Linear Dynamics of the Multiple-Vortex Phenomenon in Tornadoes

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ABSTRACT

The reason for existence of two separate unstable modes, previously described by Gall for flows in vortex simulators, is explored. When the energy equation for an unstable disturbance is considered, it is clear that the most unstable wave must be centered inside the maximum in the vertical vorticity of the basic state if this vorticity has a radial distribution that is triangular-shaped and this triangle is near the center of the vortex. When this vorticity is at large radius, the most unstable wave can be centered near or even outside the basic-state vorticity maximum. This suggests different modes when the triangular profile of vorticity is near or far from the center and that the transition from one to another mode should be gradual. These notions are verified by a careful analysis of the stability properties of the triangular-shaped vorticity profile.

It is shown that the triangular-shaped vorticity profile closely resembles the vorticity distribution in the vortex simulator after a downdraft has been established along the centerline of the vortex. In fact, the stability properties of these triangular profiles closely resemble the stability properties of the simulated vortex when the scale of the triangular profile is comparable to the vorticity distribution in the simulators.

Square-shaped vorticity profiles, which have been considered in the past, have significantly different stability properties, as compared to the triangular profiles. In particular, there is only one unstable mode, and the instability is extinguished as the vorticity region approaches the center of the basic vortex. The reason for this is easily explained by considering the perturbation energy equation.

1. Introduction

In a recent paper (Gall, 1983), the author described a linear stability analysis of a series of computed axisymmetric states from a model of the tornado simulators (Church et al., 1979). This analysis showed that the multiple-vortex phenomenon observed in those simulators could be mostly explained by the linear instability of the basic axisymmetric vortex. That study also showed that two different modes of unstable waves were present: a mode which appeared at low swirl ratio and low wavenumber, and a second mode which appeared at moderate swirl ratio and somewhat higher wavenumber. A number of properties distinguished the two modes. In particular, the maximum perturbation streamfunction was centered inside the vorticity maximum of the basic state for mode 1, while for mode 2 it was centered outside the vorticity maximum. In addition, mode 1 received most of its energy from the radial shear of the vertical wind, while mode 2 received most of its energy from the radial shear of the tangential wind.

The reason for the existence of these two modes was not determined. An understanding of the dynamics of these modes would not only promote a better understanding of the instability that leads to the multiple vortices but also would determine whether the results reported in Gall are particular to the model used or are really representative of the vortices in the simulators. In this paper, we will demonstrate the basic dynamics of the two modes.

In Gall (1983) it was shown that, while mode 1 receives most of its energy from the radial shear of the vertical wind, evidence of this mode is present in results from the inviscid nondivergent model of Staley and Gall (1979), where the effects of the vertical component of the wind are eliminated. This suggests that, even though the vertical wind of the axisymmetric state plays a very important role in the dynamics of this mode, the fundamental reason for the existence of this mode is present in the simple nondivergent model. Thus we will use the model of Staley and Gall to perform this preliminary study of the dynamics of the two modes. The reason why the vertical motion plays such a significant role in mode 1 awaits further research.

2. Theoretical discussion

a. The basic wave structure

Consider the energy equation for perturbations in a nondivergent, incompressible, inviscid flow in cylindrical coordinates,

$$\frac{\partial K_e}{\partial t} = -\nabla' \cdot \left[ \nabla \bar{u} - \frac{\bar{v}}{r} \right],$$

where $K_e$ is the kinetic energy contained in the perturbation, the overbar denotes an average over
azimuth and the prime a deviation from this average. It is, of course, well-known that this equation implies that the wave will grow where \( v' \) and \( v'' \) are positively correlated and the bracketed part is negative and vice versa. For a shear zone at large radius, this in turn implies that the phase lines of streamfunction bend upstream with increasing radius in regions of positive shear and downstream in negative shear.

For basic states that are strongly curved, these conditions for instability are not quite so simple, since \( v/r \) can change the sign of the bracketed term in (1) in some regions. This description for instability, however, is only what is sufficient for instability; there are still a large number of possibilities for the actual structure. We can, using simple graphical arguments actually get a little closer to a picture of just what the unstable waves should look like.

In Gall 1983, we reported that the basic vortex is unstable when its vertical vorticity is contained in an annulus. In Fig. 1a, the vertical solid line represents the line along the vorticity maximum of the basic state, while the dashed line delineates the region where the vorticity of the basic state is nonzero. Although these lines are actually curved in the vortex, we draw them straight for simplicity and only a section of the annulus is shown. We assume for our discussion that this vertical vorticity is positive and the mean flow is toward the top of the figure. Now introduce infinitesimal disturbances in the streamfunction. The thin solid lines show these disturbances as alternating regions of high and low streamfunction along the basic state vorticity maximum. Of course, in introducing the perturbation streamfunction, we are introducing perturbation vorticity which by

\[
\nabla \times \psi' = \xi'
\]

implies that a positive (negative) perturbation of \( \xi' \) is associated with low (high) streamfunction.

Such a configuration of perturbation streamfunction won’t last; the perturbation flow will distort the mean vorticity field, causing additional perturbation vorticity to develop in the regions shown in Fig. 1a, where \( P \) implies positive perturbation vorticity develops, while \( N \) implies negative perturbations develop. Once the vorticity is perturbed in this way, the perturbation streamfunction will be altered. We can determine how it is altered by noting that wherever a maximum in \( \xi' \) appears, Eq. (2) implies there will be a tendency for minima in \( \psi' \) (and vice versa; see Holton, 1979, p. 131). Thus the streamfunction field shown in Fig. 1a will elongate, with low streamfunction elongating toward the new positive vorticity perturbations and high streamfunction elongating toward the new negative perturbations. Figure 1b shows schematically these modifications.

