An Alternative Form of Andrews' Conservation Law for Quasi-geostrophic Waves on a Steady, Nonuniform Flow

R. ALAN PLUMB

CSIRO Division of Atmospheric Research, Aspendale, Australia, 3195

2 May 1984

ABSTRACT

This note reports the derivation of an alternative form of Andrews' conservation law for small-amplitude, quasi-geostrophic, transient eddies on a steady but spatially nonuniform flow.

1. Introduction

Recently, Andrews (1983) derived a conservation law for small-amplitude, quasi-geostrophic waves on a steady but zonally nonuniform flow. This law parallels the generalized Eliassen-Palm theorem (e.g., Edmon *et al.*, 1980) for zonally averaged statistics on a zonal mean flow; the departure from zonal symmetry opens the possibility of using analogous diagnostic procedures for, say, time-averaged statistics.

In this note, an alternative form of Andrews' relation is derived from the eddy energy and enstrophy equations. While the expression derived here is equivalent to that of Andrews, the fluxes involved differ; that defined here is linked in a more obvious way with the energy fluxes which appear in conventional and in transformed Eulerian-mean formalisms. Implications of the differences are discussed.

2. The eddy energy budget

The momentum and thermodynamic equations in log-pressure coordinates (with $z = -H \ln p$, where H is a constant) may be written, for quasi-geostrophic flows on a beta-plane with Coriolis parameter $f = f_0 + \beta y$,

$$\frac{d\mathbf{u}}{dt} - \beta y \mathbf{k} \times \mathbf{u}_a - f_0 \mathbf{k} \times \mathbf{u}_a = \mathbf{X}, \qquad (2.1)$$

$$\frac{d\theta}{dt} + w_a \frac{\partial \theta}{\partial z} = Q, \qquad (2.2)$$

where $\mathbf{u} = (u, v, 0)$ is the geostrophic wind velocity (as defined below), $\mathbf{u}_a = (u_a, v_a, w_a)$ the ageostrophic wind, θ potential temperature, $\mathbf{k}$ the unit vector in the vertical direction, $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$, $\mathbf{X}$ is the frictional force per unit mass and Q is proportional to the diabatic heating rate per unit mass. Through the geostrophic and thermal wind relations, $\mathbf{u}$ and θ are defined in terms of the geostrophic streamfunction $\psi = \phi/f_0$ (where ϕ is the geopotential) by

$$\left.\begin{aligned} \mathbf{u} &= \mathbf{k} \times \nabla\psi \\ \theta &= \frac{f_0 H}{R} p^{-\kappa} \frac{\partial \psi}{\partial z} \end{aligned}\right\}, \qquad (2.3)$$

where R is the gas constant and $\kappa = R/c_p$. The geostrophic wind $\mathbf{u}$ is, of course, horizontally nondivergent while the ageostrophic component satisfies the log-pressure mass continuity equation

$$\nabla\cdot(p\mathbf{u}_a) = 0. \qquad (2.4)$$

If we now look at small amplitude [formally $O(\delta)$, say] perturbations ψ', $\mathbf{u}'$, $\mathbf{u}_a'$, θ' to a steady basic state $\bar{\mathbf{u}}(x, y, z)$, $\bar{\theta}(x, y, z)$ (where the overbar denotes time-averaging) it is straightforward to linearize (2.1) and (2.2), take the dot product of $\mathbf{u}'$ with the former and the product of θ' with the latter, time-average and then linearly combine the results to derive an eddy energy equation, correct to $O(\delta^2)$, in the form

$$p\frac{D\epsilon}{Dt} + \nabla\cdot\hat{\mathbf{E}} = C + s_\epsilon, \qquad (2.5)$$

where

$$\epsilon = \frac{1}{2}\overline{\mathbf{u}'\cdot\mathbf{u}'} + \frac{Rp^\kappa\overline{\theta'^2}}{2H\partial\bar{\theta}/\partial z} \qquad (2.6)$$

is the perturbation energy density, $D/Dt = \partial/\partial t + \bar{\mathbf{u}}\cdot\nabla$ is the time derivative following the mean geostrophic flow,

