Calculation of Airflow over an Isolated Heat Source with Application to the Dynamics of V-Shaped Clouds

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ABSTRACT

The stably stratified airflow over a three-dimensional elevated heat source is investigated using the linearized equations of motion. A low-level upward motion can be produced for airflow over a prescribed, isolated heat source for a wide variety of mean wind speeds. Above the heating layer, a V-shaped region of upward displacement is formed by the action of the mean wind on the upward propagating waves. The horizontal pattern of the heat source is important in determining the formation of the V-shaped region of upward displacement. A high-pressure region is produced in the vicinity of the heat source at the top of the heating layer. The response of a hydrostatic airflow to a transient heating is a V-shaped region of upward displacement with an embedded region of downward displacement above the heated layer. The whole system advects downstream with a slower speed than the mean wind and eventually disperses. A region of strong divergence is associated with the region of upward displacement above the heated layer. In relation to the thunderstorm generated V-shaped clouds, the cold (warm) area can be explained by the adiabatic cooling (warming) associated with the upward (downward) displacement. In addition, the upwind displacement of the cold area in the upper level may be explained as a gravity wave type phenomenon.

1. Introduction

Curious V-shaped clouds at the top of thunderstorms have been observed in a number of studies (Adler et al., 1981; Negri and Adler, 1981; Negri, 1982; Fujita, 1982; Heymsfield et al., 1983a,b; McCann, 1983; Mack et al., 1983). A V-shaped cloud is a "V"-shaped area of low minimum equivalent blackbody temperature (T_{BB}) at the cloud top of a thunderstorm with the "V" pointed upwind. Associated with this cold area are downwind regions of higher T_{BB}, which may also form a V-shaped pattern (Fujita, 1982). The major characteristics of the V-shaped clouds may be summarized as follows: 1) there exist absolute or relative ambient winds in the upper level; 2) the major cold area and close-in warm areas are located on the upwind and downwind edges, respectively, of the high radar reflectivity core; 3) there exists a large temperature difference between the warm and cold areas which may range from 7°C to 17°C; 4) the high radar reflectivity core of the storm has considerable upwind tilt with height in the upper levels; and 5) storms with "V" features are tropopause penetrating at some time during their lifetime. The V-shaped cold feature and cold–warm couples have been shown (McCann, 1983; Adler et al., 1985) to be correlated with the occurrence of severe weather. The study of the dynamics of the V-shaped clouds may improve the severe weather forecasting.

A number of possible explanations of the warm spot have been proposed: 1) adiabatic warming by the subsidence of ambient air on the lee side of the cloud dome (e.g., Negri, 1982; Heymsfield et al., 1983a,b); 2) internal cloud subsidence on the downwind side in association with mixing with the environment (Schlesinger, 1984; Adler and Mack, 1986); 3) subsidence associated with the cloud top collapsing (Fujita, 1978); 4) mixing of warmer stratospheric air with updraft air ejected from the cloud dome (e.g., Fujita, 1974, 1982; Mills and Astling, 1977); and 5) emissivity difference between the overshooting top of the storm with its strong updrafts and the surrounding cirrus anvil (Mills and Astling, 1977). In studying the formation mechanism of the V-shaped clouds, Fujita (1978) suggests that an overshooting top acts to block the wind and diverts the flow around it. The enhanced-V signature debris associated with the flow erodes the updraft summit while being diverted around and past the remainder of the cloud top. Heymsfield et al. (1983a,b) propose a conceptual model analogous to airflow over mountains, in which the air parcel experiences an upward and downward motion when passing over the cloud dome. This model offers a possible explanation of the close-in warm area by having the air parcel subside adiabatically. In order to investigate the dynamics and the flow structure of the V-shaped cloud in more detail, it is useful to build up a simple mathematical model along this line.

To avoid having to treat the details of the thunderstorm system, we may assume that the latent heat released by it can be represented by a prescribed heat
source. This helps to reduce the problem to an airflow over a prescribed heat source fixed in a moving reference frame, since absolute or relative ambient winds always exist in the upper level. This problem is related to a number of mesoscale problems such as moist convection, heat island problems, sea-land breeze circulations and orographic rain dynamics. The response of a stably stratified airflow to a prescribed heat source or sink has been studied theoretically by a number of authors (e.g., Malkus and Stern, 1953; Smith, 1957; Olfe and Lee, 1971; Raymond, 1972; Thorpe et al., 1980; Smith and Lin, 1982—hereafter SL; Lin and Smith, 1986—hereafter LS; Raymond, 1986), but in order to simplify the problem mathematically, only a few have avoided assumptions such as two-dimensionality.

A corresponding mathematical problem is the airflow past an isolated mountain, which has been studied extensively (e.g., Wurtele, 1957; Crapper, 1962; Smith, 1980). For a stratified hydrostatic flow past an isolated mountain, the flow aloft is composed of vertically propagating mountain waves, while the flow near the ground tends to go around the mountain (Smith, 1980). The airflow over an elevated heat source may behave differently. For example, SL found that the phase relationship between the heating and the induced vertical displacement in a two-dimensional steady flow may be negative depending upon the location of heating and the basic flow structure. A similar result has been found in a study of tropical circulation (Hayashi, 1976).

In this paper, we will investigate the response of a stably stratified airflow over a three-dimensional isolated heat source with application to the dynamics of the V-shaped clouds. Using this approach, we can extend the work of SL and LS to include three-dimensionality. The solution will be obtained numerically by using a Fast Fourier Transform (FFT) algorithm. Section 2 describes the governing equations and the relevant Green's function. This solution may be used to study airflow over a shallow heat source. In section 3 the solution will be extended to cases of a deep heat source with a discontinuous and continuous vertical profile. The transient response of airflow to a time-dependent diabatic heating is investigated in section 4. In these two sections, the solutions are also applied to explain the formation of the thunderstorm-generated V-shaped clouds. In section 5, a summary of the results and some possible extensions of the idealized model are discussed.