Although we don’t illustrate it, this perturbation streamfunction field in Fig. 1b will further change the distribution of the perturbation vorticity. In particular, the maxima and minima of \( \xi' \) to the left of the line of maximum basic state vorticity will be displaced upward in the figure and those to the right, downward. This, of course, implies further modification in the perturbation streamfunction from Fig. 1b, though only slight. We can imagine that further iterations of our schematic picture would suggest progressively smaller changes. This exercise essentially illustrates the selection of the most unstable normal mode from the initial perturbation. This mode will indeed grow, since the circulation implied by \( \psi' \) will increase the perturbation vorticity, which will, in turn, increase the amplitude of \( \psi' \) which implies increased perturbation flow increasing the rate at which the basic state vorticity is distorted, and so forth. Note also that other initial perturbation states different from that shown in Fig. 1a will also eventually begin to look like Fig. 1b.

The distribution of the amplitude and the phase lines of \( \psi' \) associated with the growing wave inside the region of nonzero basic state vorticity can easily be deduced from Fig. 1b. In the region where \( \xi = 0 \), there will be no \( \xi' \neq 0 \), except that introduced by the initial perturbation, which if small enough will imply \( \xi' = 0 \) compared to elsewhere after the wave has grown for some time. It is easy to show that, in these regions, if we assume \( \xi' = 0 \) at \( r = 0 \) and \( r = \infty \), \( \psi' \) will decrease rapidly away from the regions where \( \xi' \neq 0 \), and the phase lines will be parallel to radii. This variation of phase is illustrated in Fig. 1c, where the dashed lines are phase lines.

Thus, with these simple graphical arguments, we can illustrate the main features we shall see in the unstable waves: 1) The phase lines of \( \xi' \) and \( \psi' \) will undergo a rapid change in azimuthal position as they cross the \( \xi \) maximum line with the phase lines of streamfunction shifting upstream with increasing radius, while the phase lines must be parallel to radii.

Fig. 1. Schematic to illustrate the structure of the unstable waves in the tornado simulator. See text for a complete explanation.
in the regions where $\xi' = 0$; 2) There should be regions of large $|\xi'|$ on opposite sides of $\xi$ maximum; and 3) $\psi'$ amplitude may also show a tendency for a double maximum, one on either side of the $\xi$ maximum, since maxima of $\psi'$ often are closely associated with minima in $\xi'$ and vice versa.

The reader is encouraged to compare Fig. 1b with Figs. 6, 7, and 8 in Gall (1983), which show $\psi'$ and $\xi'$ for the unstable waves found in that study. The simple picture of the unstable waves presented here will aid in understanding the development of the different modes described later.

b. Wave energetics

Perhaps the simplest way to explain the existence of the two separate modes of instability is to consider the eddy energy equation, Eq. (1). While it is not immediately obvious that this equation can explain the presence of the two modes, we found after trial and error that it indeed does. Since,

$$ v_\theta' = \frac{\partial \psi'}{\partial r}; \quad v_r' = -\frac{1}{r} \frac{\partial \psi'}{\partial \theta}, $$

(3)

if we let

$$ \psi' = A(r) \sin[k \theta + \phi(r)], $$

(4)

then

$$ -\frac{v_\theta' v_r'}{r} = \frac{k}{2r} A^2 \frac{\partial \phi}{\partial r}. $$

(5)

Insertion of (5) into (1) gives

$$ \frac{\partial K_e}{\partial t} = \frac{k}{2r} A^2 \frac{\partial \phi}{\partial r} \left( \frac{\partial \bar{\psi}_\theta}{\partial r} - \frac{\bar{\psi}_\theta}{r} \right). $$

(6)

With the following assumptions, we can simplify this equation. If we assume that the disturbance with the maximum growth rate is the one where the wavelength $L$ is proportional to the width of the shear zone, as was discussed in Gall (1983) (We will show further evidence later that this is the case), then

$$ k = \frac{2\pi r_m}{L} \approx \frac{2C_r}{\Delta \theta}, $$

(7)

where $r_m$ is the radius of the maximum vertical vorticity, $\Delta \theta$ the shear zone width and $C$ a constant that contains the proportionality between $L$ and $\Delta \theta$. With (7), (6) becomes

$$ \frac{\partial K_e}{\partial t} \approx \frac{CA^2}{\Delta \theta} D, $$

(8)

where

$$ D = \frac{r_m}{r} \left( \frac{\partial \bar{\psi}_\theta}{\partial r} - \frac{\bar{\psi}_\theta}{r} \right). $$

(9)

The significance of Eq. (8) will become more apparent in the next subsection. It suffices here to point out the following. In Eq. (8), $D$ depends only on the geometry of the shear zone, while $A^2(\partial \phi/\partial r)$ is given by the structure of the unstable wave. Where $D$ is nonzero, it is necessary for instability that the disturbance amplitude and the phase variation of the streamfunction be nonzero. When $D$ is positive, wave growth occurs when $\partial \psi/\partial \theta > 0$. In the problems to be discussed in the next section, $D$ is mostly positive. Thus the phase lines of streamfunction of the unstable waves must bend upstream with increasing radius. Recall from Fig. 1 that this was the variation of phase that we postulated.

c. Special cases

1) Triangular vorticity profile

Figure 2 shows the radial distribution of vertical vorticity from the axisymmetric model of Gall (1982) at various swirl ratios, $S$.

