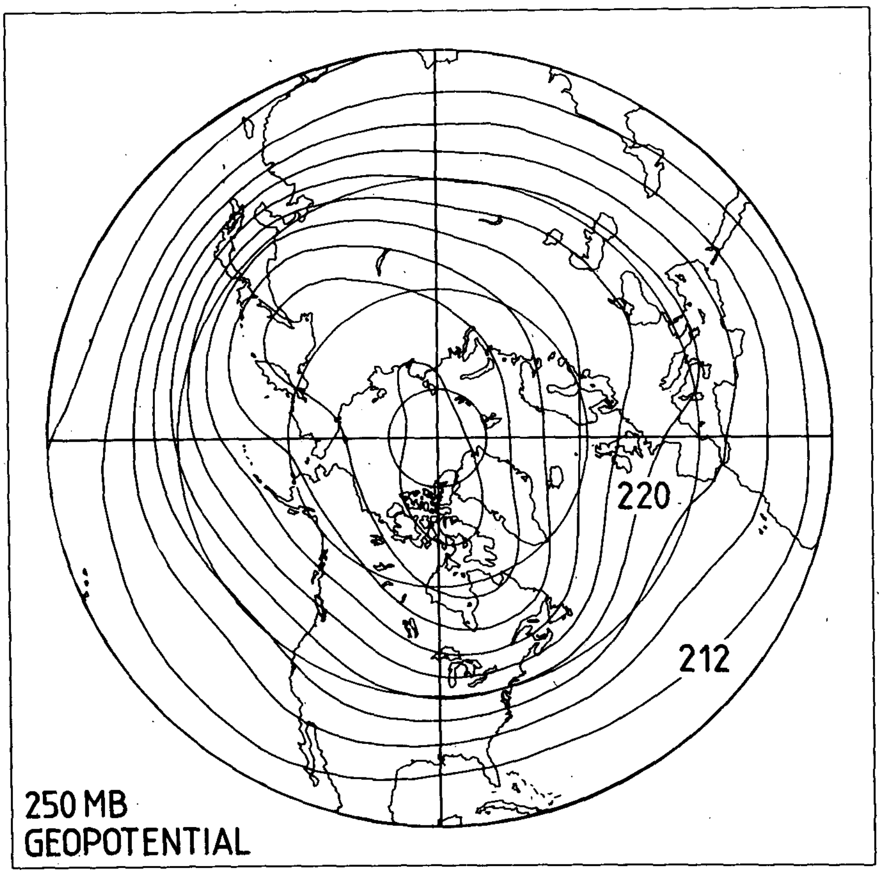

FIG. 3. Time-mean geopotential height field (dam) for northern winter as determined by the data set analyzed in Section 7.

