Probability Distribution of Modal Amplitudes in Interacting Triads with Arbitrary Random Forcing

PHILIP D. THOMPSON

National Center for Atmospheric Research, Boulder, CO 80307*

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ABSTRACT

This paper deals with a statistical-mechanical approach to the problem of calculating the statistics of randomly forced triads of modes in the two-dimensional flow of a viscous fluid. We first construct the probability distribution of modal amplitudes as an approximate solution of the Fokker–Planck equation. The net nonlinear energy-transfer and kinetic energy spectrum are then calculated directly from the probability distribution. These theoretical results agree very well with the statistics of a large ensemble of numerical integrations of the original evolution equations, particularly for only moderate departures from "white-noise" forcing.

1. Introduction

In a recent paper (Thompson, 1985), we have considered some simple aspects of the nonlinear transfer of energy between different scales of motion in two-dimensional nondivergent flow of a viscous fluid, driven by random infusions of energy. The underlying aim was to find how rapidly the kinetic energy released by sporadic conversion of available potential energy is transferred from the scales of maximum baroclinic instability to other scales and to calculate the resulting energy spectrum.

It is natural to approach these problems via a spectral representation of the flow and the evolution equations for the amplitudes of interacting modes in that representation. In the absence of forcing and dissipation, the change in amplitude of any particular mode is a sum over all "triad" interactions in which that mode takes part—i.e., mutual nonlinear interactions between three modes. Triad interactions are peculiar in that a change in amplitude of each member of the triad results from its interaction with the other two, but in such a way that the total enstrophy and kinetic energy of the entire triad are conserved. Indeed, the simplest nontrivial spectral representation consists of a single interacting triad, and it is useful to think of higher-order systems as aggregates of coupled triads.

In the paper referred to above, we have considered moment closures for calculating the net nonlinear energy transfer and partition of energy between isolated triads of modes, in a two-dimensional viscous flow with random forcing. The main conclusion was that a "sixth moment discard" closure, somewhat reminiscent of the "eddy-damped, quasi-normal" approximation, is accurate to within about 5% over a wide range of external conditions extending well into the strongly nonlinear regime. Encouraging as those results are, however, one is uncomfortably unsure about what is missing from any more or less arbitrary truncation of the moment equations.

In the same paper, it was also pointed out that the whole question of closure could be circumvented by solving the corresponding Fokker–Planck equation for the probability distribution in a phase space whose coordinates are the modal amplitudes. Our present purpose is to explore this alternative further to see if it leads to results that are better or worse than those attainable by moment closures of comparable complexity.

The solution of the Fokker–Planck equation is found to be a product of two functions. One is symmetrical and is just the Gaussian probability distribution for a triad whose modes receive random injections of energy at the same rate. The other factor of the solution is an asymmetrical function that reflects the nonlinear transfer of energy between modes. The latter function is represented as a truncated series of products of Hermite polynomials; with the omission of terms containing polynomials of third or higher degree, we find a closed system of seven equations for the coefficients in the series. This system is of about the same complexity as the moment equations with sixth moment discard. A considerably more accurate and only slightly more complicated representation of the probability distribution is based on a series of products of Hermite polynomials, from which polynomials of fourth or higher degree have been excluded, and is discussed in the Appendix.

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The energy-transfer and kinetic energy spectra have been calculated from the resulting probability distribution. Those are compared with statistics constructed directly from a large ensemble of numerical integrations of the original evolution equations for the modal amplitudes. Such comparisons indicate that statistics derived from the truncated probability distribution are very accurate if the forcing spectrum does not depart too far from that of "white noise" but are increasingly inaccurate as the probability distribution becomes strongly non-Gaussian. It should be added, however, that the modal kinetic energies are accurate to within about 3% even when the net nonlinear energy transfer between modes is comparable with the rate at which energy is imparted by random forcing.

2. Statistical–hydrodynamical formulation of the problem

Our starting point is the truncated spectral form of the vorticity equation for two-dimensional, incompressible viscous flow. Its derivation from the Navier–Stokes equations is outlined in earlier papers (Lorenz, 1960; Thompson, 1972). In the notation of the latter, it is

$$\frac{dA_k}{dt} = \sum_{i=1}^{N}\sum_{j=1}^{N} \beta_{ijk}\alpha_j^2 A_i A_j - \nu\alpha_k^2 A_k + f_k(t)$$

$$(k = 1, 2, 3, \cdots, N), \qquad (1)$$

in which A_k is the streamfunction amplitude of the kth mode, multiplied by the eigenvalue α_k corresponding to that mode. The nonlinear interaction coefficients β_{ijk} are given by

$$\beta_{ijk} = \frac{1}{\alpha_i\alpha_j\alpha_k}\int_A \phi_k J(\phi_i, \phi_j)dA.$$

The normalized eigenfunctions ϕ_k are solutions of $\nabla^2\phi_k = -\alpha_k^2\phi_k$ for $\phi_k = 0$ on the fixed boundary of some closed area domain A. The integration is taken over A. It is readily shown that the ϕ corresponding to two distinct eigenvalues are orthogonal and that the β_{ijk} have the following relevant properties: β_{ijk} vanishes if any two indices are equal, reverses sign under any noncyclic permutation of indices, and remains unchanged under any cyclic permutation.

The forcing functions $f_k(t)$ are stationary stochastic processes, representing the effect of randomly varying sources of vorticity. The kinematic molecular viscosity ν is taken to be constant.