The reasons for the variation of the vorticity as a function of $S$ are discussed by Gall. This figure is presented here to emphasize that the vorticity profile takes on a roughly triangular distribution in that model. The triangular shape, however, is not particular to Gall's model; the reader is referred also to the vertical vorticity distribution reported by Rotunno (1977, Fig. 6.8) in his two-dimensional axisymmetric model.

Based on Fig. 2 and some arguments presented in the Introduction, we choose to examine the stability properties of a basic flow which contains no azimuthal vorticity and where the vertical vorticity distribution is given by

$$ \xi = \xi_m \begin{cases} \begin{array}{ll} r - r_0 & , \quad r_0 \leq r \leq r_m \\ r - r_1 & , \quad r_m \leq r \leq r_1 \end{array} \end{cases} $$

(10)

and zero elsewhere; $r_0 < r_m < r_1$, and $\xi_m$ gives the maximum of the vorticity distribution which is located at $r_m$. Since

$$ \xi = \frac{1}{r} \frac{\partial \psi_0}{\partial \theta}, $$

(11)

for the basic state, then from (10) we have

$$ \bar{\psi}_0 = \frac{\xi_m r_0}{\Delta r_1} \left[ \frac{r^2}{3r_0} - \frac{r}{2r_0} + \frac{r_0}{\Delta r_1} \right], \quad r_0 \leq r \leq r_m, $$

(12)

$$ \bar{\psi}_0 = -\frac{\xi_m r_1^2}{\Delta r_1} \left[ \frac{r^2}{3r_1} - \frac{r}{2r_1} - \frac{r_m}{r_1} \left( \frac{r_m}{r_1} - \frac{1}{2} \right) \right] + \frac{\bar{\psi}_0 r_m}{r}, \quad r_m \leq r \leq r_1, $$

(13)

where $\bar{\psi}_m$ is given by (13) evaluated at $r_m$, $\Delta r_1 = r_m$

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1 $S = \Gamma \psi/2Q$, where $\Gamma$ is the circulation at the radius $r_0$, which is usually the radius of the updraft hole in the simulators, and $Q$ is the volume flow rate.
- \( r_0 \) and \( \Delta r_2 = r_1 - r_m \). At this point, it is convenient to let \( r^* = r/r_0 \), \( \Delta r^*_1 = \Delta r_1 / r_0 \), etc., and then insert (12) and (13) into the bracketed term of (9). This gives

\[
D = \frac{r_m^*}{\Delta r^*_1 r_0 r^* \Delta r^*_1} \left[ \frac{r^*}{3} - \frac{1}{3 r^*_1^2} \right], \quad 1 \leq r^* \leq 1 + \Delta r^*_1,
\]

(14)

\[
D = -\frac{r_m^*}{\Delta r^*_1 r_0 r^*} \left[ \frac{r^*}{3 r^*_1^2} + \frac{2 r_m^*}{r^*_1^2} \left( \frac{r_m^*}{3 r^*_1^2} - \frac{1}{2} \right) \right]
\]

\[
+ \frac{2 \bar{u}_{m r^*_1} r_m^*}{r^*_1^2} \right] 1 + \Delta r^*_1 \leq r^* \leq 1 + \Delta r^*_1,
\]

(15)

where \( \Delta r^* = \Delta r^*_1 + \Delta r^*_2 \). Note that (14) and (15) depend only on the characteristics of the basic flow. In particular, the parameters that describe the flow are \( \Delta r_1 \), \( \Delta r_2 \), which give the width of the triangular region of vorticity; \( r_0 \), which gives the distance the triangular profile is displaced from the center of the vortex; and \( \xi_m \), which gives the value of the maximum. In the discussion that follows, we will express \( r_0 \) as multiples of some measure of the width of the vorticity profile. Since both \( \Delta r_1 \) and \( \Delta r_2 \) can be varied, we arbitrarily choose to use \( r_0 = R_0 \Delta r_2 \). Thus \( R_0 \) expresses how far the nonzero region is away from the center as multiples of the width of the outer portion of the triangle. Small values of \( R_0 \) imply a thin ring of nonzero vorticity near the axis or a wide ring far away.

Figure 3 shows the radial distribution of \( D \) from \( r = r_0 \) for various values of \( R_0 \). Note that this family of curves would be identical for other values of \( \xi_m \) except for a constant multiplier and that other values of \( \Delta r_1 \) and \( \Delta r_2 \) simply stretch or compress the curves, as the case may be. At large values of \( R_0 \), which would be equivalent to flow without curvature, \( D \) takes on the shape of the vorticity profile, a triangle. This is because at large \( r \), the horizontal deformation of the basic state and its vertical vorticity are identical. As \( R_0 \) is decreased to zero (the inner edge of the triangle of vorticity reaches the center of the basic vortex), \( D \) takes on a square profile, and values greater than zero fit almost entirely inside the radius of maximum vorticity \( r_m \).