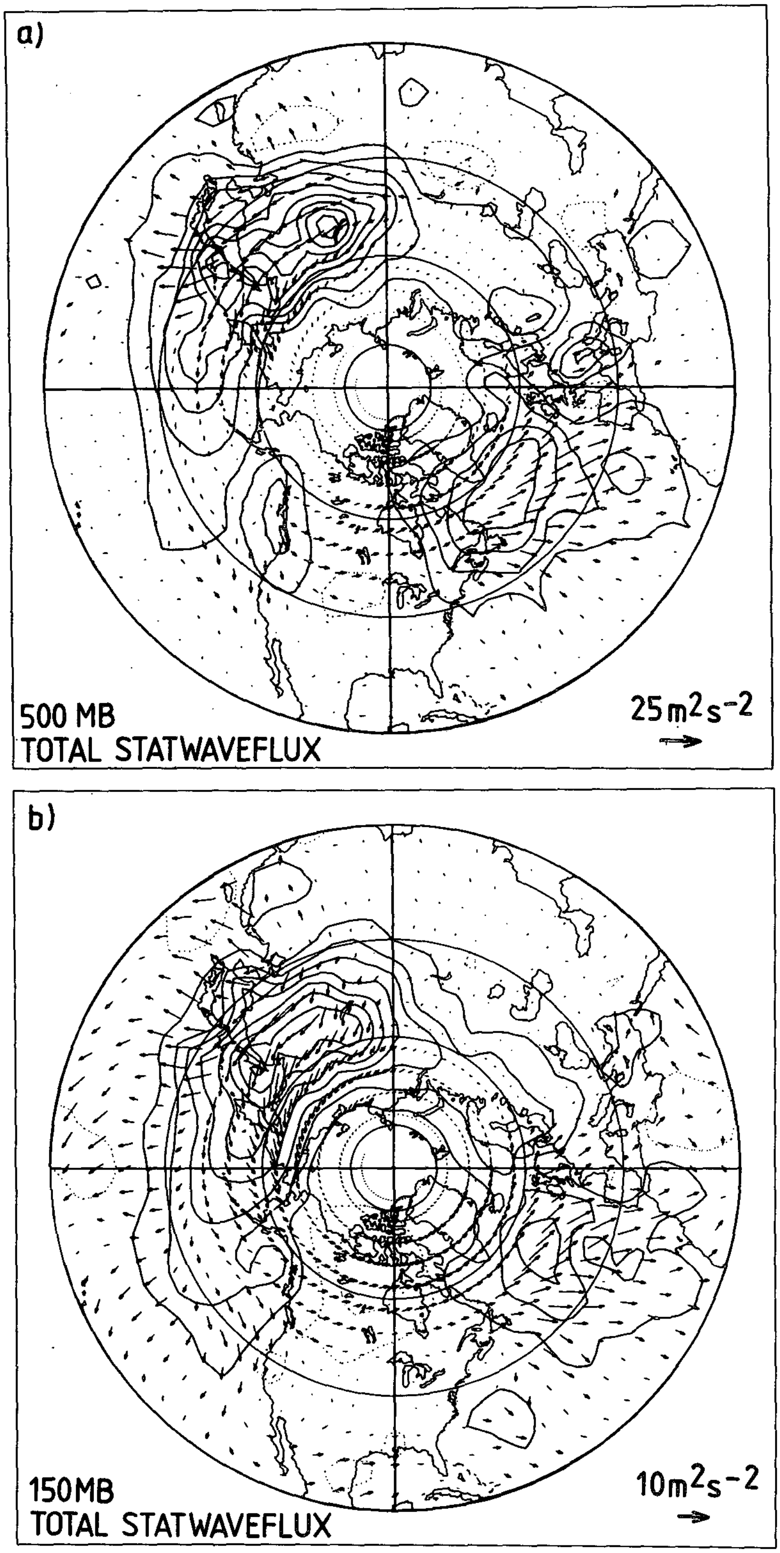

FIG. 4. Activity flux $\mathbf{F}_s$ for the Northern Hemisphere winter stationary wave field at (a) 500 mb, $\Delta = 4.12 \times 10^{-2}$ m^2 s^{-2}, and (b) 150 mb, $\Delta = 0.75 \times 10^{-2}$ m^2 s^{-2}; polar stereographic projection. Arrows, horizontal component; contours, vertical component. Contours are plotted at values $(n + \frac{1}{2})\Delta$ where Δ is the contour interval; contours are solid for $n > 0$. The scales for the horizontal fluxes, which have been mapped onto the polar stereographic projection according to (6.8), are shown in the insets.

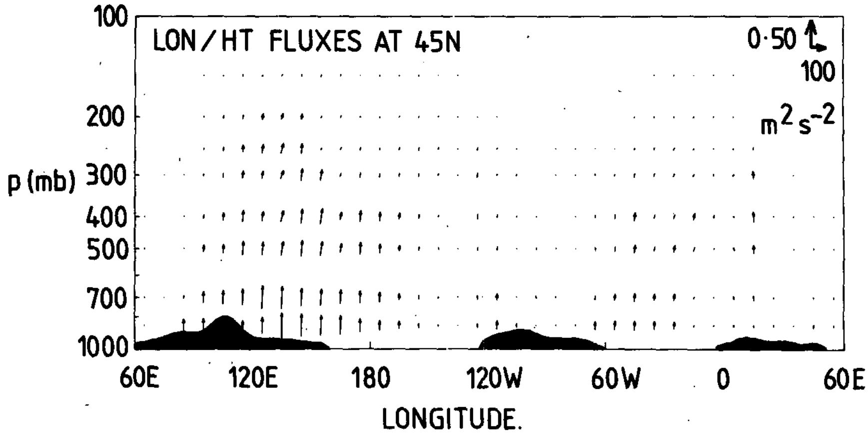

FIG. 5. Longitude–height projection of $\mathbf{F}_s$ at 45°N. The topography at 45°N is shown at the bottom of the figure.