In particular, the evolution equations for a single, isolated triad are given by (1) as

$$\frac{dA_1}{dt} = \beta_{123}(\alpha_2^2 - \alpha_3^2)A_2A_3 - \nu\alpha_1^2A_1 + f_1(t)$$

$$\frac{dA_2}{dt} = \beta_{123}(\alpha_3^2 - \alpha_1^2)A_1A_3 - \nu\alpha_2^2A_2 + f_2(t)$$

$$\frac{dA_3}{dt} = \beta_{123}(\alpha_1^2 - \alpha_2^2)A_1A_2 - \nu\alpha_3^2A_3 + f_3(t). \qquad (2)$$

For convenience and to facilitate comparison with earlier results, we introduce dimensionless amplitudes X_1, X_2, X_3, defined by

$$A_1 = \left(\frac{\mu_1}{\nu\alpha_1^2}\right)^{1/2} X_1$$

$$A_2 = \left(\frac{\mu_2}{\nu\alpha_2^2}\right)^{1/2} X_2,\ A_3 = \left(\frac{\mu_3}{\nu\alpha_3^2}\right)^{1/2} X_3 \qquad (3)$$

where

$$\mu_k = \int_0^\infty \langle f_k(t)f_k(t+\tau)\rangle d\tau = \langle f_k^2\rangle T.$$

The angle brackets denote the ensemble average or "expected value," and T is the "integral time scale" of the f_k.

Similarly, we scale the f_k so that

$$f_1 = \left(\frac{\mu_1}{T}\right)^{1/2} g_1,\quad f_2 = \left(\frac{\mu_2}{T}\right)^{1/2} g_2,\quad f_3 = \left(\frac{\mu_3}{T}\right)^{1/2} g_3 \qquad (4)$$

from which it follows that

$$\langle g_k^2\rangle = 1.$$

Introducing (3) and (4) into (2), we see that the evolution equations for the scaled variables become

$$\dot{X}_1 = \frac{dX_1}{dt} = \nu\alpha_1^2[C_1X_2X_3 - X_1 + (\nu\alpha_1^2T)^{-1/2}g_1]$$

$$\dot{X}_2 = \frac{dX_2}{dt} = \nu\alpha_2^2[C_2X_1X_3 - X_2 + (\nu\alpha_2^2T)^{-1/2}g_2]$$

$$\dot{X}_3 = \frac{dX_3}{dt} = \nu\alpha_3^2[C_3X_1X_2 - X_3 + (\nu\alpha_3^2T)^{-1/2}g_3], \qquad (5)$$

in which

$$C_1 = \frac{\beta_{123}(\alpha_2^2 - \alpha_3^2)}{\mu_1\alpha_1\alpha_2\alpha_3}\left(\frac{\mu_1\mu_2\mu_3}{\nu^3}\right)^{1/2}$$

$$C_2 = \frac{\beta_{123}(\alpha_3^2 - \alpha_1^2)}{\mu_2\alpha_1\alpha_2\alpha_3}\left(\frac{\mu_1\mu_2\mu_3}{\nu^3}\right)^{1/2}$$

$$C_3 = \frac{\beta_{123}(\alpha_1^2 - \alpha_2^2)}{\mu_3\alpha_1\alpha_2\alpha_3}\left(\frac{\mu_1\mu_2\mu_3}{\nu^3}\right)^{1/2}.$$

We now regard X_1, X_2, X_3 as the coordinates of a point in a three-dimensional phase space. The state of a particular realization at one instant is then represented by the point (X_1, X_2, X_3), and its evolution is represented by a curve of such points, each corresponding to a particular time. Since its components of velocity in the X_1, X_2 and X_3 directions are just $\dot{X}_1$, $\dot{X}_2$, and $\dot{X}_3$, its trajectory along that curve is given by the parametric equations of motion (5). Moreover, if we consider a very large ensemble of realizations originating at the same time, conservation of realizations requires that

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial X_1}(\rho\dot{X}_1) + \frac{\partial}{\partial X_2}(\rho\dot{X}_2) + \frac{\partial}{\partial X_3}(\rho\dot{X}_3) = 0, \qquad (6)$$

in which ρ is the number density of realizations in the

phase space. If ρ is normalized in such a way that its integral taken over the entire phase space (i.e., over all possible contingencies) is unity, it is just the probability distribution. Thus the probability that a realization lies within the volume element $dX_1dX_2dX_3$ is $\rho dX_1dX_2dX_3$, and the expected value of any function $F(X_1, X_2, X_3)$ is

$$\langle F\rangle = \int_V \rho F dV,$$

where $dV = dX_1dX_2dX_3$ is a volume element of the phase space V.

Substituting the expressions for $\dot{X}_1, \dot{X}_2, \dot{X}_3$ given by (5) into (6), we obtain the Liouville equation, which formally determines the probability distribution. It is

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial X_1}[\nu\alpha_1{}^2(C_1X_2X_3 - X_1)\rho + \nu\alpha_1{}^2(\nu\alpha_1{}^2T)^{-1/2}g_1\rho] + \frac{\partial}{\partial X_2}[\nu\alpha_2{}^2(C_2X_1X_3 - X_2)\rho + \nu\alpha_2{}^2(\nu\alpha_2{}^2T)^{-1/2}g_2\rho] + \frac{\partial}{\partial X_3}[\nu\alpha_3{}^2(C_3X_1X_2 - X_3)\rho + \nu\alpha_3{}^2(\nu\alpha_3{}^2T)^{-1/2}g_3\rho] = 0. \quad (7)$$