From the arguments presented earlier, the phase shift in \( \varphi \) should be restricted to the region of nonzero vorticity and should be such that the phase lines tilt upstream in this region. This statement is also made with the advantage of hindsight, since it was later found that this is indeed true for the triangular vorticity profiles. This being true, then when \( R_0 \) is small, \( A \) in Eq. (8) should be largest inside \( r_m \) for the most rapidly growing mode; if the opposite were true, the energy tendency would be negative or at least much smaller because \( D \) becomes negative beyond \( r_m \) and the phase variation must be as in Fig. 1c. At large \( R_0 \), \( A \) could be large on either side of (or centered at) \( r_m \), since \( D \) in (8) is positive throughout the region of nonzero vorticity and the growth rate could be large. Modes centered inside \( r_m \) where \( R_0 \) is large would not be as efficient in extracting energy from the basic state as those centered near \( r_m \) because the distribution of \( D \) in Eq. (8) is as maximum at \( r_m \). Thus two separate modes of instability are suggested: one where the streamfunction amplitude is a maximum inside \( r_m \), which would be most unstable at small \( R_0 \), and a second where the amplitude outside \( r_m \) is large, which would be most unstable when \( R_0 \) is large. The transition between these two modes should be gradual, since at \( R_0 \approx 3 \) or so the profile of \( D \) displays a resemblance to both the square-shaped profile at small \( R_0 \) and the triangular profile at large \( R_0 \). We will show in a moment that this

**Fig. 2.** Radial distribution of vertical vorticity at various swirl ratios given by the axisymmetric model described by Gall (1982). The numbers at the peak of each curve indicate the swirl ratio.

**Fig. 3.** Radial distribution of \( D \) [Eq. (9)] as a function of \( R_0 \) (indicated above each curve). A triangular distribution of vorticity is used. Units are arbitrary.
speculation, which is based solely on the energy equation, is indeed the case.

Before we do this, it should be pointed out that two modes of instability, very similar to those described above, were found in Gall (1983). In mode 1, the streamfunction amplification was greatest inside the radius of maximum vorticity, and this mode was most unstable at small swirl ratio. The second mode appeared at large swirl ratio and tended to have a double maximum in the streamfunction amplitude centered about the vorticity maximum. From Fig. 2, we can compute an $R_0$ for each of the profiles displayed by extending the largest slope of each of the curves to the axis. When we do this, we can plot $R_0$ versus $S$ for the experiment of Fig. 1. This is shown in Fig. 4. With this figure, individual curves in Fig. 3 can be related to swirl ratio.

Also shown in Fig. 4 is a quantity we call "P". This is simply the ratio of $D$ at $r_m$ to its value at $r^* = \Delta r_1/5 + r_0$. It is a measure of how triangular or square the profile of this term appears. The maximum value of the P is 5 for $R_0 = \infty$ when the profile is triangular, and the minimum value is 1 at $R_0 = 0$ when the profile is square. Values in between represent curves with characteristics of both the square profile and the triangular profile. For values of P greater than about 2.5 (at $R_0 = 3$), the curves are essentially triangular (see Fig. 3), where modes centered near or even outside $r_m$ may be most unstable. As Fig. 4 shows, P is 1 for $S < 1$ and increases rapidly to above 3 at $S > 2$. This curve suggests that modes centered inside $r_m$ would be most unstable at $S < 1$. For $S > 1$, there should be a rapid transition to a mode centered near $r_m$, being most unstable. This was actually what was found in Gall (1983).

2) SQUARE PROFILES

Before we proceed to show that the speculation discussed above is true, we present a similar analysis for square-shaped vorticity profiles. These profiles, while they are not realistic for real vortices, do have historical significance. Because a constant distribution of vorticity (in a region) has relatively simple velocity distributions, it is possible to obtain analytic solutions to the stability problem. Examples of studies where square profiles have been used are Rayleigh (1880), Michalke and Timme (1967), and Snow (1978). Thus much is known about these profiles and their stability. However, it is easy to show that the square profile has very significant differences from the triangular profile.

If we let 
\[ \xi = \xi_c, \quad r_0 \leq r \leq r_1, \] (16)
and zero elsewhere, then
\[ D = \frac{\xi_c r_m^*}{r_0 \Delta r^* r_m^{*3}}, \quad 1 \leq r^* \leq r_m^*. \] (17)

Here $r_m = (r_1 + r_0)/2$.

Figure 5 shows the distribution of this term for the square profile; it should be compared with Fig. 3 for the triangular profile. Note that the difference between the two profiles is dramatic when viewed in the context of the energy equation. As with the triangular profile, when $R_0$ is large, this term resembles the vorticity distribution. As $R_0$ becomes small, however, $D$ is positive and significantly different from zero over a very small region near $r_0$ and is negative beyond $r_1$. Using arguments similar to those above for the triangular vorticity profile, we might expect two different modes for the square profile, as we do for the triangular profiles. One mode would be centered near the middle of the nonzero vorticity region when $R_0$ is large; one would be centered near $r_0$ when $R_0$ is small. However, the region where $D > 0$ is infinitesimally small as $R_0 \rightarrow 0$. Since $r_0$ is the edge of the $\xi \neq 0$ region, where $\partial \phi/\partial r > 0$ and $\partial \phi/\partial r_0 > 0$ must coincide with positive regions of $D$ for instability,
it becomes increasingly unlikely that unstable waves will be found as \( R_0 \to 0 \). In fact, at \( R_0 = 0 \), the profile should be stable. This suggests that a mode that becomes dominant as \( R_0 \to 0 \) will not be found with the square profile. We will show later that there is only one mode of instability with the square profile and that the unstable waves disappear as \( R_0 \to 0 \), unlike the triangular profile.