2. The governing equation and the relevant Green's function

The small amplitude equation of vertical velocity for a stratified, incompressible, Boussinesq flow in a rotating system may be written (Bretherton, 1966)

\[
(D/Dt + \nu)^2 \nabla^2 w - (D/Dt + \nu)(U \partial w/\partial x + V \partial w/\partial y) \\
+ N^2 \nabla_H^2 w + f^2 w_{zz} = (g/c_p \bar{T}) \nabla_H^2 q
\]

where

\[
\begin{align*}
D/Dt &= \partial/\partial t + U \partial/\partial x + V \partial/\partial y \\
\nabla^2 &= \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 \\
\nabla_H^2 &= \partial^2/\partial x^2 + \partial^2/\partial y^2.
\end{align*}
\]

The definition of symbols can be found in appendix A. The buoyancy frequency \(N\) is assumed to be constant with height in this study. Notice that \(\nu\) is the coefficient of both the Rayleigh friction and the Newtonian cooling which have been added into the system to avoid the problem of net heating in a steady-state flow (SL). For a steady state flow with no vertical wind shear and no rotational effect, Eq. (1) becomes

\[
(U \partial/\partial x + \nu)^2 \nabla^2 w + N^2 \nabla_H^2 w = (g/c_p \bar{T}) \nabla_H^2 q.
\]

In the above equation, the mean flow is assumed to be independent of \(y\). To solve Eq. (2), we determine the relevant Green's function similar to that in LS. Taking the double Fourier transform in \(x\) and \(y\) \((x \rightarrow k, y \rightarrow \ell)\) of Eq. (2), we have

\[
\hat{w}_{zz} + \left[ N^2 - (Uk - i\ell)^2 \right] \hat{w} = \frac{gk^2}{(Uk - i\ell)^2} \hat{\delta}(z)
\]

where \(\kappa = (k^2 + \ell^2)^{1/2}\) is the magnitude of the horizontal wave vector.

Consider a bell-shaped heat source with circular contours

\[
q(x, y, z) = Q_0 ((r/b)^2 + 1)^{-3/2} \delta(z)
\]

\[
r = (x^2 + y^2)^{1/2}
\]

Taking the double Fourier transform of Eq. (4) and substituting into Eq. (3) we get

\[
\hat{w}_{zz} + \lambda^2 \hat{w} = \left\{ gQ_0 b^2 \kappa^2 / [2\pi c_p \bar{T} (Uk - i\ell)^2] \right\} e^{-b_k \delta(z)}
\]

where

\[
\lambda = \left[ N^2 - (Uk - i\ell)^2 \right]^{1/2} \kappa / (Uk - i\ell).
\]

An appropriate set of lower and upper boundary conditions are \(\hat{w} = 0\) at \(z = -H\) and the radiation condition, i.e., \(\hat{w} \sim \exp(i\lambda z)\) as \(z \rightarrow \infty\). At the interface \(z = 0\), one condition is that \(\hat{w}\) is continuous across the interface. Integrating Eq. (5) across the interface yields another condition that \(\hat{w}_z\) is continuous. Thus the solution of Eq. (5) can be obtained

\[
\hat{w}(k, \ell, z) = \frac{igQ_0 b^2 \kappa e^{-b_k}}{4\pi c_p \bar{T} (Uk - i\ell)[N^2 - (Uk - i\ell)^2]^{3/2}}(e^{i\lambda z} - e^{i\lambda |z|})
\]

The vertical displacement, \(\eta\), defined by \(w = D\eta/Dt\), may be written as
\[ \eta = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{8Q_0 b^2 \kappa e^{-b\kappa}(e^{b(z+2H)} - e^{b|\eta|})}{4\pi c_p \Gamma(Uk - iv)\sqrt{N^2 - (Uk - iv)^2}} \times e^{k(x+iy)}dkdl \quad (7) \]

\[ z \geq -H \]

Now let us introduce nondimensional variables
\[ (\tilde{x}, \tilde{y}) = (x/b, y/b); \quad (\tilde{\kappa}, \tilde{\eta}, \tilde{\zeta}, \tilde{H}) = (bk, bl, b\kappa); \]
\[ \tilde{\nu} = \nu b/U; \quad (\tilde{\eta}, \tilde{\zeta}, \tilde{H}) = (\eta N/U, zN/U, HN/U); \quad (8) \]
\[ \tilde{Q}_0 = Q_0 gb/(c_p \tilde{T}U^3) \]

The unit of \( Q_0 \) is J m kg\(^{-1} \) s\(^{-1} \) for a vertical heating profile of Eq. (4). Thus Eq. (7) becomes (the tilde is dropped)
\[ \eta = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\tilde{\varrho}(\kappa)(e^{\tilde{\kappa}(z+2\tilde{H})} - e^{\tilde{\kappa}|\tilde{\eta}|})}{2k(k - iv)\sqrt{1 - \tilde{M}^2(k - iv)^2}} e^{(kx+iy)}dkdl \quad (9) \]

where \( \tilde{\varrho}(\kappa) = (Q_0/2\pi) \exp(-\kappa) \). Notice that the non-hydrostatic effect is represented by a nondimensional number \( \tilde{M} (= U/bN) \), which is proportional to the ratio of the period of a buoyancy oscillation \( (2\pi/N) \) to the time it takes for an air parcel to cross the heat source \( (b/U) \). This is similar to the mountain wave problem in which the horizontal scale is measured by the mountain width (Smith, 1979). For simplicity, we will assume the flow is hydrostatic \( (M \ll 1) \) in most cases. The nonhydrostatic effect will be investigated in section 3. The solution of Eq. (9) may be regarded as an airflow over a shallow heat source in which the depth of the heating is much smaller than the vertical wavelength of the basic flow (i.e., \( U/N \)).