This equation, of course, simply expresses the fact that any net flux of realizations across the faces of a small (but finite) volume element of the phase space must be exactly balanced by a change of number density within that volume. If we imagine that the volume element is large enough that many realizations cross each face of the volume per unit time, then $\nu\alpha_1{}^2(\nu\alpha_1{}^2T)^{-1/2}g_1\rho$ is to be interpreted as the expected net rate of flux of realizations in the X_1-direction, due solely to the random part $\nu\alpha_1{}^2(\nu\alpha_1{}^2T)^{-1/2}g_1$ of the speeds of many realizations. For simplicity we suppose that the latter vary on a time scale much shorter than the dissipation time, so that the "forced" flux of realizations may be calculated from the current density distribution and the statistics of the ensemble of g_1. It has been shown (cf. de Groot and Mazur, 1962) that, if g_1 has a Gaussian distribution, the random transport of realizations is a diffusive process, described by

$$\nu\alpha_k{}^2(\nu\alpha_k{}^2T)^{-1/2}\overline{g_k\rho} = -\frac{\nu^2\alpha_k{}^4}{\nu\alpha_k{}^2T}\int_0^\infty \langle g_k(t)g_k(t+\tau)\rangle \times d\tau\,\frac{\partial\rho}{\partial X_k} = -\nu\alpha_k{}^2\langle g_k{}^2\rangle\frac{\partial\rho}{\partial X_k} = -\nu\alpha_k{}^2\frac{\partial\rho}{\partial X_k},$$

where the overbar denotes the average over a large subensemble. Finally, introducing these results into (7), we arrive at a Fokker–Planck equation:

$$\frac{\partial\rho}{\partial t} + \nu\alpha_1{}^2\frac{\partial}{\partial X_1}\left[\rho(C_1X_2X_3 - X_1) - \frac{\partial\rho}{\partial X_1}\right] + \nu\alpha_2{}^2\frac{\partial}{\partial X_2}\left[\rho(C_2X_1X_3 - X_2) - \frac{\partial\rho}{\partial X_2}\right] + \nu\alpha_3{}^2\frac{\partial}{\partial X_3}\left[\rho(C_3X_1X_2 - X_3) - \frac{\partial\rho}{\partial X_3}\right] = 0. \quad (8)$$

At this point it should be noted that, since X_1, X_2, X_3 are now independent variables and ρ is the sole dependent variable, (8) is a linear partial differential equation whose solution (when $\partial\rho/\partial t$ is set to zero) is just the equilibrium probability distribution.

3. Solution of the Fokker–Planck equation

Our next concern is to construct an approximate solution of the equilibrium form of (8). Setting $\partial\rho/\partial t$ to zero in (8) and letting

$$\rho = \phi(X_1, X_2, X_3)\exp\left[-\tfrac{1}{2}(X_1{}^2 + X_2{}^2 + X_3{}^2)\right],$$

we may write (8) as

$$L\{\phi\} = \alpha_1{}^2\left(\frac{\partial^2\phi}{\partial X_1{}^2} - X_1\frac{\partial\phi}{\partial X_1}\right) + \alpha_2{}^2\left(\frac{\partial^2\phi}{\partial X_2{}^2} - X_2\frac{\partial\phi}{\partial X_2}\right) + \alpha_3{}^2\left(\frac{\partial^2\phi}{\partial X_3{}^2} - X_3\frac{\partial\phi}{\partial X_3}\right) - \alpha_1{}^2C_1X_2X_3\frac{\partial\phi}{\partial X_1} - \alpha_2{}^2C_2X_1X_3\frac{\partial\phi}{\partial X_2} - \alpha_3{}^2C_3X_1X_2\frac{\partial\phi}{\partial X_3} + \epsilon X_1X_2X_3\phi = 0 \quad (9)$$

where $\epsilon = \alpha_1{}^2C_1 + \alpha_2{}^2C_2 + \alpha_3{}^2C_3$. From the definition of the C_k we see that, if the μ_k are all the same, ϵ vanishes. In this case (9) implies that ϕ is constant and $\rho = \gamma\exp[-\frac{1}{2}(X_1{}^2 + X_2{}^2 + X_3{}^2)]$. The constant γ is fixed by the condition that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho dX_1dX_2dX_3 = 1.$$

Thus, in the special case of "white noise" forcing, the probability distribution is Gaussian. In consequence, owing to the symmetry of ρ, all odd moments of ρ vanish, and there is no net nonlinear transfer of kinetic energy to or from any mode.

In general, of course, the forcing spectrum is not uniform, the probability distribution is not symmetrical, and the equilibrium kinetic energy spectrum is determined by the balance between the rates of energy injection by random forcing, viscous dissipation and nonlinear transfer of energy between modes. The problem, then, is to solve (9) for $\epsilon \neq 0$.

Inspecting (9), we observe that the operator $L\{\ \}$ is linear but is not separable. It is convenient, however, to represent ϕ as a truncated series of products of Hermite polynomials, the eigenfunctions of that operator's separable part. That is, we write

$$\phi = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} A_{ijk}H_i(X_1)H_j(X_2)H_k(X_3) \quad (10)$$

$$L\{\phi\} = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} A_{ijk}L\{H_i(X_1)H_j(X_2)H_k(X_3)\} = 0 \quad (11)$$

in which $H_n(X)$ is the Hermite polynomial of degree n. It is a suitably normalized solution of

$$\frac{d^2H_n}{dX^2} - X\frac{dH_n}{dX} = -nH_n \tag{12}$$

and has the properties that

$$\frac{dH_n}{dX} = nH_{n-1} \quad XH_n(X) = H_{n+1}(X) + nH_{n-1}(X). \tag{13}$$

For reference, the first five polynomials are

$$H_0(X) = 1, \quad H_1(X) = X, \quad H_2(X) = X^2 - 1$$
$$H_3(X) = X^3 - 3X, \quad H_4(X) = X^4 - 6X^2 + 3.$$

The choice of the $H_n(X)$ as basis functions is suggested by the similarity of the first three terms of (9) to the left-hand side of (12).