3. Stability analysis

In this section, we will present calculations of the linear stability of the profiles described by (10) and (16). This is accomplished with the inviscid nondivergent model of Staley and Gall (1979). In all the calculations described below, we will retain the geometry of the simulator used in the calculations of Gall (1983). For the calculations at low \( R_0 \), the axisymmetric flow will be confined inside a radius of 62 cm and the perturbation streamfunction will vanish at \( r = 0 \) and \( r = 62 \) cm. Thus, for moderate values of \( R_0 \) (approximately 6, using values of \( \Delta r_1 \) and \( \Delta r_2 \) defined below), the outer edge of the region of nonzero vorticity approaches this maximum radius. We retain the geometry of the simulator, so we can compare the results here with those described by Gall. For the calculations at large \( R_0 \), the nonzero vorticity region is contained at the center of the region of calculation, which is still 62 cm wide; the boundary conditions on perturbation streamfunction are the same as for small \( R_0 \); and the inner edge of the calculation is at large radius instead of at the center of the vortex. In all the calculations, 92 grid points were used between the inner and outer edges of the computational domain.

a. Stability where \( R_0 = \infty \)

Figure 6 shows the growth rate as a function of wavelength for various profiles of vorticity, where \( R_0 \) is large. The profiles with \( \Delta r_1 = 10.4 \) cm and \( \Delta r_2 = 5.2 \) cm are of comparable width to the profiles shown in Fig. 2, which are all about the same width. In what follows, this particular triangular profile will be the one most frequently considered, since it most nearly approximates the distribution of vertical vorticity found in the basic states considered by Gall (1983). This will allow a good comparison between the idealized results of the present study and the more complete results. In Fig. 6, \( \xi_m = 10 \) s\(^{-1} \) was used for all profiles. Other values of \( \xi_m \) merely change the growth rate by a constant multiple, such that doubling \( \xi_m \) doubles the growth rate.

Changing the shape of the triangular profile by varying \( \Delta r_1 \) and \( \Delta r_2 \) but keeping their sum constant has no significant effect on the growth rate curves. However, doubling the width of the zone where \( \xi \neq 0 \) doubles the wavelength of maximum growth. The shortwave cutoff of the instability also doubles.

When the square profiles are considered, the wavelength of maximum instability doubles from the triangular profile of the same width, while the shortwave cutoff quadruples.

In Gall (1983), we speculated that the wavelength of maximum growth would be related in a simple way to the width of the shear zone or rather the width of the region of nonzero vorticity. For the square region of vorticity, Rayleigh (1880) showed that the ratio of half the wavelength to this width is about 4 (as it is in Fig. 6), while for the hyperbolic tangent wind profile this ratio is about 1.4. (This is only approximate, since there are no zero vorticity regions in the hyperbolic tangent profile.) For the triangular profile of this study, this ratio is about 1.8, which is very nearly equal to that of the hyperbolic tangent wind profile.

Finally, for both the triangular and square vorticity profiles, there is only a single unstable mode when \( R_0 \) is large. This is to be contrasted with the triangular profile when \( R_0 \) is small, where we find two unstable modes.

b. Triangular profile for small \( R_0 \)

Figure 7 shows growth rate and phase speed curves as a function of wavelength for various values of \( R_0 \). For most of the \( R_0 \) shown, there are two sets of curves. This is because, at those values of \( R_0 \), there were two unstable modes for certain wavelengths. The solid curve, which for a given wavelength has the higher phase speed of the two modes, will refer to mode 1; while the dashed curve, which corresponds to the remaining mode, refers to mode 2. As we will see, this terminology corresponds to that used in Gall (1983).

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**Fig. 6.** Growth rate as a function of wavelength for triangular and square vorticity profiles at great radius. The curves denoted by T refer to triangular profiles and by S to square profiles. The dimensions of the profile in cm are indicated in the lower left of the figure. The two numbers for the triangular profile are for \( \Delta r_1 \) and \( \Delta r_2 \), respectively.
Mode 1 always is most unstable at wave 2. (For some reason, the nondivergent model never shows wave 1 to be unstable.) At very low values of $R_0$, the mode 1 instability extends toward high wavenumbers, as high as wave 7 for $R_0 = 1$. As $R_0$ increases, the short wavelength cutoff of this mode moves toward lower wavenumbers, and by $R_0 = 3$, only wave 2 is unstable, and the growth rate of even this wave has decreased by a factor of 2. Beyond $R_0 = 4$, mode 1 no longer appears as an unstable wave.

At low values of $R_0$, mode 2 is less unstable than mode 1 and is most unstable at two wavenumbers (waves 4 and 6). The bimodal distribution of growth rate of mode 2 at low $R_0$ suggests different modes at low and high wavenumber, but a careful examination of wave structure shows no obvious differences. The reason for the bimodal distribution is unknown. As $R_0$ is increased, the maximum growth rate of mode 2 increases rapidly, so that at $R_0 = 1.5$ it is more unstable than mode 1. By $R_0 = 5$, it is the only unstable mode and is most unstable at wave 5.

The lower portion of Fig. 7 shows the phase speed for the two modes. The phase speed of mode 1 at a given $R_0$ is nearly independent of wavenumber and exceeds the phase speed of mode 2 at all wavenumbers. As $R_0$ increases, the phase speed of this mode decreases.

At most $R_0$, the phase speed of mode 2 is a strong function of wavenumber, such that wave 2 is the most slowly moving wave. The phase speed thus indicates one very distinct difference between the two modes. At $R_0 = 5$, the curve for mode 2 has become almost flat. We feel that this may be due to the fact that the outer edge of the triangular region of vorticity is approaching the edge of the simulator and the long wavelengths (which span a greater region of $r$ than the short wavelengths) are beginning to "feel" this edge. As with mode 1, the phase speeds of mode 2 decrease with increasing $R_0$.