A two-dimensional FFT algorithm is employed to solve Eq. (9). A summary of the numerical technique can be found in Smith (1980). Figure 1 shows an example of a hydrostatic flow over a shallow heat source with \( \tilde{H} = \pi \). The dimensional parameters may be considered as \( U = 10 \text{ m s}^{-1}, N = 0.01 \text{ s}^{-1}, b = 5 \text{ km}, \) and \( H = 3.14 \text{ km} \). The response of the fluid to the heating

**Fig. 1.** Vertical displacement of the hydrostatic flow over an isolated heat source which is added at \( z = 0 \). The dashed circle is the heating contour at \( r = b \). The basic flow is directed from left to right in the \( x \) direction. The solution is given by Eq. (9) with \( \tilde{H} = \pi, \tilde{M} = 0, \nu = 0.2 \). The four levels shown are (a) \(-\pi/2\), (b) 0, (c) \(\pi/2\), and (d) \(\pi\).
at the heating level, \( z = 0 \), is a downward displacement upstream of the prescribed heat source followed by an upward displacement downstream. This is similar to the two-dimensional flow as studied in SL. The negative-phase relationship between heating and displacement can be explained either by an energy argument or the development of downward displacement near the heat source which is associated with a region of growing positive displacement moving downstream (LS). The disturbance of the flow at the heating level is almost confined in the direction perpendicular to the mean wind.

The region of disturbance widens in general as we move aloft and beneath the heating level. A V-shaped pattern in the region of upward displacement forms above the heating center at the level of \( z = \pi/2 \). This region of upward displacement is shifted upstream as we move further aloft as required by the radiation condition. At the level of \( z = \pi \), a new region of downward displacement forms just downstream of the V-shaped area of upward displacement. The hydrostatic flow is almost periodic in the vertical with a wavelength of \( \pi \) (e.g., comparing Fig. 1a, c) like that in the mountain wave theory (Queney, 1947). The amplitude of the vertical displacement decreases vertically, which is mainly due to the divergence above the heating region and the viscosity.

The vertical cross section along the x-axis for the foregoing case is plotted in Fig. 2a. The upstream (downstream) phase tilt of the disturbance in the layer above (below) the heating level (\( z = 0 \)) indicates that the wave energy propagates upward (downward) (Eliassen and Palm, 1960). The term \( \exp(2H) \) in the numerator of Eq. (9) is associated with the reflected wave from the ground, which may cancel the direct outgoing wave, i.e., the term \( \exp(Iz) \), above the heating level with some special values of \( H \). This is similar to the two-dimensional flow (SL). One example with \( H = 2\pi \) is shown in Fig. 2b in which the disturbance above the heating level (\( z = 0 \)) is much smaller than the case of Fig. 2a.

Similar to the mountain wave theory (Smith, 1980), the formation of the V-shaped pattern of the vertical displacement can be explained by the group velocity argument. At any height \( z \) above the heating level, the wave energy associated with the forced perturbation is concentrated near the parabola \( y^2 = Nzbx / U \) for airflow over an isolated heat source. At higher altitudes this parabola becomes wider in the direction perpendicular to the mean wind and the vertex of the parabola displaces farther upstream. The horizontal scale in the y-direction of the parabola is determined by the buoyancy frequency \( N \), the speed of the mean wind \( U \) and the horizontal scale of the heat source.

**3. Flow over a deep heat source**

In the real atmosphere, the regions of latent heating tend to be distributed in a layer instead of being concentrated at one level, as in section 2. For simplicity, we first consider a case with heating distributed uniformly in the vertical. A more complicated heating profile will be considered later. The solution can be obtained by applying the Green's function method to the solution of Eq. (9). The result may be written as

\[
\begin{align*}
\eta_1 &= \int_{-\infty}^{\infty} \hat{q}(k) \sin \lambda z \left( e^{i k z_2} - e^{i k z_1} \right) e^{i(kx+y)d} dkdl, \quad 0 < z < z_1 \\
\eta_2 &= \int_{-\infty}^{\infty} -i\hat{q}(k) e^{i k z} \left( \cos \lambda z - \cos \lambda z_1 \right) e^{i(kx+y)d} dkdl \\
&\quad + \int_{-\infty}^{\infty} \hat{q}(k) \sin \lambda z \left( e^{i k z_2} - e^{i k z_3} \right) e^{i(kx+y)d} dkdl, \quad z_1 < z < z_2 \\
\eta_3 &= \int_{-\infty}^{\infty} -i\hat{q}(k) e^{i k z} \left( \cos \lambda z_2 - \cos \lambda z_3 \right) e^{i(kx+y)d} dkdl, \quad z_2 < z. \quad (10)
\end{align*}
\]

The heating layer extends from \( z_1 \) to \( z_2 \). The level \( z_1 \) may represent the cloud base or the top of the moist boundary layer where the surface air becomes unstably buoyant in a cumulus convection (Lindzen, 1974). The variables are nondimensionalized according to Eq. (8) except \( \hat{Q}_0 = Q_{ab}/(c_T U^2 N) \).

**a. Vertical motion at the heating base**

For prescribed heating as used in this study, it is important that the vertical motion be consistent with the heating at the heating base in order to support the existing convection. The vertical velocity can be obtained immediately from the dimensional relationship \( w = U \hat{\nu}_s / \partial x \) for a steady flow and Eq. (10). In the heating layer, we have

\[
\begin{align*}
w_z &= \int_{-\infty}^{\infty} \hat{q}(k) e^{i k z} \left( \cos \lambda z - \cos \lambda z_1 \right) e^{i(kx+y)d} dkdl \\
&\quad + \int_{-\infty}^{\infty} i\hat{q}(k) \sin \lambda z \left( e^{i k z_2} - e^{i k z_3} \right) e^{i(kx+y)d} dkdl. \quad (11)
\end{align*}
\]

Notice that \( w_z \) is a function of \( z_1 \) and \( z_2 \). The reciprocal of \( (z_2 - z_1) \) is equivalent to the heat induced Froude number as defined in LS, which has a dimensional form of \( F = U / N(z_2 - z_1) \).