$$\hat{\mathbf{E}} = p\overline{\mathbf{u}_a'\phi'} \qquad (2.7)$$

is the conventional eddy energy flux,

$$C = -p\overline{\mathbf{u}'\cdot(\mathbf{u}'\cdot\nabla)}\bar{\mathbf{u}} - \frac{pf_0}{\partial\bar{\theta}/\partial z}\mathbf{k}\cdot\overline{\mathbf{u}'\theta'} \times \frac{\partial\bar{\mathbf{u}}}{\partial z} \qquad (2.8)$$

is the mean-to-eddy energy conversion and

$$s_\epsilon = p\overline{\mathbf{u}'\cdot\mathbf{X}'} + \frac{R_p{}^{1+\kappa}\overline{\theta' Q'}}{H\partial\bar{\theta}/\partial z} \qquad (2.9)$$

is the nonconservative source of eddy energy. (Here a "nonconservative" quantity is defined to be one which vanishes for conservative flow, i.e., when $\mathbf{X}$ and Q are both zero).

Introducing the quasi-geostrophic potential vorticity

$$q = f_0 + \beta y + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + \frac{f_0}{p}\frac{\partial}{\partial z}\left(\frac{p\theta}{\partial\bar{\theta}/\partial z}\right), \quad (2.10)$$

it is straightforward, using the geostrophic relations (2.3), to show that

$$\left.\begin{aligned} \overline{u'q'} &= \frac{\partial}{\partial x}(\overline{u'v'}) + \frac{\partial}{\partial y}(\epsilon - \overline{u'^2}) + \frac{\partial}{\partial z}\left(\frac{f_0\overline{u'\theta'}}{\partial\bar{\theta}/\partial z}\right) \\ \overline{v'q'} &= \frac{\partial}{\partial x}(\overline{v'^2} - \epsilon) - \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}\left(\frac{f_0\overline{v'\theta'}}{\partial\bar{\theta}/\partial z}\right) \end{aligned}\right\};$$

hence, with some manipulation of (2.7) and (2.8), (2.4) may be recast into a form compatible with transformed Eulerian-mean energetics (Plumb, 1983), viz.

$$p\frac{D\epsilon}{Dt} + \nabla\cdot\mathbf{E} = \overline{\mathbf{u}'q'}\cdot\nabla\bar{\psi} + s_\epsilon \quad (2.11)$$

where

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = p\begin{pmatrix} \overline{u'_a\phi'} + \bar{u}(\epsilon - \overline{v'^2}) + \bar{v}\overline{u'v'} \\ \overline{v'_a\phi'} + \bar{u}\overline{u'v'} + \bar{v}(\epsilon - \overline{u'^2}) \\ \overline{w'_a\phi'} - \dfrac{f_0}{\partial\bar{\theta}/\partial z}[\bar{u}\overline{v'\theta'} - \bar{v}\overline{u'\theta'}] \end{pmatrix}. \quad (2.12)$$

3. The eddy enstrophy budget and the conservation law

The quasi-geostrophic potential vorticity equation

$$\frac{dq}{dt} = s \quad (3.1)$$

(where s is the nonconservative source/sink of potential vorticity) yields, after linearization, multiplication of q' and averaging, the eddy enstrophy equation [correct to $O(\delta^2)$]

$$\frac{De}{Dt} + \overline{\mathbf{u}'q'}\cdot\nabla\bar{q} = \overline{s'q'} \quad (3.2)$$

(e.g., Rhines and Holland, 1979) where $e = \frac{1}{2}\overline{q'^2}$ is the eddy potential enstrophy density.