We next apply the operator $L\{\ \}$ to a deliberately chosen sequence of products of polynomials, making use of (12) and (13). Carrying out the differentiations indicated in (9), we obtain

$$L\{1\} = \epsilon X_1X_2X_3 = \epsilon H_1(X_1)H_1(X_2)H_1(X_3) \tag{14}$$

$$\begin{aligned}L\{X_1X_2X_3\} &= -(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)X_1X_2X_3 \\ &+ \alpha_1^2C_1H_2(X_1) + \alpha_2^2C_2H_2(X_2) + \alpha_3^2C_3H_2(X_3) \\ &+ (\alpha_2^2C_2 + \alpha_3^2C_3)H_2(X_2)H_2(X_3) + (\alpha_1^2C_1 + \alpha_3^2C_3) \\ &\times H_2(X_1)H_2(X_3) + (\alpha_1^2C_1 + \alpha_2^2C_2)H_2(X_1)H_2(X_2) \\ &+ \epsilon H_2(X_1)H_2(X_2)H_2(X_3)\end{aligned} \tag{15}$$

$$\begin{aligned}L\{H_2(X_1)\} &= -2\alpha_1^2H_2(X_1) + 2(\alpha_2^2C_2 + \alpha_3^2C_3)X_1X_2X_3 \\ &+ \epsilon H_3(X_1)H_1(X_2)H_1(X_3)\end{aligned} \tag{16}$$

$$\begin{aligned}L\{H_2(X_2)\} &= -2\alpha_2^2H_2(X_2) + 2(\alpha_1^2C_1 + \alpha_3^2C_3)X_1X_2X_3 \\ &+ \epsilon H_1(X_1)H_3(X_2)H_1(X_3)\end{aligned} \tag{17}$$

$$\begin{aligned}L\{H_2(X_3)\} &= -2\alpha_3^2H_2(X_3) + 2(\alpha_1^2C_1 + \alpha_2^2C_2)X_1X_2X_3 \\ &+ \epsilon H_1(X_1)H_1(X_2)H_3(X_3)\end{aligned} \tag{18}$$

$$\begin{aligned}L\{H_2(X_2)H_2(X_3)\} &= -2(\alpha_2^2 + \alpha_3^2)H_2(X_2)H_2(X_3) + 4\alpha_1^2C_1X_1X_2X_3 \\ &+ 2(\alpha_1^2C_1 + \alpha_3^2C_3)H_1(X_1)H_1(X_2)H_3(X_3) \\ &+ 2(\alpha_1^2C_1 + \alpha_2^2C_2)H_1(X_1)H_3(X_2)H_1(X_3) \\ &+ \epsilon H_1(X_1)H_3(X_2)H_3(X_3)\end{aligned} \tag{19}$$

$$\begin{aligned}L\{H_2(X_1)H_2(X_3)\} &= -2(\alpha_1^2 + \alpha_3^2)H_2(X_1)H_2(X_3) + 4\alpha_2^2C_2X_1X_2X_3 \\ &+ 2(\alpha_1^2C_1 + \alpha_2^2C_2)H_3(X_1)H_1(X_2)H_1(X_3) \\ &+ 2(\alpha_2^2C_2 + \alpha_3^2C_3)H_1(X_1)H_1(X_2)H_3(X_3) \\ &+ \epsilon H_3(X_1)H_1(X_2)H_3(X_3)\end{aligned} \tag{20}$$

$$\begin{aligned}L\{H_2(X_1)H_2(X_2)\} &= -2(\alpha_1^2 + \alpha_2^2)H_2(X_1)H_2(X_2) + 4\alpha_3^2C_3X_1X_2X_3 \\ &+ 2(\alpha_1^2C_1 + \alpha_3^2C_3)H_3(X_1)H_1(X_2)H_1(X_3) \\ &+ 2(\alpha_2^2C_2 + \alpha_3^2C_3)H_1(X_1)H_3(X_2)H_1(X_3) \\ &+ \epsilon H_3(X_1)H_3(X_2)H_1(X_3)\end{aligned} \tag{21}$$

$$\begin{aligned}L\{H_2(X_1)H_2(X_2)H_2(X_3)\} &= -2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) \\ &\times H_2(X_1)H_2(X_2)H_2(X_3) + [\text{terms involving } H_3(X)].\end{aligned} \tag{22}$$

We now consider a low-order truncation from which terms involving Hermite polynomials of third or higher degree are excluded. We then determine the coefficients A_{ijk} in (10) by multiplying (14)–(22) by A_{000}, A_{111}, A_{200}, A_{020}, A_{002}, A_{022}, A_{202}, A_{220} and A_{222}, respectively, and adding them to form the series in (11). Thus, equating the coefficients of like terms to zero, in order to satisfy $L\{\phi\} = 0$, we find that

$$\begin{aligned}&\epsilon A_{000} - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)A_{111} + 2(\alpha_2^2C_2 + \alpha_3^2C_3)A_{200} \\ &+ 2(\alpha_1^2C_1 + \alpha_3^2C_3)A_{020} + 2(\alpha_1^2C_1 + \alpha_2^2C_2)A_{002} \\ &+ 4\alpha_1^2C_1A_{022} + 4\alpha_2^2C_2A_{202} + 4\alpha_3^2C_3A_{220} = 0\end{aligned} \tag{23}$$