the North Atlantic; for example, the vertical flux through the 500 mb surface is about twice as large for the former than for the latter. Another difference is that the North Pacific wavetrain shows evidence of the splitting which is characteristic of some ray-tracing calculations (Hoskins and Karoly, 1981) with one branch propagating slightly north of eastward and the other south-eastward, while the North Atlantic wavetrain is directed almost entirely south-eastward without much evidence of a bifurcation. This is shown more clearly in meridional cross-sections of $\mathbf{F}_s$ at 30°W and 150°E, just downstream of the regions of maximum upward flux of the North Atlantic and North Pacific wavetrains, respectively (Fig. 6). The former (Fig. 6a) radiates upward from 45–65°N, refracting equatorward in much the same way as the zonally-averaged EP flux (Edmon *et al.*, 1980; recall that the latitudinal and vertical components of $\mathbf{F}_s$ reduce to the EP flux when zonally averaged). Very little activity propagates up beyond 200 mb. In the North Pacific region, however, the splitting evident in Fig. 4 shows up clearly in the meridional cross-section of Fig. 6b, at around 45°N and 250 mb, with the northern portion of the wavetrain radiating slightly poleward and upward into the stratosphere. In fact, it appears from Fig. 4b that the flux of stationary wave activity into the stratosphere is dominated by the North Pacific wavetrain (more so than at midtropospheric levels).

The locations of the apparent origins of the two major wavetrains do not appear to be entirely consistent with a purely orographic generation process. The North Pacific wavetrain (see Figs. 4a and 5) appears to originate primarily on the northeastern slopes of the Tibetan plateau, rather than on the highest orography. This in itself is not inconsistent with orographic generation—recall Fig. 2 and the discussion thereof in Section 6. However, a significant portion of the wave activity seems to originate so far north and, especially, so far east over the western North Pacific that it is difficult to relate this directly to orography. The difficulty is more acute for the North Atlantic wavetrain which, apart from the weak contribution propagating across North America from the west, appears to originate entirely over the ocean and the east coast of Canada, well away from the Rocky mountains. Nonlinearity may be playing some part in this. It is conceivable, for example, that an orographically forced disturbance may be highly nonlinear in the weak low-level flow and that the effective source for the linear propagation mechanism (which, after all, is what is manifested in $\mathbf{F}_s$) may appear some distance downstream of the orography. However, it is difficult to find positive evidence for such effects.

Perhaps the most obvious alternative forcing mechanism is diabatic heating. We have a reasonably good idea of its distribution; an estimate of the time-mean diabatic heating rate at 700 mb for this data set is presented in Fig. 21 of Lau (1979). While calculation of this quantity is indirect and therefore prone to error, its structural features are confirmed by independent observational analyses of Brown (1964) and Geller and Avery (1978) (see the discussion in Lau, 1979) and by heating rates in numerical models (e.g., Held, 1983). These features are broad heating maxima over the western oceans (presumably arising from latent heat release in the storm tracks) and cooling over continental North America and Eurasia. In fact there seems to be good correspondence between the regions of diabatic heating and cooling and $\mathbf{F}_s$ to the extent that the regions of strong upward flux in the lower and middle troposphere over the western North Pacific and North Atlantic correspond closely with the regions of strong gradient in the diabatic heating to the north of the heating maxima. Further, even the local maximum of the vertical component of $\mathbf{F}_s$ at 500 mb (Fig. 4a) over Siberia at about (50°N, 110°E) coincides with a strong gradient between weak