Figure 3 shows the vertical velocities at the heating base \( z_1 \) for \((z_1, z_2) = (2, 18), (1, 9), (0.5, 4.5), (0.25, 2.25), \) and \((0.125, 1.125) \). The dimensional parameters may be considered as \( N = 0.01 \text{ s}^{-1} \), \( z_1 = 1 \text{ km} \), \( z_2 = 9 \text{ km} \)
km, $b = 5$ km, and $U = 5, 10, 20, 40, 80$ m s$^{-1}$. For a fixed heating depth (dimensional), a smaller $z_2 - z_1$ corresponds to a higher mean wind speed. In Fig. 3a, the region of positive vertical velocity has an elongation in the direction perpendicular to the mean wind. There are two regions of weak downward motion on the upwind and downwind sides. This pattern of vertical velocity is caused by the advection of the mean wind since it should respond to the axisymmetric shape of the heat source in a quiescent fluid. The advection effect is more significant for cases with larger mean winds (Fig. 3c–e). In Fig. 3d, e, both regions of downward and upward velocity extend farther downstream compared with Fig. 3b and 3c and form a V-shaped pattern. Except for strong winds such as the cases of Fig. 3d and e, the vertical velocities are upward in the region of $r < b$. An upward motion at low level such as $z = z_1$ may be satisfied for a flow with a wide variety of the mean wind speeds in the present model. As the wind speed increases, the heating layer becomes a heating plane. Thus the vertical velocity field shown in Fig. 3e should approach the $x$-derivative of the vertical displacement field shown in Fig. 1b.

The response of airflow over an elongated or a two-dimensional heat source is very different from that over an isolated heat source. To elucidate this point, we calculate the vertical velocities at the heating base for a case of an elongated heat source. The width of the heat source in the $y$-direction is elongated 20 times more than that in the $x$-direction. Figure 4 displays the vertical velocity at $z = z_1$ along the $x$ axis. In the region of $|x| < b$, the air parcel may experience either an upward or a downward motion depending upon the strength of the mean wind. For example, the air parcel has a downward velocity upwind of the heating center followed by an upward velocity for cases of Fig. 4b, d. The result of the two-dimensional flow of SL is recomputed and plotted in Fig. 5 for comparison. For airflow over a two-dimensional steady heat source, the consistency between the heating and low-level upward motion can be satisfied only for certain values of the heating-induced Froude number (Raymond, 1986).

b. Flow response in a hydrostatic atmosphere

Figure 6 shows the vertical displacement for the case of Fig. 3b; i.e., the heating is distributed uniformly from $z_1 = 1$ to $z_2 = 9$. The dimensional flow parameters considered are $N = 0.01$ s$^{-1}$, $z_1 = 1$ km, $z_2 = 9$ km, $b = 5$ km, and $U = 10$ m s$^{-1}$. The nondimensional parameter $M$ in this case is 0.2, which may be in the weakly nonhydrostatic regime. The nonhydrostatic effect will be investigated in subsection 3c. The vertical velocity at the heating base is shown in Fig. 3b. The vertical displacement is related to the vertical velocity by the steady state relationship $w = U \partial \eta / \partial x$. In the heating layer, the response of the airstream to the thermal forcing is a downward displacement upstream of the heating center followed by an upward displacement downstream. Notice that the phase tilt of the vertical displacement is not significant in the heating layer. This may be explained by the equal strength of the heating rate in the layer. As we move aloft to $z = 14$ (Fig. 6d), V-shaped regions of upward and downward displacement are formed and located on the upwind and downwind sides of the heating center, respectively. The V-shaped regions are formed by the action of the mean wind on the direct and reflected upward propagating gravity waves.

Figure 7a–c shows the lateral displacement, the lateral velocity, and the perturbation pressure fields at the level of $z = 5$ for the case shown in Fig. 6. The equations for these fields in the Fourier space can be derived from the basic equations in a hydrostatic atmosphere (see appendix B) and may be written

$$\hat{x} = \frac{i \hat{q}(k)}{k} k(k-i\alpha)^2 [\cos \lambda z e^{i\alpha z_2} - \cos \lambda z e^{i\alpha z_1}]$$  (12a)

$$\hat{\theta} = i k \hat{x},$$  (12b)
\[ \hat{\rho} = (-ik(k - iv)/l) \hat{\xi}, \text{ for } z_1 < z < z_2. \]  \hspace{1cm} (12c)

Again the FFT algorithm is employed to transform these fields back to the physical space. As an air parcel to the right of the x-axis approaches the heating region, it curves to the left first and then to the right immediately after it passes the heating center (Fig. 7a). This lateral displacement is related to the lateral velocity field (Fig. 7b) according to a dimensional relationship of \( v = U\theta_\xi/\partial x \). The lateral velocity field can be explained by the positive (negative) pressure gradient (Fig. 7c) in the y-direction upstream (in the vicinity) of the heating center. Along the x-axis, the air parcel is not curved because there exists no lateral pressure gradient. The flow behaves differently from that over an isolated mountain in which an air parcel curves away from the x-axis upstream of the mountain (Smith, 1980). Notice that the magnitude of the lateral displacement is almost one order smaller than that of the vertical displacement (Fig. 6b). However, the lateral displacement becomes

Fig. 3. Vertical velocity at \( z = z_1 \) of the hydrostatic flow over a heat source which is distributed uniformly from \( z_1 \) to \( z_2 \). The solution is given by Eq. (11) with \( M = 0 \) and \( \nu = 0.2 \). The four cases of different \((z_1, z_2)\) shown are: (a) \((2, 18)\), (b) \((1, 9)\), (c) \((0.5, 4.5)\), (d) \((0.25, 2.25)\), and (e) \((0.125, 1.125)\).
larger for larger mean winds, which may affect the flow field significantly.

On top of the heating layer, there exists a high pressure region in the vicinity of the heating center (Fig. 7d). This is consistent with observations (Fujita, 1974). The horizontal pressure gradient of the mesohigh aloft acts to slow down the air parcel in the vicinity of the heating center according to the x-component momentum equation

\[ U_{x} = (-1/\rho)p_{x} - \nu u. \]

As in the vertical displacement field, the pressure field (Fig. 7d, e) shows an upstream phase tilt which indicates an upward propagation of the wave energy. This implies that the vertical transport of the horizontal momentum is downward (Eliassen and Palm, 1960).

In a region with no thermal forcing such as \( z > z_{2} \), the density anomaly is mainly caused by the thermally generated vertical motion according to the thermodynamics equation (B3) and is related to the vertical displacement assuming the viscous effect is negligible (see appendix B):

\[ \rho = (\rho N^{2}/g)\eta. \] (13)

Thus the V-shaped regions of the positive and negative vertical displacement shown in Fig. 6d correspond to the V-shaped regions of cold and warm air, respectively. In the region of positive (negative) vertical displacement, the cold (warm) air is produced by adiabatic cooling (warming). The upwind displacement of the cold region with height (Fig. 6d, e) can then be explained by the upward propagating gravity waves generated by the latent heating in a moving airstream.