$$\alpha_1^2C_1A_{111} - 2\alpha_1^2A_{200} = 0 \tag{24}$$

$$\alpha_2^2C_2A_{111} - 2\alpha_2^2A_{020} = 0 \tag{25}$$

$$\alpha_3^2C_3A_{111} - 2\alpha_3^2A_{002} = 0 \tag{26}$$

$$(\alpha_2^2C_2 + \alpha_3^2C_3)A_{111} - 2(\alpha_2^2 + \alpha_3^2)A_{022} = 0 \tag{27}$$

$$(\alpha_1^2C_1 + \alpha_3^2C_3)A_{111} - 2(\alpha_1^2 + \alpha_3^2)A_{202} = 0 \tag{28}$$

$$(\alpha_1^2C_1 + \alpha_2^2C_2)A_{111} - 2(\alpha_1^2 + \alpha_2^2)A_{220} = 0 \tag{29}$$

$$\epsilon A_{111} - 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)A_{222} = 0. \tag{30}$$

This procedure is exactly equivalent to projecting a set of orthogonal eigenfunctions onto $L\{\phi\}$.

Examining the linear system (23)–(30), we first note that A_{222} appears only in (30) and is thus determined by the solution of (23)–(29). Further, since $L\{\phi\}$ is homogeneous, the seven equations (23)–(29) determine only ratios of the remaining eight A_{ijk} to A_{000}. Thus, we may set A_{000} to unity, and later multiply ϕ by an arbitrary constant. Equations (23)–(29) then comprise a complete system of linear algebraic equations. Substituting from (24)–(29) into (23) and solving the resulting equation for A_{111}, we find that

$$A_{111} = (\alpha_1^2 C_1 + \alpha_2^2 C_2 + \alpha_3^2 C_3)\Big[(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) - C_1(\alpha_2^2 C_2 + \alpha_3^2 C_3)\Big(1 + \frac{2\alpha_1^2}{\alpha_2^2 + \alpha_3^2}\Big) - C_2(\alpha_1^2 C_1 + \alpha_3^2 C_3)\Big(1 + \frac{2\alpha_2^2}{\alpha_1^2 + \alpha_3^2}\Big) - C_3(\alpha_1^2 C_1 + \alpha_2^2 C_2)\Big(1 + \frac{2\alpha_3^2}{\alpha_1^2 + \alpha_2^2}\Big)\Big]^{-1}. \quad (31)$$

Once A_{111} is determined from (31), the remaining A_{ijk} are readily calculated from (24)–(30). All other coefficients in the series (10) whose indices are two or less necessarily vanish, since they are solutions of a nonsingular system of homogeneous equations.

The approximate solution of the equilibrium form of (8) is then

$$\rho = \gamma e^{-(1/2)(X_1^2+X_2^2+X_3^2)}[1 + A_{111}X_1X_2X_3 + A_{200}(X_1^2 - 1) + A_{020}(X_2^2 - 1) + A_{002}(X_3^2 - 1) + A_{022}(X_2^2 - 1)(X_3^2 - 1) + A_{202}(X_1^2 - 1) \times (X_3^2 - 1) + A_{220}(X_1^2 - 1)(X_2^2 - 1) + A_{222}(X_1^2 - 1)(X_2^2 - 1)(X_3^2 - 1)] \quad (32)$$

in which the A_{ijk} are as determined above. The value of γ required to normalize the probability distribution is $(2\pi)^{-3/2}$.[1] It will be noted that in the "white noise" case when $(\alpha_1^2 C_1 + \alpha_2^2 C_2 + \alpha_3^2 C_3) = 0$, A_{111} and hence all other coefficients vanish, except in the leading term; the probability distribution is then Gaussian, as it should be. In general, however, the probability distribution is not symmetrical around the coordinate axes, owing to the presence of $A_{111}X_1X_2X_3$. Indeed, that term reflects the nonlinear transfer of energy between modes.

As indicated in the Introduction, we follow an exactly analogous procedure to determine the coefficients A_{ijk} in the series (10) for $N = 3$. The simultaneous system of linear equations that determine the A_{ijk} is displayed in the Appendix.

Knowing the probability distribution, we may now calculate the transfer moment $\langle X_1X_2X_3\rangle$ as

$$\langle X_1X_2X_3\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(X_1, X_2, X_3)X_1X_2X_3 dX_1 dX_2 dX_3.$$

Let us first observe that all terms of (32) except the second are even functions, so that the equation above reduces to

$$\langle X_1X_2X_3\rangle = (2\pi)^{-3/2}A_{111}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_1^2X_2^2X_3^2 \times e^{-(1/2)(X_1^2+X_2^2+X_3^2)}dX_1dX_2dX_3 = A_{111}. \quad (33)$$

Thus the transfer moment is given by (31).

The modal kinetic energies $\langle X_1^2\rangle$, $\langle X_2^2\rangle$, $\langle X_3^2\rangle$ may be calculated directly from (33), by making use of the energy balance conditions. With our present scaling, they are

$$\langle X_1^2\rangle = 1 + C_1\langle X_1X_2X_3\rangle \quad (34)$$

$$\langle X_2^2\rangle = 1 + C_2\langle X_1X_2X_3\rangle \quad (35)$$

$$\langle X_3^2\rangle = 1 + C_3\langle X_1X_2X_3\rangle. \quad (36)$$

Given the molecular viscosity ν and the variances μ_k of the random forcing functions, (31) and (33)–(36) provide a means of estimating the rate of energy transfer between modes with eigenvalues α_k, as well as the partition of energy among them.