Now, following a procedure similar to that of Andrews (1983) define

$$L = \frac{\nabla_2\bar{\psi}\cdot\nabla_2\bar{q}}{|\nabla_2\bar{q}|^2}, \quad (3.3)$$

where ∇_2 is the horizontal gradient operator. Then, given that the steady background state satisfies, from (3.1),

$$\bar{\mathbf{u}}\cdot\nabla\bar{q} = \bar{s}. \quad (3.4)$$

A little manipulation shows that

$$L\nabla_2\bar{q} = \nabla_2\bar{\psi} + \frac{\bar{s}}{|\nabla_2\bar{q}|}\mathbf{k}\times\nabla\bar{q}. \quad (3.5)$$

Therefore (2.11) and (3.2) may be linearly combined to give

$$\frac{D\mathscr{E}}{Dt} + \nabla\cdot\mathbf{E} = S_E \quad (3.6)$$

where

$$\mathscr{E} = p(\epsilon + Le), \quad (3.7)$$

$$S_E = s_\epsilon + Lp\overline{s'q'} - \frac{\bar{s}}{|\nabla_2\bar{q}|}(\mathbf{k}\cdot\overline{\mathbf{u}'q'}\times\nabla\bar{q}) + e\frac{DL}{Dt}. \quad (3.8)$$

Here S_E is a purely nonconservative term; clearly this is true of the first three terms of (3.8) since s and s_ϵ are nonconservative. That DL/Dt also vanishes in the conservative limit follows from (3.4); when $\bar{s} = 0$ this becomes $\bar{\mathbf{u}}\cdot\nabla\bar{q} = 0$, whence $\bar{\psi} = \bar{\psi}(q, z)$. Then $L = \partial\bar{\psi}/\partial\bar{q} = L(\bar{q}, z)$ whence $\nabla_2 L = (\partial L/\partial\bar{q})\nabla_2\bar{q}$ and therefore $DL/Dt \equiv \bar{\mathbf{u}}\cdot\nabla_2 L = (\partial L/\partial\bar{q})\bar{\mathbf{u}}\cdot\nabla_2\bar{q} = 0$. Therefore (3.6) is of conservation form D(Density)/Dt + DIV(Flux) = (Nonconservative terms), with $\mathbf{E}$ representing a radiative (i.e., relative to the local mean flow) flux of wave activity. The relation may perhaps be more usefully written

$$\frac{\partial\mathscr{E}}{\partial t} + \nabla\cdot\mathbf{E}_T = S_E, \quad (3.9)$$

where

$$\mathbf{E}_T = \mathbf{E} + \bar{\mathbf{u}}\mathscr{E} \quad (3.10)$$

is the total (radiative plus advective) flux. (Note that of course $\partial\mathscr{E}/\partial t$ vanishes for time-averaged statistics.)

In the limit of a purely zonal mean flow $\bar{\mathbf{u}} = (\bar{u}, 0, 0)$, the y and z component of $\mathbf{E}_T$ reduce to

$$\mathbf{E}_T = p\overline{\mathbf{u}'_a\phi'} - \bar{u}\mathbf{F}, \quad (3.11)$$

where $\mathbf{F}$ is the quasi-geostrophic Eliassen-Palm flux (e.g., Edmon *et al.* 1980); (3.11) is just the eddy energy flux which appears in the transformed Eulerian-mean energetics derived in Plumb (1983).

Equation (3.9) is an alternative statement of the law obtained by Andrews [1983, Eqs. (2.18) and (A1)], who discusses its formal analogy with the statement of pseudoenergy conservation in generalized Lagrangian-mean theory (Andrews and McIntyre, 1978) and its relationship with the generalized stability criterion of Arnold (1965). While the density $\mathscr{E}$ is identical to that which appears in Andrews' expression, the flux $\mathbf{E}_T$ differs from his $\rho_0\mathscr{F}$ (in his notation), even in the conservative limit. In fact, it can be shown by using the momentum equations (2.1) and thermodynamic equation (2.2) to eliminate the ageostrophic terms, followed by some lengthy but straightforward manipulation, that in the conservative limit

$$(\rho_0 \mathcal{F}) = \mathbf{E}_T + \nabla \times \mathbf{G}, \tag{3.12}$$

where

$$\mathbf{G} = \frac{1}{2} p \begin{pmatrix} \frac{f_0^2}{N^2}\left(\bar{v}\frac{\partial}{\partial z}(\overline{\psi'^2}) - \frac{\partial \bar{v}}{\partial z}\overline{\psi'^2}\right) \\ \frac{-f_0^2}{N^2}\left(\bar{u}\frac{\partial}{\partial z}(\overline{\psi'^2}) - \frac{\partial \bar{u}}{\partial z}\overline{\psi'^2}\right) \\ \bar{u}\frac{\partial}{\partial y}(\overline{\psi'^2}) - \bar{v}\frac{\partial}{\partial x}(\overline{\psi'^2}) - \frac{\partial \bar{u}}{\partial y}\overline{\psi'^2} + \frac{\partial \bar{v}}{\partial x}\overline{\psi'^2} \end{pmatrix}. \tag{3.13}$$