Figure 8 shows a case similar to the case of Fig. 6 except the heat source is elongated 20 times wider in the y direction. The vertical velocity at the heating base is shown in Fig. 4b. Above the heating layer, there is no V-shaped area of vertical displacement produced, as opposed to the previous case (Fig. 6). The result indicates that the horizontal pattern of the heat source is important for the formation of a V-shaped region of vertical displacement.

c. Application to the dynamics of V-shaped clouds

Studies of the Wave-CISK mechanism, in which the heating is parameterized in terms of the low-level moisture convergence, indicate that the airflow is sensitive to the vertical profile of the diabatic heating (e.g., Stevens and Lindzen, 1978). In this problem with a prescribed heating, we consider a case here with continuous heating profile in the vertical:

\[ f(z) = A \sin(\pi(z - z_{1})/(z_{2} - z_{1}))e^{-\alpha(z - z_{0})} \]

\[ = 0 \quad \text{for} \quad 0 < z < z_{1} \quad \text{and} \quad z > z_{2} \] (14)

where \( A \) is a factor for adjusting the vertical integration of the heating function to be \( 2(z_{2} - z_{1})/\pi \) so that the total heating rate is the same as that for \( \alpha = 0 \). A similar profile has been used for simulating the convective heating in the problem of tropical circulation (e.g., DeMaria, 1985). Similar to the case of uniformly distributed heating, the nondimensional form of the solution for \( z > z_{2} \), (which we are most interested in) may be written

\[ \eta = \int_{-\infty}^{\infty} \frac{Aq(\kappa)\kappa e^{\rho z}}{2Dk\{(1 - M^{2}(k - i\nu))^{2}/(k - i\nu)^{2}\}}^{1/2} \]

\[ \times \left[ \frac{e^{i\lambda z_{1}}}{(i\lambda - \alpha)^{2} + (\pi/D)^{2}(e^{i\lambda - \alpha})D + 1} \right. \]

\[ \left. - \frac{e^{-i\lambda z_{1}}}{(-i\lambda - \alpha)^{2} + (\pi/D)^{2}(e^{-i\lambda - \alpha})D + 1} \right] \times e^{i(kx + ly)} dk dl \] (15)
where

\[ D = z_2 - z_1. \]

Figure 9 shows the vertical displacement for a case with \( M = 0.2, \alpha = 0, z_1 = 1 \text{ km}, z_2 = 9 \text{ km}, b = 5 \text{ km}, \) and \( U = 10 \text{ m s}^{-1}. \) At the top of the heated layer (Fig. 9a) a V-shaped region of upward displacement is developed and the whole pattern is shifted farther upstream compared with the case with uniform heating (Fig. 6c) in which there is no V-shaped pattern developed at the heating top. This is because the gravity waves generated by the diabatic heating are dominated by the forcing term near the level of maximum heating \((z = 5)\). The disturbance also depends on the asymmetry of the heating profile (i.e., \( \alpha \)). At levels of \( z = 11.5 \) and \( z = 14 \) (Fig. 9b, c), a V-shaped region of upward displacement is formed and shifted farther upstream.

In application to the dynamics of V-shaped clouds, the prescribed heating may represent the latent heat released by the storm, which is fixed in a reference frame of the storm. Notice that only the latent heat of condensation in the troposphere is considered since the latent heat of fusion in the stratosphere is small and may be ignored. As implied by Eq. (13), the V-shaped regions of positive (negative) vertical displacement correspond to the cold (warm) area. The ascent (subsidence) of an air parcel could produce a cold (warm) area by adiabatic cooling (warming), which is consistent with the observations (Heymsfield et al., 1983a).

Downwind of the major V-shaped region of upward displacement there exist a major region of downward displacement and a trail of damped oscillations (Fig. 9c). The major region of downward displacement also appears to have a V-shape. This may correspond to a V-shaped warm area as sometimes observed (Fujita, 1982). The trail of damped oscillations may not be visible in the atmosphere if there is no sufficient anvil cirrus material present. In addition, the disturbance has a narrower extent in the \( y \)-direction than that of the case of Fig. 6. Similar to the mountain wave problem, the damped oscillations and the narrower extent
in the y-direction of the disturbance is mainly caused by the nonhydrostatic effect. The effect of the continuous heating is the upwind shifting of the whole pattern at the heating top (Fig. 9a). In the Lahoma storm (storm B1 in Heymsfield et al., 1983a), there are two warm areas observed downwind of the major cold area. The close-in warm area moves with the environmental winds. Furthermore, the trajectories of cold and warm regions seem to be related to the internal dynamics of the storm (Schlesinger, 1984; Adler and Mack, 1986). Subsidence due to collapsing tops may also generate warm regions (Fujita, 1978). However, the present

Fig. 7. (a) Lateral displacement, and (b) lateral velocity at \( z = 5 \) for the case of Fig. 6. Three levels of the perturbation pressure field are shown in (c) \( z = 5 \), (d) \( z = 9 \), and (e) \( z = 14 \).
Fig. 8. As in Fig. 6 except the heat source is 20 times wider in the y direction. The two levels of the vertical displacement shown are (a) 1, (b) 14.

study indicates that the distant warm area may correspond to a region of downwind displacement in the trail if the nonhydrostatic effect is stronger. The repeating of the cold and warm regions may be more pronounced if a more realistic wind profile and stratification are considered. For example, a trapped wave

Fig. 9. Vertical displacement of a flow over an isolated heat source according to Eq. (14) with $z_1 = 1$ to $z_2 = 9$. The solution is given by Eq. (15) with $M = 0.2$, $r = 0.2$ and $\alpha = 0$. The three levels shown are (a) 9, (b) 11.5 and (c) 14. The divergence at $z = 9$ is shown in (d).
downstream, similar to the mountain lee waves, may occur if the Scorer parameter decreases rapidly with height (Scorer, 1949).