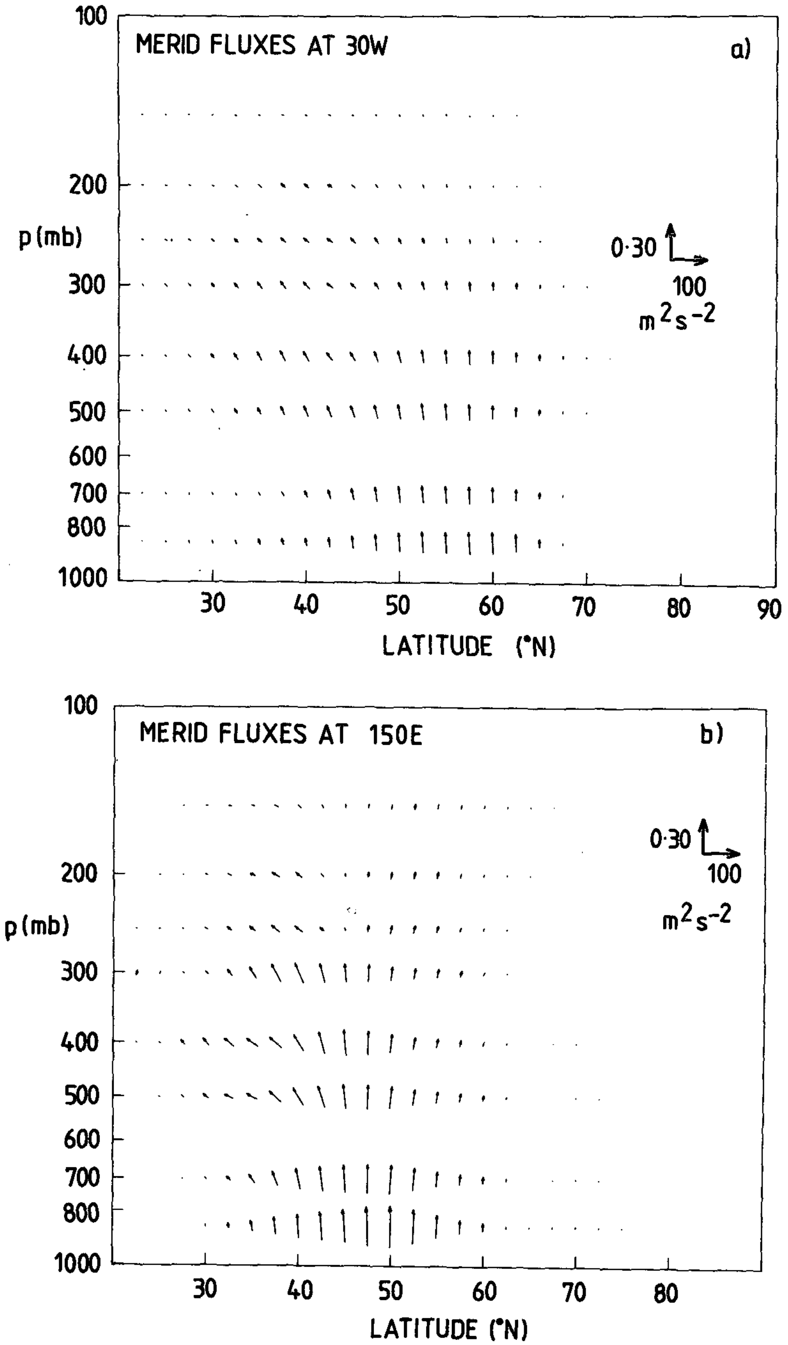

FIG. 6. Latitude–height projections of $\mathbf{F}_s$ at (a) 30°W and (b) 150°E.

heating to the south and strong cooling to the north (cf. Fig. 4a and Fig. 21 of Lau, 1979). As in the case of orographic forcing, there is no reason to expect the maximum flux generation to coincide with the maximum heating, although neither is there any obvious reason to expect a relation with heating gradient. Nevertheless, these coincidences suggest that there may be a substantial role for diabatic heating in the forcing of the stationary wave field.