4. Comparison with Monte Carlo calculations

To judge the accuracy of estimates derived from the truncated probability distribution, we have carried out a series of Monte Carlo calculations, consisting of many numerical integrations of the original evolution equations for the modal amplitudes, with independently generated random forcing functions. Ensemble and time averages of modal kinetic energy and nonlinear energy transfer were then constructed directly from the assemblage of numerical integrations. Such statistics are here compared with those derived from (31) and (33)–(36).

The values of the dimensionless constants that enter into the finite-difference form of (5) were taken to be

$$\nu\alpha_1^2\Delta t = 0.0996, \quad \nu\alpha_2^2\Delta t = 0.1914,$$

$$\nu\alpha_3^2\Delta t = 0.3060, \quad C_1 = -0.2775n^{1/2}$$

$$C_2 = 0.4995n^{-1/2}, \quad C_3 = -0.2220n^{1/2}, \quad T = \frac{\Delta t}{2},$$

in which Δt is the length of the time step, and n is a measure of the unscaled intensity of forcing in the mode of intermediate scale. The scaled random functions g_1, g_2 and g_3 have magnitude unity, but their signs were chosen independently and by random choice at each time stage. The values given above are appropriate for a triad consisting of the lowest three modes of two-dimensional flow in a rectangular domain, with aspect ratio 2/3.

The evolution equations were integrated over 10 000 time steps by a predictor–corrector method. An ensemble of 300 independent integrations were first calculated for $n = 1$, a case in which the average rate of energy input to each mode by random forcing is the same for all modes. This case, "white-noise forcing," is one for which the exact statistics are known: the

[1] Here and in what follows, it is important to note that

$$\int_{-\infty}^{\infty} e^{-(1/2)X^2}dX = \int_{-\infty}^{\infty} X^2 e^{-(1/2)X^2}dX.$$

probability distribution is Gaussian, and there is no net nonlinear transfer of energy to any mode. The time and ensemble averages $\langle X_1^2 \rangle$, $\langle X_2^2 \rangle$, $\langle X_3^2 \rangle$ and $\langle X_1 X_2 X_3 \rangle$ were constructed from the ensemble of 300 integrations and compared with the exact statistics. In this case, of course, such comparisons are primarily tests of numerical procedure and the randomness of a random sign generator.

To isolate the effect of nonuniform forcing on the nonlinear transfer of energy, the calculations described above were repeated for integer values of n up to 10. These give the equilibrium statistics corresponding to an n-fold increase of the rate at which energy is imparted to the mode of intermediate scale but with no change in the forcing of the lowest and highest modes. In this regime, the ordering parameters C_1, C_2, C_3 range from smaller (but not much smaller) than unity to almost unity. In the most extreme case when $n = 10$, the rate of nonlinear energy transfer becomes comparable with the rate at which energy is imparted by random forcing.

For comparison, we also calculated $\langle X_1 X_2 X_3 \rangle$, $\langle X_1^2 \rangle$, $\langle X_2^2 \rangle$ and $\langle X_3^2 \rangle$ from (31) and (33)–(36), for external

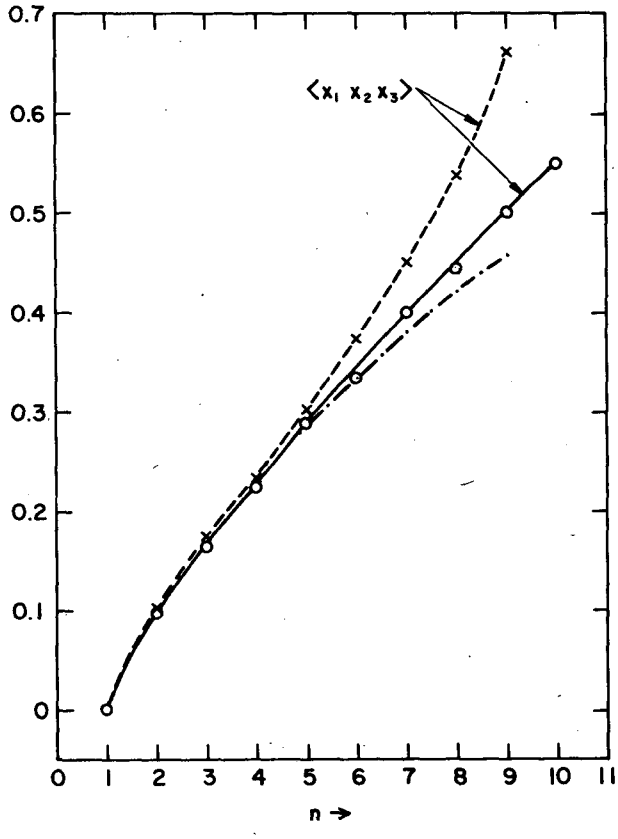

FIG. 1. Energy transfer moment $\langle X_1 X_2 X_3 \rangle$ plotted against the rate of energy input n to the intermediate mode. Results derived from an ensemble of numerical integrations are shown by the solid curve, second-degree truncation by the dashed curve, and third-degree truncation by the dashed–dotted curves.