The divergence field can be calculated from Eq. (15) and the continuity equation with the steady state relationship $w = \frac{U}{\partial z/\partial x}$:

$$\nabla_H \cdot V = FT^{-1}[k\lambda \eta(k, l, z)]$$  \hspace{1cm} (16)

Figure 9d shows the divergence field corresponding to Fig. 9a at the top of the heating layer ($z = 9$). The divergence and vertical displacement fields shown in Figs. 10 and 11 suggest the existence of a gravity wave where the fields of divergence and the vertical velocity are in phase at all levels.

4. Transient response

The time evolution of an airflow over an isolated heat source can be studied by solving an initial value problem. This is important because the thermal forcing associated with the latent heating of a thunderstorm cloud is not stationary. The variations of the cold and warm regions with time are often observed (e.g., Figs. 1 and 2 of Negri, 1982). For a hydrostatic flow with no vertical wind shear and no rotational effect, Eq. (1) reduces to

$$\left(\frac{\partial}{\partial t} + \frac{U}{\partial x}\right)^2 w_{zz} + N^2 \nabla_H^2 w = \left(\frac{g}{c_p T}\right) \nabla_H^2 q$$ \hspace{1cm} (17)

Notice that the Rayleigh friction and the Newtonian cooling are excluded in the system since there is no net heating problem in solving an initial value problem (LS). First, let us consider a simple case of a pulse of heat in a half plane,

$$q(t, x, y, z) = Q_0 [(t^2 - 1)]^{-3/2} \delta(z) \delta(t), \quad -H < z. \hspace{1cm} (18)$$

Taking the Fourier transforms in $x$, $y$ and $t$ ($t \to \omega$) of Eqs. (17) and (18) and following the procedure of the steady state problem in section 2, the relevant Green’s function can be found.

![Fig. 10. Time evolution of the vertical displacement at $z = 2.78$ of a hydrostatic flow over an isolated heat source which is described by Eq. (21) with $z_1 = 0.28$ and $z_2 = 2.78$. The solution is given by Eq. (22) with $Q_0 = 1$, $\alpha = 0$ and $\beta = 0.25$. The three time steps shown are (a) 0, (b) 2 and (c) 4. The storm center is assumed to be located at $X$ at $t = 4$. The divergence field corresponding to c is shown in d. Actual magnitudes are those indicated times $10^{-2}$.](image-url)
\[
\eta = \int \int g(\omega, \kappa) e^{i(\omega z + 2H)} e^{-i\alpha y} e^{-\beta x} e^{-i(\omega x + \kappa y)} d\omega d\kappa
\]

where

\[
\begin{align*}
\lambda &= \kappa N/(\omega + Uk), \\
\hat{g}(\omega, \kappa) &= Q_0 b^2 (8\pi^2 \alpha)^{-1/2} \exp(-b\kappa) \exp(-\omega^2/4\alpha).
\end{align*}
\]

The inverse Fourier transform of \(\omega\) can be performed leaving

\[
\eta = \int \int \hat{g}(\omega, \kappa) e^{-ib\kappa} e^{-i\beta x} e^{-i(\omega x + \kappa y)} d\omega d\kappa
\]

\[
(z + 2H)^{1/2} - J_1(2\sqrt{\kappa N}/|z|)|z|^{1/2} e^{i(\kappa x + \beta y)} d\kappa d\beta
\]

where \(J_1\) represents the Bessel function of the first kind and of order one. Equation (20) reduces to the solution of LS for a two-dimensional flow. An asymptotic solution of the integrand of the above equation can be obtained for large time. The response of the fluid to a pulse of heat is an outwardly moving ring of updraft in the reference frame moving with the mean wind. The gravity waves produced by a pulse of heat in an unsheared flow are axisymmetric about its center and impart no net momentum flux to the flow. The vertical displacement associated with the disturbance will reach a maximum height which is proportional to the maximum heating rate added to the flow. Basically the response is similar to a two-dimensional flow (LS).

Now let us consider a more complicated heating function

\[
q(t, x, y, z)
\]

\[
\begin{align*}
AQ_0[(r/\beta)^2 + 1]^{-3/2} \sin[\pi(z - z_1)/(z_2 - z_1)] \\
\times \exp[-\alpha(z - z_1)] \exp(-\beta t^2)
\end{align*}
\]

(21)

for \(z_1 < z < z_2\),

0 for \(0 < z < z_1\) and \(z > z_2\).

The above heating function may represent the latent heat released by a cumulus cloud lasting for a certain period. The symbol \(\alpha\) is a slope parameter for adjusting the asymmetry of the vertical profile of the heating and
\( \beta \) is a slope parameter for the time scale. Similar to the steady-state problem, the nondimensional form of the solution for 0 > k may be obtained

\[
\eta \approx \int_{-\infty}^{\infty} Aq(\omega, \beta, z) \left( \frac{D}{\alpha} \right)^{\frac{1}{2}} \left( \frac{\lambda}{\alpha} \right)^{\frac{1}{2}} \left( 1 + \frac{1}{\alpha^2} \right) e^{i\beta \omega} \left( 1 + \frac{1}{\alpha^2} \right) e^{-i\beta \omega} \right) d\omega dkd\lambda,
\]

(22)

where

\[
q(\omega, \beta) = Q_0 \exp(-\kappa) \exp(-\omega^2/\lambda), \quad \lambda = \kappa/(\alpha + \beta).
\]

Solution of the above equation can be obtained numerically using a three-dimensional FFT. Figures 10 and 11 display a case with \( \alpha = 0, \beta = 0.25, z_1 = 0.28 \) and \( z_2 = 2.78 \). The corresponding dimensional parameters may be considered as \( z_1 = 1 \text{ km}, z_2 = 10 \text{ km}, \beta = 5 \text{ km}, N = 0.01 \text{ s}^{-1} \) and \( U = 36 \text{ m s}^{-1} \). Notice that the heating starts from \( t = 0 \), reaches its maximum at \( t = 0 \), and then decreases to its \( \tau \)-fold value at \( t = 4 \). The dimensional heating \( \tau \)-fold time is 30 min. A rainfall rate \( R \) lasting for a period of \( \Delta t \) is associated with a latent heating at the center of the rain source

\[
Q_0 = \pi LR\Delta t \rho_w/[2\rho_d(z_2 - z_1)].
\]

(23)

For \( R = 1 \text{ mm h}^{-1} \) and the latent heat of condensation \( L = 5 \times 10^5 \text{ J kg}^{-1}, Q_0 \) is approximately 7000 J kg\(^{-1}\) for a rainfall period of 1 h. The choice of a nondimensional heating \( Q_0 = 1 \) represents a precipitation rate of about 27 mm h\(^{-1}\) lasting for 1 h. This is not unrealistic compared to the estimated rainfall rates from radar echo reflectivity, which may range from 3–5 mm h\(^{-1}\) for weak cells to 49 mm h\(^{-1}\) for strong cells (Negri and Adler, 1981).