The phase speeds of the waves decrease as $R_0$ increases because for the curves in Fig. 7 we do not change $\xi_m$ as $R_0$ is increased. Since $\xi_m$ is fixed, the maximum tangential velocities remain nearly the same as $R_0$ increases. Hence the maximum angular velocities in the basic vortex must decrease as the radius of the maximum velocity increases. The phase speeds are limited by this maximum angular velocity in the basic state; in fact, they are always less. Thus, the phase speed must likewise decrease. The increasing phase speed with increasing wavenumber in mode 2 can probably be explained by the tendency for the radial width of the wave to decrease as wavenumber increases. We will describe this tendency more clearly in Section 4. Since the waves are centered near the radius of maximum angular velocity of the basic state (just inside the radius of maximum tangential velocity, which is near $r_1$), as the waves become narrower in $r$, they feel less of the slower angular velocity in the basic state, and hence their phase velocity approaches the maximum. Unfortunately, we cannot think of a reason why mode 1 travels more rapidly than mode 2 and why its phase speed is independent of wavenumber.

In Gall (1983) it was pointed out that the transition from mode 1 being most unstable to mode 2 being most unstable occurred at about $S = 1.4$. In Fig. 7a, this transition occurs at $R_0 = 1.5$, which, according to Fig. 4, corresponds to a swirl ratio of about 1.2. Thus there is a good correspondence between the transition from mode 1 to mode 2 in Gall (1983) and in this study. This is not entirely surprising, since $\Delta r_1$ and $\Delta r_2$, used to compute Fig. 7, were chosen to be typical of the curves shown in Fig. 2.

c. Square profile at small $R_0$

Figure 8 corresponds to Fig. 7a but for the square profile with the same width as the triangular profile of Fig. 7 and where $\xi_c$ is the same as $\xi_m$ in that figure. It is clear that the growth rate diagrams for the square and the triangular profiles are very different, as we suggested that they would be. For the square profile, over a wide range of $R_0$, only waves 2 and 3 are unstable for the values of $R_0$ considered; and for $R_0$ less than 2, only wave 2 is unstable, and its growth rate is decreasing with decreasing $R_0$. This result has
been discussed already by Michalke and Timme (1967). At high values of $R_0$, wave 3 is most unstable, and the transition from 2 to 3 being most unstable occurs at $R_0 \sim 3$. According to Fig. 4, this would correspond to a swirl ratio of 1.5 in Gall (1983). Thus, even though quantitatively there is a great difference between the stability properties of the square and triangular profiles, qualitatively, at least in the sense of the transition of the most unstable wavenumber to the next higher number as $R_0$ increases, the two profiles are not very different. This is true only through wave 3. At higher values of $R_0$, up to the point where the nonzero vorticity region reaches the edge of the computational domain, waves 4 and 5 are never unstable, yet the transitions to these wavenumbers are found for the triangular profile. They are also found in the simulator. Thus, while the square profile and its stability properties do bear some resemblance to reality, generally the agreement is not good. The triangular profile and its stability properties do bear a strong resemblance to the observations.

Finally, there is only one unstable mode with the square profile. In Section 2, we speculated that this would be the case. This and the fact that the instability is extinguished at small $R_0$ are other substantial differences between the square and the triangular profiles.

d. Other profiles

We examined other triangular profiles with different values of $\Delta r_1$ and $\Delta r_2$, and, although there were significant quantitative changes, such as the growth rate at a particular value of $R_0$ and wavenumber, qualitatively the changes were small. That is, in each case, there were two modes of instability, just as in Fig. 7, and the behavior of these modes as $R_0$ changed was similar. Figure 9 illustrates the effect of doubling the width of the triangular region. This figure is to be compared with Fig. 7a. The two cases are the same, except $\Delta r_1$ and $\Delta r_2$ in Fig. 9 are exactly twice those used in Fig. 7a. The main differences in the growth-rate spectra between the two cases were that the mode 1 instability extended to much higher wavenumbers at low $R_0$ when the width of the shear zone was doubled and that the most rapidly growing wavenumber of mode 2 was 3 instead of 4 at intermediate values of $R_0$. The latter difference suggests that, as $R_0$ is increased with the wider shear zone, the most unstable wavenumber will shift from 2 to 3 rather than 2 to 4 as in Fig. 7. In other words, since the shear zone is wider, the second mode is most unstable at a longer wavelength for the reasons cited earlier. Note that, if we had included viscosity in this analysis, wave 3 would have clearly been more unstable than wave 4 in Fig. 9 for $R_0 = 2$, since the higher wavenumbers would be more strongly affected by viscosity than lower wavenumbers. This result may explain why wave 3 was never most unstable in Gall (1983). Perhaps that study simply did not use a high enough value of the mixing coefficient, which would have widened the shear zone.

The results of Section 2 show that $D$ is independent of the choice of $\Delta r_1$ and $\Delta r_2$ (as long as their ratio is constant), and depends only on $R_0$. Thus, one might expect that the growth-rate spectra would be similarly independent of $\Delta r_1$ and $\Delta r_2$ and depend only on the choice of $R_0$. However, as $\Delta r_2$ is increased, the $\rho_0$ increases for a given value of $R_0$, and the effect of curvature on the perturbation is different. Therefore, while there is a qualitative resemblance of Figs. 7 and 9, quantitatively they differ.

4. Wave structure

a. Triangular profile at large $R_0$

Figure 10 shows the distribution of streamfunction amplitude as a function of $r$ when $R_0$ is very large. Also indicated is the total phase variation across the
streamfunction of mode 1 is inside the vorticity maximum of the basic state for all wavenumbers, while for mode 2 this maximum is outside the basic-state vorticity maximum. At all wavelengths, the total phase shift is much greater for mode 1 than for mode 2. In fact, for the low wavenumbers, the phase shift for mode 1 is nearly 180°. In addition, the radial width of the disturbance (defined as we did above) decreases with increasing wavenumber.