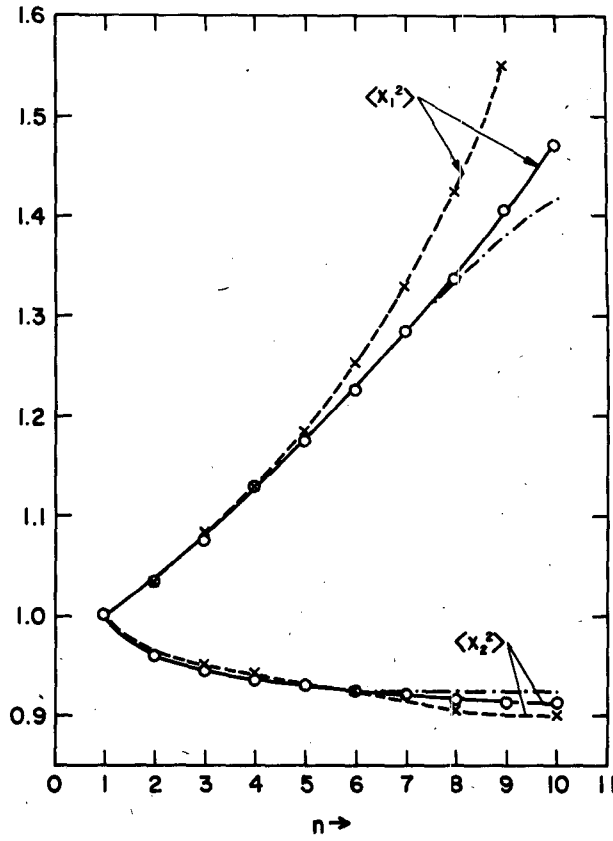

FIG. 2. Scaled kinetic energy $\langle X_1^2 \rangle$ contained in the lowest mode and the energy $\langle X_2^2 \rangle$ in the intermediate mode, both plotted against n.

conditions identical to those of the numerical integrations. Improved theoretical estimates, based on the truncated probability distribution for $N = 3$, are also shown.

The results of both the numerical and theoretical calculations are summarized in Figs. 1, 2 and 3. Figure 1 shows $\langle X_1 X_2 X_3 \rangle$, a measure of the net rate of nonlinear energy transfer between modes, plotted against n. The theoretical estimates based on the second-degree truncation of the probability distribution (dashed curve) and on the third-degree truncation (dashed–dotted curve) agree very well with the numerical results (solid curve) up to about $n = 5$, then gradually diverge as n is increased further. The accuracy of the third-degree truncation of the probability distribution is considerably greater than that of the second degree truncation and, in fact, greater than that of the "sixth moment discard" closure. It appears that the expansion in Hermite polynomials not only converges, but also converges fairly rapidly.

The scaled kinetic energies $\langle X_1^2 \rangle$ and $\langle X_2^2 \rangle$ are shown in Fig. 2 as functions of n, and $\langle X_3^2 \rangle$ is displayed in Fig. 3. As might be expected, the agreement between the numerical results and the theoretical estimates, based on the third-degree truncation of the probability

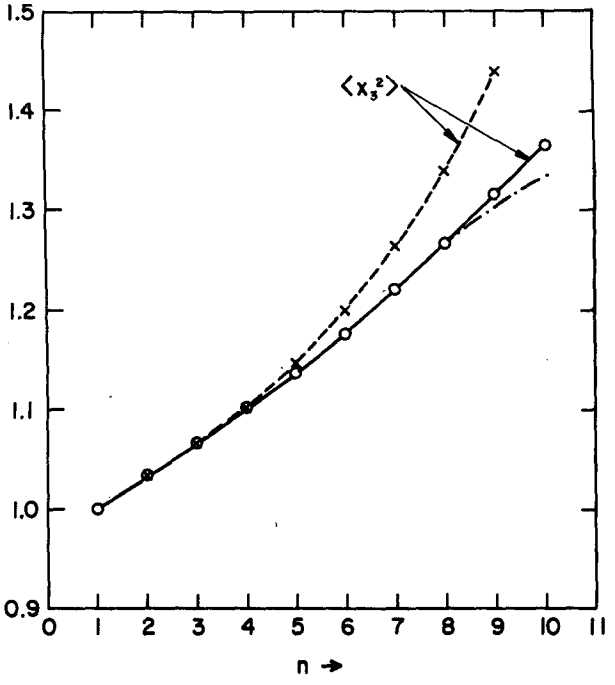

FIG. 3. Scaled kinetic energy $\langle X_3^2 \rangle$ contained in the highest mode, also plotted against n.

distribution, is very good—simply because (34)–(36) are exact. The error is about 3% for $n = 10$, when the rate of nonlinear energy transfer is comparable with the rate of energy input, and within 1% over most of the range in which the nonlinear interactions are weaker.

Similar figures, comparing Monte Carlo calculations with estimates derived from the "sixth moment discard" closure, are shown in Thompson (1985). On balance, statistics derived from the truncated probability distribution appear to be more accurate and have the advantage of not requiring solution of nonlinear equations.

5. Summary and conclusions

In an effort to avoid the closure approximations inherent in moment equations, we have here sought to construct the equilibrium probability distribution as the solution of the classical Fokker–Planck equation of statistical mechanics. In order to compare the results of different methods of comparable complexity, we have represented the solution as a highly truncated expansion in Hermite polynomials. One multiplicative factor of the solution is the Gaussian probability distribution typical of systems forced by "white noise." The other factor is an asymmetrical function, one of whose terms determines the nonlinear energy transfer between the modes of a randomly forced triad.

From the truncated probability distribution, we then calculate the transfer moment and the resulting kinetic energy spectrum. These are compared with statistics derived from a large ensemble of numerical integrations of the original evolution equations for the modal amplitudes. Such comparisons show that our theoretical estimates of modal kinetic energy are accurate to within about 3% when the rate of energy transfer is comparable to the rate of random energy injection, but to within 1% over most of the regime of weaker nonlinear interaction. On the whole, estimates derived from the truncated probability distribution appear to be more accurate than those based on higher moment closures, such as "sixth moment discard."

Acknowledgments. I wish to recognize and thank Mr. Arthur Mizzi for his valuable assistance with the numerical calculations described in section 4.