A second alternative source of excitation, given that much of the propagating stationary wave activity appears to originate in the region of the two major storm tracks, is interaction with the transient eddies. To some extent this interaction has been included in the previous paragraph, since much of the diabatic heating is associated with latent heat release in traveling systems. Additionally, however, there are the effects of heat and momentum transport by the transients. In fact it will be argued in a following paper (Plumb, 1985b) that the stationary flow in the region of the storm tracks shows features which are qualitatively in accord with the expected response to transient eddy forcing. Further, a number of studies (e.g., Opsteegh and Vernekar, 1982) have demonstrated the quantitative impact of transient eddy forcing on the stationary flow. Unfortunately, the present analysis of the *location* of the apparent sources of stationary wave activity does not permit an assessment of the *nature* of the forcing within the storm tracks, so that we are unable on this basis to discriminate between latent heating and eddy transport mechanisms.

8. Discussion

The apparently successful application of $\mathbf{F}_s$ to general circulation data lends credence to the theoretical expectation that this quantity will be a useful diagnostic of three-dimensional Rossby-like propagation of stationary waves. The results of Section 7 present a clearer picture of the sources and sinks of wave activity than has been possible hitherto and indeed, these results have raised, and suggested answers to, questions that are central to our understanding of the maintenance of the stationary wave field.

One such result is the apparent absence of any significant propagation out of the tropics into middle latitudes in this climatology of ten Northern Hemisphere winters. Even given that the observational coverage in low latitudes over the oceans is relatively poor and therefore that the data may be in error in such regions, it seems unlikely that a substantial stationary wavetrain of large spatial scale could be missed altogether. It is perhaps possible that, for example, a wavetrain propagating across the Pacific Ocean from Indonesia could be masked by the more intense disturbance emanating from eastern Asia but, given the structure of the results in Figs. 4 and 6, this also seems unlikely. In any case, the absence of any signal in this long-term mean data does not of course preclude the existence of such wavetrains in anomalous seasons nor, indeed, does it preclude teleconnections between the tropics and higher latitudes via processes other than Rossby-like wave propagation.

The geographical location of the source regions of the midlatitude wavetrains gives some clues as to their generation mechanisms. Thus the results suggest nonorographic effects are a major source for the North Atlantic wavetrain and a substantial contributor to that in the North Pacific region. While the results are consistent with a predominantly orographic forcing in the latter case, the Rocky mountains do not appear to contribute much to the former.

If the transient eddies are an important source of stationary wave activity (either via direct interaction or via latent heat release) this of course arises because of the longitudinal asymmetry of the storm tracks; the processes *ultimately* responsible for this component of the stationary wave field could then rest with the mechanisms which induce these asymmetries. The present interpretation thus leads us to endorse the statement of Held (1983) that "A theory of the thermally forced stationary eddies thus requires a theory for the location of the storm tracks" with the proviso that "thermally forced" may need to be extended to include the influence of transient eddy transports.

It is important to note, however, that assessments such as these of the relative importance of various generation mechanisms depend on how one measures the response. For example, it is quite possible that the response to thermal forcing might dominate the field of $\mathbf{F}_s$ while orographic forcing dominates the geopotential height response. In fact, on the basis of GCM experiments with and without mountains, Held (1983) argued that these two forcing effects are of comparable importance, as measured by the geopotential height field. Even there, there are suggestions in his comparison (his Fig. 6.2a) that nonorographic forcing dominates the geopotential response over the North Atlantic.

In any case, one must beware of placing too much emphasis on the details of $\mathbf{F}_s$ at this stage, given lack of experience with this diagnostic procedure. It is not clear in practice just how locally $\mathbf{F}_s$ may be interpreted (but note that locating effective sources and sinks through $\nabla \cdot \mathbf{F}_s$ or boundary fluxes in independent of the manipulations applied to make $\mathbf{F}_s$ phase-independent). In fact it must be realised that $\mathbf{F}_s$ is not strictly a local quantity at all, since the stationary wave field is defined as a departure from the zonally-averaged flow. Like any diagnostic quantity, the potential value of $\mathbf{F}_s$ is that it will provide a discriminating vehicle for the comparison of theoretical predictions with observed reality. Future application of this procedure to results of theoretical stationary wave models and controlled general circulation model experiments will hopefully confirm or invalidate these conclusions.

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