Both modes show weak secondary maxima in the amplitude. In mode 1 there is perhaps a slight tendency for the secondary maxima to increase with increasing wavenumber. (This was more clearly evident at other values of $R_0$.) For mode 2 this secondary maximum clearly decreases with increasing wavenumber. The primary maxima of the two modes are centered about equidistant from the radius of maximum basic-state vorticity.

Of the characteristics described above, the location of the primary maxima of the amplitude of each mode, the tendency for a secondary maximum, the existence of two separate modes and the location of the phase shift are all more or less explained using the arguments presented in Section 2. The variation of the phase shift and the secondary maxima with wavenumber have not been explained, and as yet we do not really understand the reasons for them.

Figure 12 shows how the structure of two modes varies as $R_0$ changes. This figure shows the structure of wave 2 for $R_0$ ranging from 1 to 6. Note that at low $R_0$ the two modes are distinctly different. As $R_0$ is increased, the two modes seem to merge (at least with respect to amplitude) into a single mode. By $R_0$

shear zone. We do not show the curve for the phase angle since it is constant outside the region of nonzero vorticity and varies nearly linearly within that region. We present this figure for comparison with the structure as $R_0$ becomes small. Note that the streamfunction amplitude shows a double maximum for long wavelengths. This is the structure we predicted in Section 2. As the wavelength decreases, the two maxima more or less merge into a single maximum near $r_m$ and the phase shift decreases. In addition, as the wavelength decreases, the radial width of the disturbance (defined for example by the region where the normalized value of the amplitude in Fig. 10 exceeds 1) decreases. This is not surprising if the disturbance is to remain more or less circular as the wavelength decreases.

b. Wave structure at small $R_0$

Figure 11 shows the radial distribution of the amplitude of the streamfunction at $R_0 = 1$ for various wavenumbers. The solid curves are for mode 1, while the dashed curves are for mode 2. Also the total phase shift in the streamfunction is shown on the diagrams. As with solutions for large $R_0$ shown above, all of the phase shift occurs within the region of nonzero vorticity, where it is nearly a linear function of $r$.

Figure 11 clearly displays the difference in structure of the two modes. The maximum amplitude of the

![Fig. 10. Radial distribution of the amplitude of the perturbation streamfunction for the triangular profile with $\Delta r_1 = 10.4$ cm and $\Delta r_2 = 5.2$ cm at large radius. Radius increases toward the right. The number on the upper right of the figure indicates the azimuthal wavelength of the disturbance. The three arrows from left to right indicate $r_0$, $r_m$, and $r_1$, respectively. The number on the upper left gives the change in the phase angle of the disturbance across this region.

![Fig. 11. Radial distribution of the amplitude of the perturbation streamfunction at $R_0 = 1$ for various wavenumbers for the triangular profile with $\Delta r_1 = 10.4$ and $\Delta r_2 = 5.2$ cm. The number in the upper portion of each panel indicates azimuthal wavenumber. The solid curve is for mode 1, and the dashed curve, mode 2. The phase shifts for modes 1 and 2 (denoted M1 and M2, respectively) are indicated in the lower right portion of the figure. The arrow indicates the radius $r_m$.](image-url)
as \( r_m \) moves outward. The two dashed lines indicate the most unstable wavenumber of each mode at the various \( R_0 \) from Fig. 7a. For mode 2 at low \( R_0 \), where there is a double maximum in the growth-rate curve, we have indicated only the lower, most unstable wavenumber.

At \( R_0 = 6 \) the most unstable wavenumber of mode 2 has a wavelength of 53 cm, which is close to the most unstable wavelength at large \( R_0 \). As \( R_0 \) decreases, however, the most unstable wavenumber of this mode does not decrease rapidly enough for the most unstable wavelength to remain near 53 cm. In fact, by \( R_0 = 1 \) the most unstable wavelength of this mode has decreased to 25 cm. This is probably because the region of positive values of \( D \) outside \( r_m \) shrinks as \( R_0 \) decreases. Thus, the most unstable wave of mode 2 must shrink in radial width as well, otherwise the negative region outside \( r_m \) will contribute to decay of the wave or at least decrease the growth rate. Mode 1, on the other hand, is always most unstable at wave 2 which, for large values of \( R_0 \) is much longer than 53 cm, but at \( R_0 = 1.5 \) has a length of about 60 cm. Recall that at \( R_0 = 1.5 \) the most unstable mode changes from mode 2 to mode 1 as \( R_0 \) decreases. At this value of \( R_0 \), mode 2 has a most unstable length of 30 cm. Thus the tendency for the most unstable wavelength to be determined by the width of the shear zone is clearly there although the relationship is not constant.

6. Conclusions

By considering the energy equation for perturbations, the reason for the existence of the two modes of instability that were discussed in Gall (1983) is apparent. Because of the curvature of the flow in the basic state, the part of the energy conversion integral that depends on the basic state takes on a radically

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5. Wavelength versus shear zone width

It is not obvious that the relation between the most unstable wavelength and the shear zone width that was discussed briefly in Section 3 for large values of \( R_0 \) should continue as the curvature of the basic flow increases. This relation is fairly well-established for a number of profiles at large radius, although it is a function of the nature of the profile.