In the early stages the response of the fluid to the transient heating is an upward displacement at the top of the heated layer (Fig. 10a). When the heating reaches its maximum, i.e., \( t = 0 \), the disturbance is still growing. For wave growing with time, the peak of the upward displacement propagates downstream with a slower speed than the mean wind (LS). At \( t = 2 \) (Fig. 10b), the upward displacement has a maximum at \( x = 0.8 \), with a stronger gradient upstream of it. There is a region of weak downward displacement developed upstream of the region of upward displacement. This is due to the compensating downdraft associated with the major region of updraft. The disturbance develops to a V-shaped region of upward displacement with two maxima and spreads out to a larger region at \( t = 4 \) (Fig. 10c). The downstream side of the region of upward displacement may extend far downstream due to the advection of the mean wind. The maximum height (=0.046) becomes smaller as the energy is distributed to a larger area of the disturbance compared with that of the earlier time (Fig. 10b). The system is dispersive and advecting farther downstream as the heating weakens.

Similar to the steady-state problem [Eq. (16)], the divergence field may be obtained

\[
\nabla H \cdot V = F T^{-1} [\eta(\omega, \beta, \xi, \psi)].
\]

(24)

Figure 10d shows the divergence field corresponding to Fig. 10c. Associated with the region of upward displacement, there exists a region of strong divergence. This is consistent with the steady state flow as studied in section 3. Such divergent regions above storms have been observed (Fujita, 1982; Heymsfield et al., 1983a,b). Two regions of convergence, located upward and downstream of the region of divergence, are associated with regions of downward displacement at a later time (not shown).

Moving further aloft to \( z = 3.89 \) (Fig. 11), the evolution of the disturbance is similar to that at \( z = 2.78 \) (Fig. 10) except there exists a close-in region of downward displacement embedded downstream of the more pronounced V-shaped region of upward displacement. The closed-in warm (downward displacement) region evolves with time (Fig. 11b, c), which does not exist at \( t = 0 \). This is consistent with observations (e.g., Figs. 1 and 2 of Negri, 1982). The upstream phase tilt of the disturbance indicates that there are upward propagating waves produced by the transient heating. If one assumes the storm center is located at the maximum of the upward displacement along \( y = 0 \) at the top of the heated layer (denoted by \( "X" \) in Fig. 10c), then the close-in warm area at \( z = 3.89 (z = 14 \text{ km}) \) is displaced downstream about 8 km (\( x = 0.8 \)) while the V-shaped cold area is displaced upstream about 12 km (\( x = 1.2 \)). This model may offer an explanation for the large displacement of the cold air upstream to the storm top and the downstream movement of the cold and warm areas. The vertical displacement from the highest to lowest point at \( z = 3.89 (z = 14 \text{ km}) \) can be estimated to be about 220 m (\( \Delta H \approx 0.06 \)). In the EMC sounding of the Lahoma storm (see Fig. 2 of Heymsfield et al., 1983a), the air parcel would be heated to temperatures 12\(^\circ\)C higher than the environment by subsiding adiabatically 500 m at the level of 14–16 km. For an air parcel subsiding 500 m in the present case, it would produce a temperature difference of about 5.3\(^\circ\)C higher than the ambient air. The divergence field at \( z = 3.89 \) (Fig. 11d) shows a region of divergence associated with the region of upward displacement similar to that at \( z = 2.78 \) (Fig. 10d). Notice that there exists a region of convergence downstream of the region of divergence.

5. Conclusions

1) The response of a stably stratified airstream over an isolated, prescribed heat source was investigated using linear theory. The solutions were obtained numerically by the Fast Fourier Transform algorithm. In the heating layer, the air parcel experiences a downward displacement upstream of the heating center followed by an upward displacement downstream. This pheno-
nomenon is similar to the two-dimensional flow. Above the heating layer, a V-shaped region of upward displacement is formed. The formation mechanism of the V-shaped regions of upward and downward displacement could be explained by the advection of the mean wind on the upward propagating waves generated by the latent heating. Similar to the mountain wave theory, the wave energy is concentrated in a V-shaped region trailing downstream along the parabolas \( y^2 = N z b x / U \).

2) One interesting finding of this study was that the vertical velocities at the base of a prescribed isolated heat source are very different from that of an elongated or a two-dimensional heat source. The vertical velocities at the heating base are almost always positive in the region of \( r < b \) for a wide variety of mean wind speeds. This is because the air is allowed to deflect laterally in passing the three-dimensional isolated heat source. For prescribed heating as used in this study, it is important that the vertical motion be consistent with the heating at the heating base in order to support the existing convection.

3) The basic pattern of airflow over a heat source with continuous vertical profile is mainly dominated by the gravity waves generated from the level of maximum heating. It was found that the horizontal pattern of the heat source is important in determining the formation of the V-shaped region of vertical displacement. A high pressure region is produced in the vicinity of the heat source at the top of the heating layer. The response of a nonhydrostatic flow to a stationary heat source has a trail of damped oscillations and the disturbance is confined in a narrower region in the direction perpendicular to the mean wind.

4) The response of a hydrostatic airflow to a transient heating is a V-shaped region of upward displacement formed upward with an embedded V-shaped region of downward displacement above the heated layer. The region of downward displacement is formed in the later stages. The whole system advects downstream with a slower speed than the mean wind and eventually disperses. The transient heating is able to generate upward propagating waves. A region of strong divergence is associated with the region of upward displacement above the heated layer.