Since the two fluxes differ by a nondivergent term, $\mathbf{E}_T$ and $(\rho_0 \mathcal{F})$ are interchangeable in (3.9). (In the nonconservative case the extra, nonconservative, terms which appear in (3.12) can be absorbed into the term S_E).

This difference exemplifies the nonuniqueness discussed by Andrews, viz. that, as determined by (3.9), the definition of $\mathbf{E}_T$ is arbitrary to the extent that any vector whose divergence is nonconservative may be added to $\mathbf{E}_T$. Note that this arbitrariness cannot be resolved by demanding that, in the WKB limit of weakly dissipated waves on a slowly varying mean flow, the flux be parallel to the group velocity $\mathbf{c}_g$, since both $\mathbf{E}_T$ and $\rho_0 \mathcal{F}$ can be shown to equal $\mathbf{c}_g \mathcal{E}$ in this limit [when the term $\nabla \times \mathbf{G}$ becomes negligible in (3.12)].

4. Discussion

The derivation presented here of the result (3.9) has led to the definition of a conservable flux of wave activity $\mathbf{E}_T$ for perturbations to a steady but nonzonal flow. As compared with Andrews' definition, the present form avoids the, perhaps conceptually unappealing, presence of time-derivatives [cf. Andrews 1983, Eq. (2.20)] although the price paid for this is the involvement of ageostrophic terms. In turn, however, this illustrates more clearly the relationship of $\mathbf{E}_T$ with the conventional "energy flux" $p\mathbf{u}_a'\phi'$ and shows that, in the special case of waves on a purely zonal flow, it reduces to an expression for "energy flux" that arises naturally in transformed Eulerian-mean theory. While the ageostrophic terms may make this form difficult to evaluate from observational data, its application to general circulation model results should be straightforward.

The difference between $\mathbf{E}_T$ and Andrews' flux $\rho_0 \mathcal{F}$ exemplifies the comments of Edmon *et al.* (1980) and of Andrews (1983) on the nonuniqueness of fluxes defined on the basis of conservation laws or, as in the case of the Eliassen-Palm flux, of the equations describing the interaction between waves and mean flows where only the flux divergence is involved. This raises general questions as to whether one may interpret such fluxes in general as indicators of the direction of eddy activity propagation (it seems unlikely that this can be true both of $\mathbf{E}_T$ and $\rho_0 \mathcal{F}$) or whether one must implicitly invoke slowly-varying assumptions in order to do so. However, this nonuniqueness does not detract from the usefulness (or potential usefulness) of such fluxes as diagnostic tools.

REFERENCES

Andrews, D. G., 1983: A conservation law for small-amplitude quasi-geostrophic disturbances on a zonally asymmetric basic flow. *J. Atmos. Sci.*, **40,** 85–90.

——, and M. E. McIntyre, 1978: On wave action and its relatives. *J. Fluid Mech.*, **89,** 647–664.

Arnold', V. I., 1965: Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid. *Dokl. Akad. Nauk SSSR*, **162,** 975–978. [In Russian: English translation in *Soviet Math.*, **6,** 773–776.]

Edmon, H. J., B. J. Hoskins and M. E. McIntyre, 1980: Eliassen-Palm cross-sections for the troposphere. *J. Atmos. Sci.*, **37,** 2600–2616.

Plumb, R. A., 1983: A new look at the energy cycle. *J. Atmos. Sci.*, **40,** 1669–1688.

Rhines, P. B., and W. R. Holland, 1979: A theoretical discussion of eddy-driven mean flows. *Dyn. Atmos. Oceans*, **3,** 289–325.