APPENDIX

Determination of Coefficients in the Series (10), for $N = 3$

The coefficients A_{ijk} are the solutions of the linear system:

$$(\alpha_1^2C_1 + \alpha_2^2C_2 + \alpha_3^2C_3) - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)A_{111} + 2(\alpha_2^2C_2 + \alpha_3^2C_3)A_{200} + 2(\alpha_1^2C_1 + \alpha_3^2C_3)A_{020} + 2(\alpha_1^2C_1 + \alpha_2^2C_2)A_{002} + 4\alpha_1^2C_1A_{022} + 4\alpha_2^2C_2A_{202} + 4\alpha_3^2C_3A_{220} = 0$$

$$\alpha_1^2C_1A_{111} - 2\alpha_1^2A_{200} = 0$$

$$\alpha_2^2C_2A_{111} - 2\alpha_2^2A_{020} = 0$$

$$\alpha_3^2C_3A_{111} - 2\alpha_3^2A_{002} = 0$$

$$(\alpha_2^2C_2 + \alpha_3^2C_3)A_{111} - 2(\alpha_2^2 + \alpha_3^2)A_{022} + 3\alpha_3^2C_3A_{131} + 3\alpha_2^2C_2A_{113} = 0$$

$$(\alpha_1^2C_1 + \alpha_3^2C_3)A_{111} - 2(\alpha_1^2 + \alpha_3^2)A_{202} + 3\alpha_3^2C_3A_{311} + 3\alpha_1^2C_1A_{113} = 0$$

$$(\alpha_1^2C_1 + \alpha_2^2C_2)A_{111} - 2(\alpha_1^2 + \alpha_2^2)A_{220} + 3\alpha_2^2C_2A_{311} + 3\alpha_1^2C_1A_{131} = 0$$

$$(\alpha_1^2C_1 + \alpha_2^2C_2 + \alpha_3^2C_3)A_{200} + 2(\alpha_1^2C_1 + \alpha_2^2C_2)A_{202} + 2(\alpha_1^2C_1 + \alpha_3^2C_3)A_{220} + 4\alpha_1^2C_1A_{222} - (3\alpha_1^2 + \alpha_2^2 + \alpha_3^2)A_{311} = 0$$

$$(\alpha_1^2C_1 + \alpha_2^2C_2 + \alpha_3^2C_3)A_{020} + 2(\alpha_1^2C_1 + \alpha_2^2C_2)A_{022} + 2(\alpha_2^2C_2 + \alpha_3^2C_3)A_{220} + 4\alpha_2^2C_2A_{222} - (\alpha_1^2 + 3\alpha_2^2 + \alpha_3^2)A_{131} = 0$$

$$(\alpha_1^2C_1 + \alpha_2^2C_2 + \alpha_3^2C_3)A_{002} + 2(\alpha_1^2C_1 + \alpha_3^2C_3)A_{022} + 2(\alpha_2^2C_2 + \alpha_3^2C_3)A_{202} + 4\alpha_3^2C_3A_{222} - (\alpha_1^2 + \alpha_2^2 + 3\alpha_3^2)A_{113} = 0$$

$$(\alpha_1^2C_1 + \alpha_2^2C_2 + \alpha_3^2C_3)A_{022} + 2(\alpha_2^2C_2 + \alpha_3^2C_3)A_{222} - (\alpha_1^2 + 3\alpha_2^2 + 3\alpha_3^2)A_{133} = 0$$

$$(\alpha_1^2C_1 + \alpha_2^2C_2 + \alpha_3^2C_3)A_{202} + 2(\alpha_1^2C_1 + \alpha_3^2C_3)A_{222} - (3\alpha_1^2 + \alpha_2^2 + 3\alpha_3^2)A_{313} = 0$$

$$(\alpha_1^2C_1 + \alpha_2^2C_2 + \alpha_3^2C_3)A_{220} + 2(\alpha_1^2C_1 + \alpha_2^2C_2)A_{222} - (3\alpha_1^2 + 3\alpha_2^2 + \alpha_3^2)A_{331} = 0$$

$$(\alpha_1^2C_1 + \alpha_2^2C_2 + \alpha_3^2C_3)A_{111} - 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) \times A_{222} + 3(\alpha_2^2C_2 + \alpha_3^2C_3)A_{311} + 3(\alpha_1^2C_1 + \alpha_3^2C_3)A_{131} + 3(\alpha_1^2C_1 + \alpha_2^2C_2)A_{113} + 9\alpha_1^2C_1A_{133} + 9\alpha_2^2C_2A_{313} + 9\alpha_3^2C_3A_{331} = 0$$

$$(\alpha_1^2C_1 + \alpha_2^2C_2 + \alpha_3^2C_3)A_{222} - 3(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)A_{333} = 0.$$

A short Fortran program was written to evaluate the matrix elements, given α_1^2, α_2^2, α_3^2, C_1, C_2, and C_3. The A_{ijk} were calculated by the Cray I computer, using a standard library routine for matrix inversion.

A result that is convenient in calculating $\langle X_1X_2X_3\rangle$ from the truncated probability distribution is that

$$\int_{-\infty}^{\infty} XH_n(X)e^{-(1/2)X^2}dX = \int_{-\infty}^{\infty} H_1(X)H_n(X)e^{-(1/2)X^2}dX = 0 \quad \text{for} \quad n > 1.$$

This follows directly from the orthogonality of the Hermite polynomials on the interval $(-\infty, \infty)$, with the weighting function $e^{-(1/2)X^2}$. In particular, the only contributions to $\langle X_1X_2X_3\rangle$ arise from the term $A_{111}H_1(X_1)H_1(X_2)H_1(H_3)$ in the series (10).

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