5) When applied to the dynamics of V-shaped clouds, the cold (warm) area above a thunderstorm is explained by the adiabatic cooling (warming) associated with the upward (downward) displacement. The upwind displacement of the cold area in the upper level might be explained by the upward propagating gravity waves generated by the latent heating from the middle and lower layers in a moving airstream. The movement of the whole system in the direction of the mean wind is consistent with the observations.

6) The model in this paper may be extended to include a more realistic heating parameterization and stratification, nonlinearity, wind shear, and rotation. This will allow one to compare the theoretical results with observations, such as the horizontal scale separation between warm and cold regions, temperature differences, the width of V-shaped clouds, etc., in more detail. To include such factors, a numerical model might be necessary.

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APPENDIX A

List of Symbols

- \( b \): width of the heat source
- \( c_p \): specific heat capacity at constant pressure
- \( D \): depth of the heating layer (\( z_2 - z_1 \))
- \( f \): Coriolis parameter
- \( F \): heat-induced Froude number defined as \( U / N (z_2 - z_1) \)
- \( FT^{-1} \): inverse Fourier transform
- \( g \): gravitational acceleration
- \( H \): depth of the ground to the shallow heating level
- \( J_1 \): Bessel function of the first kind and of order 1
- \( k \): wavenumber in \( x \) direction
- \( l \): wavenumber in \( y \) direction
- \( L \): latent heat of condensation
- \( M \): parameter of the nonhydrostatic effect defined as \( U/b N \)
- \( N \): Brunt-Väisälä (buoyancy) frequency
- \( p \): perturbation pressure
- \( q \): heating rate per unit mass
- \( Q_0 \): amplitude of heating function
- \( r \): radial distance from the origin
- \( R \): rainfall rate
- \( t \): time
- \( \bar{T} \): incoming temperature
- \( U \): incoming velocity in \( x \) direction
- \( v \): lateral velocity
- \( V \): incoming velocity in \( y \) direction
- \( W \): perturbation horizontal wind
- \( w \): vertical velocity
- \( x \): downstream coordinate
- \( y \): lateral coordinate
- \( z \): vertical coordinate
- \( z_1 \): base of the heating layer
- \( z_2 \): top of the heating layer
- \( \alpha \): slope parameter of heating function in \( z \) direction.
\( \beta \) slope parameter of heating function in time

\( \delta \) Dirac delta function

\( \xi \) lateral displacement

\( \eta \) vertical displacement

\( \kappa \) horizontal wavenumber

\( \rho \) perturbation density

\( \rho_0 \) incoming density

\( \rho_a \) density of dry air

\( \rho_w \) density of liquid water

\( \nu \) coefficient of Rayleigh friction and Newtonian cooling

\( \omega \) wave frequency

\( \nabla^2 \) Laplacian operator

\( \nabla_H^2 \) horizontal Laplacian operator

APPENDIX B

To derive the equations for lateral displacement, lateral velocity, perturbation pressure, and perturbation density, we may consider the momentum equation in the \( y \)-direction, the hydrostatic equation and the thermodynamic equation:

\[
U \frac{\partial v}{\partial x} - \nu v = -\frac{\rho g}{\rho_0} \frac{\partial}{\partial z} \left[ \frac{N^2}{\kappa^2} \right] + \frac{\partial}{\partial z} \left( \frac{\partial p}{\partial z} \right) - \frac{\partial}{\partial z} \left( \frac{\partial p}{\partial z} \right)
\]

\[\text{(B1)}\]

Making the double Fourier transforms of the above equations, we have

\[
\hat{p} (kU - i\nu) = -i\hat{p}
\]

\[\text{(B4)}\]

\[
\hat{\rho} = -\hat{\rho}_z
\]

\[\text{(B5)}\]

\[
i(kU - i\nu) \hat{\rho} - (\hat{\rho} N^2/\kappa^2) \hat{w} = -\left( \frac{\partial}{\partial z} \frac{\partial p}{\partial z} \right)
\]

\[\text{(B6)}\]

With the relationships of \( w = U \eta \delta/\partial x \) and \( v = U \partial \xi/\partial x \), these equations may be reduced to obtain

\[
\hat{\xi} = \frac{N^2}{ik(kU - i\nu)} \hat{\eta} + \frac{gl}{C_p \frac{T}{T(kU - i\nu)^2}} \hat{\eta}
\]

\[\text{(B7)}\]

In the heating layer for flow over a deep heat source, \( \xi \) may be integrated from \( z \) to \( +\infty \),

\[
\hat{\xi} = \frac{N^2}{ik(kU - i\nu)} \int_z^{\infty} \eta(k, l, z) dz' + \int_z^{\infty} \eta(k, l, z') dz'
\]

\[\text{(B8)}\]

Assuming the vertical displacement at infinity is zero and substituting Eq. (10) into (B8), we have

\[
\hat{\xi} = \frac{N^2}{ik(kU - i\nu)} \left[ \cos \lambda z \cos \kappa z - \sin \lambda z \sin \kappa z \right].
\]

\[\text{(B9)}\]

Notice that Eq. (B9) is nondimensionalized. The relationship between \( v \) and \( \xi \) can be derived by \( v = U \partial \xi/\partial x \), which leads to the nondimensional form

\[
\hat{v} = \frac{i\lambda z}{\kappa(kU - i\nu)}.
\]

\[\text{(B10)}\]

The nondimensional pressure field can be obtained by Eqs. (B4) and (B10)

\[
\hat{p} = \frac{\partial}{\partial z} \frac{\partial p}{\partial z}
\]

\[\text{(B11)}\]

Above the heating layer, there exists no thermal forcing. Thus the homogeneous part of the thermodynamic equation [Eq. (B6)] may lead to

\[
\hat{p} = \left( \frac{\partial}{\partial z} \frac{\partial p}{\partial z} \right)
\]

\[\text{(B12)}\]

If the viscous effect is neglected, the above equation reduces to Eq. (13).

REFERENCES