Figure 14 shows the wavelength of the various wavenumbers at the radius \( r_m \) (the radius of the maximum basic-state vorticity) as a function of \( R_0 \), where \( \Delta r_1 \) and \( \Delta r_2 \) are as in Section 3a. The wavelength of a given wavenumber, of course, increases

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FIG. 12. Radial distribution of amplitude of the perturbation streamfunction for wave 2 at various values of \( R_0 \) (indicated in the upper right portion of each panel). Otherwise, as in Fig. 11.

FIG. 13. Radial distribution of amplitude and phase angle of the perturbation streamfunction for the square profile of width 15.6 cm for the values of \( R_0 \) indicated on each curve. Each curve has been superimposed so that the region of nonzero vorticity for each profile coincides. The center of the region of nonzero vorticity is indicated by the vertical line, and the peaks of the amplitude curves coincide with the edges of this region. Solid lines are amplitudes, dashed lines are phase angles.
different radial distribution when the basic-state vorticity is near the center of the vortex, as compared to when it is far out. At large radius this part of the integral has the same shape as the vorticity distribution because horizontal deformation and vertical vorticity of the axisymmetric part of the flow are the same. When the vorticity is near the center, the deformation is quite different from the distribution of vorticity. If the radial distribution of basic-state vorticity is assumed to have a triangular shape, then as this vorticity is moved toward the center, the part of the energy equation that is determined by the basic state takes on a rectangular shape and is positive (and therefore contributes to wave growth) inside the radius of the vorticity maximum of the basic state, while it is negative outside this radius. From the arguments presented in Section 2, for the triangular regions of vorticity at large radius, the unstable disturbance will tend to have a double maximum in the amplitude of the streamfunction with the minimum in between centered on the radius \( r_m \) of the basic-state vorticity maximum. At small radius, however, the maximum outside \( r_m \) would contribute to the decay of the wave. Therefore, modes where the streamfunction amplitude is a maximum inside \( r_m \) would be favored.

By selecting the width of the triangular vorticity region to approximate the width of the vorticity region in the study by Gall (1983), the transition from mode 1 being most unstable to mode 2 being most unstable was found to occur at the same value as \( R_0 \) in both studies. The \( R_0 \) in that study was computed approximately as described in Section 2 and can be related to swirl ratio; hence the transition between modes in this study occurs at approximately the same swirl ratio as in Gall (1983). Thus, the conclusion in Gall (1983) that the discontinuous transition from wave 2 to wave 4 being most unstable is due to a transition of most unstable modes is correct. Furthermore, the structures of the two modes in Gall (1983) and in this study compare very well.

Recently Walko and Gall (1984) completed a study similar to the one described by Gall (1983) but where both the axisymmetric and the linear calculations retained realistic vertical variations. The models were two-dimensional in \( r \) and \( z \), whereas in Gall (1983) the vertically integrated equations were used. The results of these two studies compared very well. In particular, in both models the most unstable wavenumber increased with the swirl ratio, and the wave structures were very similar. Because the structure of the two modes of instability discussed here is not drastically different, it was difficult to identify these two modes in the results of Walko and Gall, although they are suggested. This complication appears to arise because the radius of maximum vorticity of the basic state in the complete model increases with height.

The square profiles give quite different growth-rate spectra, and the wave structures are quite different from those found for the triangular profiles. Since the triangular profiles give results similar to those from the models of the simulators, we can conclude that the triangular profiles are a more realistic approximation to the vorticity distribution in the basic vortex than are the square profiles. This is an important point since many of the earlier studies of the stability of vortices used square profiles.

In Gall (1983) it was pointed out that the jump in the most unstable wavenumber from 2 to 4 that was found in that study as the swirl ratio increased is not observed in the simulator to which that study was to be compared. In that simulator (the Purdue simulator, see Church et al., 1979), the number of vortices progress regularly from 1 to 2 to 3 through 4, and so forth. There are a wide range of swirl ratios where three vortices are present. Figure 9, however, suggests that if the width of the zone of nonzero vorticity regions in Gall (1983) were wider, then the transition in most unstable wavenumber when the most unstable mode changed would be 2–3. In Gall, the width of the nonzero vorticity zone could have been increased by increasing the friction coefficient. In Gall, this had been done. The friction coefficient was raised to very high levels and still wavenumber 3 did not appear as most unstable. It was pointed out however, that the axisymmetric model did not contain the effect of the multiple vortices, and their effect could not be included by simply increasing the mixing coefficient. Thus, it is still possible that when the effect of unstable eddies is included in the basic state calculation, a wave 3 might still appear. The effect of the unstable eddies on the basic state will cause the width of the vorticity zone to increase and at the same time retard the outward expansion of the radius of maximum vorticity as swirl ratio increases (Gall, 1983). Both of these effects will tend to force the second mode toward lower wavenumbers at low values of \( R_0 \) (and swirl ratio).

The analysis described above neglects the vertical component of motion in the basic vortex. Since mode 1 in Gall (1983) was found to obtain most of its

![Figure 14: Wavelength (cm) of the various wavenumbers versus \( R_0 \) at the radius \( r_m \) for \( \Delta r_1 = 10.5 \) cm and \( \Delta r_2 = 5.2 \) cm. The left dashed curve indicates the most unstable wavenumber of mode 1 (which disappears at \( R_0 = 4 \)), and the left dashed curve the most unstable wavenumber of mode 2.](image-url)
energy from the shear of this part of the basic flow, the vertical motion must have a very important effect on mode 1. The obvious extension of this study would be to include the vertical motion in the analysis described herein.

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REFERENCES


